On two theorems of A. Abian

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ON TWO THEOREMS OF A. ABIAN

Dedicated to Professor Kiiti Morita on his 60th birthday

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A (non-zero) ring without non-zero nilpotent elements is called a reduced ring. Recently, in his papers [1] and [2], A. Abian proved the following:

(I) A commutative reduced ring is a direct product of fields if and only if it is orthogonally complete and hyperatomic.

(II) A commutative reduced ring is a direct product of integral domains if it is orthogonally complete and superatomic.

In this paper, we shall prove that both these are still true for non-commutative reduced rings, more precisely,

Theorem 1. The following conditions are equivalent:

(1) $R$ is a reduced ring which is orthogonally complete and hyperatomic.

(2) $R$ is a direct product of division rings.

Theorem 2. The following conditions are equivalent:

(1) $R$ is a reduced ring which is orthogonally complete and superatomic.

(2) $R$ is a direct product of integral domains and the annihilators of those integral domains exhaust the proper prime ideals of $R$.

Although Theorem 1 has been obtained in [4] and our proofs of Theorems 1 and 2 are very similar to those of (I) and (II) in [1] and [2] respectively, we are much more skilful in performing those.

1. Definitions and lemmas. In a reduced ring, as is well-known, the intersection of prime ideals equals 0, namely, every non-zero element is excluded by some prime ideal (see for instance [5, p. 56]), and every idempotent is central. In what follows, $R$ will represent always a reduced ring.

Lemma 1. Let $r$ and $s$ be elements of a reduced ring $R$.

(a) If $rs = 0$, then $sr = 0$, and for every prime ideal $P$ of $R$ either $r$ or $s$ is contained in $P$.

(b) If $r^2s = r$, then there exists one and only one element $r'$ such that $rr' = r'r$, $r'r' = r$ and $rr'^2 = r'$. ($r'$ will be called the semi-inverse
of \( r \).

**Proof.** (a) \( sr = 0 \) is clear by \((sr)^2 = 0\). Moreover, \( srR = 0 \) yields \( rRs = 0 \). Hence, \( r \in P \) or \( s \in P \).

(b) This is only a consequence of [3, Lemma 1 and Theorem 1]. However, for the sake of completeness, we give here the proof. Evidently, \((rsr - r)^2 = 0\), and so \( rsr = r \). By making use of this, we have \((sr^2 - r)^2 = 0\), which implies \( sr^2 = r \) and \( sr = sr^2 s = rs \). Then, \( r^2 = rs^2 \) satisfies the relations requested. Now, let \( r^2 r' = r'' r' \), \( r^2 r'' = r \) and \( r^2 r'' = r'' \). Then, \( r^2 r' = r = r^2 r'' \) implies \( r^2 r'' = r^2 r^2 r' = r^2 r' \). Hence, \( r^2 = r^2 r = r^2 r' = r^2 r'' = r'' = r' \).

Now, for \( x, y \in R \) we define \( x \leq y \) if and only if \( xy = x^2 \) (and \( yx = x^2 \) by Lemma 1 (a)). Then, the relation \( \leq \) is a partial order in \( R \). In fact, the reflexivity and the antisymmetry are easy, and the transitivity can be seen as follows: If \( xy = x^2 \) and \( yz = y^2 \) then \( x^2 z = xy = x^2 y = x^2 \), i.e., \( x(xz - x^2) = 0 \), which implies \((xz - x^2)x = 0\) (Lemma 1 (a)). Hence, \((xz - x^2)^2 = 0\), and eventually \( xz = x^2 \).

Following [1], \( R \) is defined to be **orthogonally complete** if for every orthogonal subset \( T \) of \( R \) (i.e., a subset \( T \) such that \( st = 0 \) for every different \( s, t \in T \) there exists \( \sup T \) with respect to \( \leq \) mentioned above. A non-zero element \( a \in R \) is called an **atom** of \( R \) if \( x \leq a \) implies \( x = 0 \) or \( x = a \). An atom \( a \) is called a **hyperatom** if \( ax \neq 0 \) (i.e., \( x \in R \)) implies always \( ax = a \) for some \( x \in R \), and \( H \) will denote the set of all hyperatoms of \( R \). \( R \) is defined to be **hyperatomic** if for every non-zero element \( r \in R \) there exists \( a \in H \) such that \( a \leq r \). Finally, an element \( a \in R \) is called a **superatom** if \( a \) is contained in every proper prime ideal except exactly one \( P(a) \), and \( S \) will denote the set of all superatoms of \( R \). (In [2], a superatom in our sense is called an atom.) Obviously, if \( a \) is in \( S \) then \( a \) is non-zero and \(-a \) is a superatom with \( P(-a) = P(a) \). \( R \) is defined to be **superatomic** if for every proper prime ideal \( P \) and every element \( r \in R \setminus P \) there exists \( a \in S \) such that \( a \in R \setminus P \) and \( a \leq r \).

**Lemma 2.** If \( a \) is a hyperatom of a reduced ring \( R \) then \( aa' \) is an idempotent hyperatom, where \( a' \) is the semi-inverse of \( a \).

**Proof.** Since \( a \in H \) and \( aa' \neq 0 \), by Lemma 1 (b) \( a \) has the semi-inverse \( a' \) and \( e = aa' \) is a central idempotent. If \( 0 \neq er = aa'r \) (\( r \in R \)) then \( a(a'r)t = a \) with some \( t \in R \), and so \( (er)(ta') = e \). It remains therefor to show that \( e \) is an atom. Assume that \( x \leq e \), i.e., \( ex = x^2 \). Then, \( ex^2 = e(ex) = x^2 \), which yields \((ex - x)^2 = 0 \). Hence,
$x = ex = x^2$. Recalling that $x$ is then central, we have $xa \leq a$. Accordingly, $(xa)a' = xe = x$ is either 0 or $aa' = e$. This proves that $e$ is an atom.

Now, let $E = \{e_r : r \in \Gamma\}$ be the set of all idempotent hyperatoms of $R$. We claim that $A_r = e_r R$ is a division ring. In fact, $e_r$ is the identity of $A_r$, and for every non-zero element $e_r r (r \in R)$ there exists an element $s \in R$ such that $e_r = e_r r s = (e_r r) (e_r s)$ (Lemma 2).

**Lemma 3.** If $R$ is a hyperatomic reduced ring, then for every non-zero element $r \in R$ there holds $r E \neq 0$.

**Proof.** By hypothesis, $ra = a^2$ with some $a \in H$. Then, by Lemma 2, $a a' \in E$ and $r a a' = a^2 a' = a \neq 0$, where $a'$ is the semi-inverse of $a$.

**Lemma 4.** Let $R$ be a reduced ring.

(a) Let $a \in S$, and $r \in R$. If $ar \neq 0$ then $ar, ra \in S$ and $P(ar) = P(ra) = P(a)$.

(b) Let $a, b \in S$. Then, $ab \neq 0$ if and only if $P(a) = P(b)$.

(c) Let $a, b \in S$. If $ab \neq 0$ and $a - b \neq 0$ then $a - b \in S$ and $P(a - b) = P(a)$.

**Proof.** (a) Immediately, $ar$ is contained in every proper prime ideal different from $P(a)$. On the other hand, there exists a proper prime ideal excluding $ar$. Hence, $ar \in S$ and $P(ar) = P(a)$. Similarly, by Lemma 1 (a) we see that $ra \in S$ and $P(ra) = P(a)$.

(b), (c) If $ab \neq 0$ then $P(a) = P(ab) = P(b)$ by (a), and $a - b$ is contained in every prime ideal different from $P(a)$. Hence, in case $a - b \neq 0$, $a - b \in S$ and $P(a - b) = P(a)$. Conversely, assume that $P(a) = P(b)$. If $ab = 0$ then by Lemma 1 (a) $a \in P(a)$ or $b \in P(b)$, a contradiction.

**Corollary 1.** In a reduced ring $R$, every superatom $a$ is an atom.

**Proof.** Assume that $x \leq a$ and $x \neq 0$. By Lemma 4 (a), $xa = x^2 \in S$ and $P(xa) = P(a)$, whence it follows $x \in P(a)$. Hence, $x(a - x) = 0$ implies $a - x \in P(a)$ (Lemma 1 (a)), and so $a(a - x) \in P(a)$. On the other hand, $a(a - x)$ is contained in every prime ideal different from $P(a)$. We obtain therefore $a(a - x) = 0$. Combining this with $x(a - x) = 0$, we readily obtain $(x - a)^2 = 0$, and hence $x = a$.

In virtue of Lemma 4 (b), we can define an equivalence relation $\sim$ in $S$, where $a \sim b$ if and only if $ab \neq 0$, or equivalently, $P(a) = P(b)$. Let
S = \bigcup_{\lambda \in I} S_{\lambda} be the partition of S into the equivalence classes with respect to \( \sim \). Obviously, \( S_{\lambda} \mapsto P_{\lambda} = P(a) \ (a \in S_{\lambda}) \) is well-defined, and \( S_{\lambda} \cap P_{\lambda} = \emptyset \). As a direct consequence of Lemma 4, we see that \( B_{i} = S_{i} \cup \{0\} \) is an ideal of \( R \) which is an integral domain and \( B_{i}P_{j} = 0 \).

In the rest of this section, we assume further that \( R \) is superatomic. Then, \( S_{i} \mapsto P_{i} \) gives a 1-1 correspondence between \( \{S_{\lambda} \mid \lambda \in \Lambda\} \) and the set of all proper prime ideals of \( R \). If \( r \) is in \( R \setminus P_{i} \), then by hypothesis there exists some \( r_{i} \in S_{i} \) with \( r_{i} \leq r \). We claim here that such \( r_{i} \) is unique. In fact, if \( \tilde{r}_{i} \leq r \) and \( \tilde{r}_{i} \in S_{i} \) then \( (r - r_{i})r_{i} = 0 = (r - \tilde{r}_{i})\tilde{r}_{i} \) implies \( r - r_{i}, \ r - \tilde{r}_{i} \in P_{i} \) (Lemma 1 (a)). Hence, \( \tilde{r}_{i} - r_{i} \in P_{i} \cap B_{i} = 0 \), namely, \( \tilde{r}_{i} = r_{i} \). On the other hand, if \( r \) is in \( P_{i} \), then there is no \( r_{i} \in S_{i} \) with \( r_{i} \leq r \). We define here the map \( g_{i} : R \rightarrow B_{i} \) by

\[
\begin{align*}
g_{i}(r) &= \begin{cases} r_{i} & \text{if } r \notin P_{i} \\ 0 & \text{if } r \in P_{i} \end{cases}
\end{align*}
\]

Lemma 5. Let \( R \) be a superatomic reduced ring. Then, \( g_{i} \) is a ring homomorphism leaving every element of the integral domain \( B_{i} \) invariant and \( \ker g_{\lambda} = P_{i} \). Accordingly, \( R = P_{i} \oplus B_{i} \) and \( P_{i} \) coincides with the annihilator of \( B_{i} \).

Proof. Since \( R \) is a reduced ring, the (right and left) annihilator of \( B_{i} \) has the intersection \( 0 \) with \( B_{i} \). It remains therefore to prove (i) \( g_{i}(r + s) = g_{i}(r) + g_{i}(s) \) and (ii) \( g_{i}(rs) = g_{i}(r)g_{i}(s) \) \( r, s \in R \). First, we consider the case \( r \in P_{\lambda} \) and \( s \in P_{\mu} \). Since \( s_{\lambda}r_{i} = 0 \), we obtain \( s_{\lambda}(r + s) = s_{\lambda}s_{\lambda} = s_{\lambda}r_{i} = s_{\lambda}r_{i} \), which means \( r + s \in S_{\lambda} \). Hence, we have (i), and readily (ii). Next, we consider the case \( r \notin P_{i} \) and \( s \notin P_{i} \). We claim that \( rs_{\lambda} = rs_{\lambda} \) and \( sr_{\lambda} = sr_{\lambda} \). In fact, by \( r_{i} \leq r \) and \( s_{\lambda} \leq s \) it follows \( r_{i}s_{\lambda} = r_{i}s_{\lambda} \). Since \( B_{i} \) is an integral domain, we have \( rs_{\lambda} = rs_{\lambda} \), and similarly \( sr_{\lambda} = sr_{\lambda} \). Hence, we have \( (rs)(rs_{\lambda}) = rs_{\lambda}, \ (rs_{\lambda}) = (rs_{\lambda})^{2} \), namely \( (rs) = (rs_{\lambda})^{2} \), proving (ii). (i) In order to see (i), we shall distinguish between two cases. (1) \( r + s \notin P_{\lambda} \). If \( r_{\lambda} + s_{\lambda} = 0 \) then \( r_{\lambda}r_{\lambda} + s_{\lambda}s_{\lambda} = r_{\lambda}s_{\lambda} = -r_{\lambda}s_{\lambda} \), i.e., \( r_{\lambda}(r + s) = 0 \), whence it follows \( r_{\lambda} \in P_{\lambda} \) or \( r + s \in P_{\mu} \) (Lemma 1 (a)). This contradiction means that \( r_{\lambda} + s_{\lambda} \in S_{\text{lem}} \) (Lemma 4). Since \( (r + s)(r_{\lambda} + s_{\lambda}) = rr_{\lambda} + rs_{\lambda} + s_{\lambda}r_{\lambda} + s_{\lambda}s_{\lambda} = r_{\lambda}^{2} + s_{\lambda}r_{\lambda} + r_{\lambda}s_{\lambda} + s_{\lambda}^{2} = (r_{\lambda} + s_{\lambda})^{2} \), we have \( (r + s) = (r_{\lambda} + s_{\lambda})^{2} \), proving (i). (2) \( r + s \in P_{\lambda} \). Since \( 0 = (r + s)(r_{\lambda} = r_{\lambda}^{2} + s_{\lambda}r_{\lambda} = r_{\lambda}^{2} + s_{\lambda}r_{\lambda} \) and \( 0 = (r + s)s_{\lambda} = rs_{\lambda} + s_{\lambda}^{2} = rs_{\lambda} + s_{\lambda}^{2} \), we obtain \( (r_{\lambda} + s_{\lambda})^{2} = 0 \), and so \( r_{\lambda} + s_{\lambda} = 0 \), proving (i). Finally, in case \( r \in P_{\lambda} \) and \( s \in P_{\lambda} \), there is nothing to prove.

2. Proofs of theorems. The notations employed in the preceding section will be used here.
Proof of Theorem 1. (1) \(\implies\) (2): Let \(f: R \to \prod_{r \in \Gamma} A_r\) be the map defined by \(f(r) = (re_r)\). Then, \(f\) is a ring homomorphism, and by Lemma 3, \(\text{Ker } f = \{r \in R \mid rE = 0\} = 0\). If \(f\) is shown to be an isomorphism, \(A_r (r \in \Gamma)\) are adapted for the division rings in (2). Now, let \((r')\) be an arbitrary element of \(\prod_{r \in \Gamma} A_r\). By \(e_r e_t = e_r\) and \(e_e\), we can easily see that \(e_r e_t = 0\) for every \(r \neq t\). Hence, \(T = \{r' \mid r \in \Gamma\}\) is an orthogonal subset of \(R\) and there exists \(r = \sup T\). We shall prove now \(r e_r = r'\) for every \(r \in \Gamma\). By \(r^t \leq r\), we obtain \(r'r e_r = (r^t)^2\), i.e., \(r^t \leq re_r\). On the other hand, to be easily seen, \(r'(r' - re_r + r) = (r^t)^2\), i.e., \(r' \leq r' - re_r + r\) for every \(r \in \Gamma\). Hence, \(r \leq r' - re_r + r\), and so \((r^t - re_r + r) r = r^t\), whence it follows \(r^t(re_r) = r^t r = r^t e_r = (re_r)^2\). Combining this with \(r^t \leq re_r\), we obtain \(re_r = r^t\).

(2) \(\implies\) (1): Let \(R\) be the direct product of division rings \(R_\alpha (\alpha \in K)\). Then, it is clear that \(R\) is orthogonally complete. If \(x = (x^t)\) is an arbitrary non-zero element of \(R\), then there exists \(\alpha \in K\) with \(x^\alpha \neq 0\). Then, we can easily see that \(x^t\) is a hyperatom and \(x^t \leq x\).

Proof of Theorem 2. (1) \(\implies\) (2): Let \(g: R \to \prod_{\lambda \in \Lambda} B_\lambda\) be the map defined by \(g(r) = (g_\lambda(r))\). Then, by Lemma 5, \(g\) is a ring homomorphism with \(\text{Ker } g = \bigcap_{\lambda \in \Lambda} \text{Ker } g_\lambda = \bigcap_{\lambda \in \Lambda} P_\lambda = 0\), and \(P_\lambda\) coincides with the annihilator of the integral domain \(B_\lambda\). If \(g\) is shown to be an isomorphism, \(B_\lambda (\lambda \in \Lambda)\) are adapted for the integral domains in (2). Now, we shall show that \(g\) is a surjection. Let \((r^t)\) be an arbitrary non-zero element of \(\prod_{\lambda \in \Lambda} B_\lambda\), \(N = \{\lambda \in \Lambda \mid r^\lambda = 0\}\), and \(M = \Lambda \setminus N\). Since the set \(T = \{r^\lambda \mid \lambda \in M\}\) is an orthogonal subset of \(R\), by hypothesis there exists \(r = \sup T\). To our end, it suffices to show that \(r \in P_\lambda\) if and only if \(\lambda \in N\). Assume first that \(r \in P_\lambda\). If \(\lambda \in M\), then \(r^\lambda = r_\lambda \in S_\lambda\), but then \(r \lambda = r_\lambda = 0\), a contradiction. Conversely, assume that \(\lambda \in N\). If \(\lambda \in P_\lambda\), then \(r^\lambda r_\lambda = 0 = r_\lambda r^\mu\) for every \(\mu \in M\). Hence, \(r^\mu (r_\lambda + r) = r^\mu r = (r^\mu)^2\), namely, \(r^\mu \leq r_\lambda + r\) for every \(\mu \in M\). This implies \(r \leq r_\lambda + r\), and so \(r (r_\lambda + r) = r^2\). However, the last contradicts \(r (r_\lambda + r) = r^2 + r^3\).

(2) \(\implies\) (1): Assume that \(R\) is the direct product of the integral domains \(R_\alpha (\alpha \in K)\) and \(\prod_{\alpha \in K} R_\alpha\) exhaust the proper prime ideals of \(R\). Then, \(R\) is orthogonally complete evidently. Moreover, if \(x = (x^t)\) is an arbitrary element of \(\prod_{\alpha \in K} R_\alpha\), then we can easily see that \(x^t\) is a superatom of \(R\) not contained in any of \(\prod_{\alpha \in K} R_\alpha\) and \(x^t \leq x\).

Corollary 2. If \(R\) is a reduced ring with 1 which is orthogonally
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**complete and superatomic, then** $R$ is a finite direct sum of integral domains.

**Proof.** In any rate, by the proof of Theorem 2, $R = \prod_{\lambda \in \Lambda} B_{\lambda}$ and $\prod_{\lambda \in \Lambda} B_{\lambda} (\alpha \in \Lambda)$ exhaust the proper prime ideals of $R$. If $\Lambda$ is infinite, then the proper ideal $\bigoplus_{\lambda \in \Lambda} B_{\lambda}$ is contained in some maximal ideal, which is a contradiction.

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**Added in proof.** A recent result of O. Goldman [J. Algebra 34 (1975), 64—73] enables us to see that the following condition is equivalent to those in Theorem 1:

(3) $R$ is a reduced ring with 1 which is complete in its intrinsic topology.