On coprimary decomposition theory for modules

Isao Mogami*  Hisao Tominaga†
ON COPRIMARY DECOMPOSITION THEORY FOR MODULES

ISAO MOGAMI and HISAO TOMINAGA

Recently, in his paper [2], D. Kirby introduced the notion of coprimary modules over a commutative ring, and obtained several results on coprimary decompositions for Artinian modules. In this note, by making use of the technique employed in [1] and [3], we shall investigate the s-coprimary decomposition theory for modules over non-commutative rings.

1. Preliminaries. Throughout, \( R \) will represent a ring, and \( M \) a non-zero left \( R \)-module. Given an ideal \( \alpha \) of \( R \), \( M^\alpha \) is defined to be the intersection \( \cap bM \), where \( b \) runs over all the finite products of ideals of \( R \) not contained in \( \alpha \). (\( M^\emptyset = M \) by definition.) As in [3], \( p(M) \) will denote the prime radical of \( l(M) = \{ x \in R | xM = 0 \} \). If \( l(M') \subseteq p(M) \), or equivalently \( p(M') = p(M) \), for every non-zero submodule \( M' \) of \( M \), 0 is defined to be a primary submodule of \( M \) (cf. [1]). Now, dualizing the notion, \( M \) is defined to be coprimary if \( l(M/M') \subseteq p(M) \), or equivalently \( p(M/M') = p(M) \), for every proper submodule \( M' \) of \( M \). In case \( M \) is coprimary and \( p = p(M) \), \( M \) will be called a \( p \)-coprimary module. If \( M \) is coprimary and \( p(M) \) is nilpotent modulo \( l(M) \), \( M \) is defined to be s-coprimary.

The next is easy, and will be freely used without mention.

Proposition 1. The following conditions are equivalent:

1. \( M \) is coprimary.
2. \( \alpha M = M \) for every ideal \( \alpha \) of \( R \) not contained in \( p(M) \).
3. \( M^{\alpha(M)} = M \).

An ideal \( \alpha \) of \( R \) is called a coassociated ideal of \( M \) if there exists a proper submodule \( M' \) such that \( M/M' \) is \( \alpha \)-coprimary. The set of all coassociated ideals of \( M \) will be denoted by \( P^*(M) \). (\( P^*(0) = \emptyset \) by definition.) If there exists an ideal \( \alpha \) in \( R \) such that \( P^*(M/M') = \{ \alpha \} \) for every proper submodule \( M' \) of \( M \) then \( M \) is called a \( P^*-module \).

Proposition 2. (1) If \( M \) is coprimary, and \( M' \) a proper submodule of \( M \), then \( l(M/M') \) is a right-primary ideal.

(2) Let \( N \) and \( M' \) be submodules of \( M \). If \( N \) is \( p \)-coprimary and not contained in \( M' \) then \( N + M' \) is \( p \)-coprimary.

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(3) If $N$ and $N'$ are $\wp$-coprimary submodules of $M$, then so is $N+N'$.

Proof. (1) Assume that there exist ideals $a, b$ of $R$ such that $ab \subseteq \iota(M/M')$ and $b \nsubseteq p(M/M')$. Then, $M' \supseteq abM = aM$, namely, $a \subseteq \iota(M/M')$.

(2) This is obvious by $N+M'/M' \cong N/N \cap M'$.

(3) Since $\iota(N+N') = \iota(N) \cap \iota(N')$, we have $p(N+N') = p(N) \cap p(N') = p$. If $a$ is an ideal of $R$ not contained in $p$, then $aN = N$ and $aN' = N'$, and hence $a(N+N') = N+N'$.

Proposition 3. (1) If $M$ is $\wp$-s-coprime then $p$ is prime and $aM \not= M$ for every ideal $a$ of $R$ contained in $p$.

(2) Let $N$ and $M'$ be submodules of $M$. If $N$ is $\wp$-s-coprime and is not contained in $M'$ then $M+M'/M'$ is $\wp$-s-coprime.

(3) If $N$ and $N'$ are $\wp$-s-coprime, then so is $N+N'$.

Proof. (2) and (3) are easy by Prop. 2 (2) and (3).

(1) If $a$ is an ideal of $R$ contained in $p$ then there exists a positive integer $h$ such that $a^hM = 0$, which means $aM \not= M$. Next, we shall prove that $p$ is prime. Let $b, c$ be ideals of $R$ such that $bc \subseteq p$. As was shown just above, there holds $bcM \not= M$. If $c \not\subseteq p$, then $M \supseteq bcM = bM$, and hence $b \subseteq p$.

Proposition 4. If $N$ is a submodule of $M$ then $P^*(M/N) \subseteq P^*(M) \subseteq P^*(N) \cup P^*(M/N)$.

Proof. Let $S$ be a proper submodule of $M$ such that $M/S$ is $\wp$-coprimary. If $S+N \not= M$ then $M/S+N$ is $\wp$-coprimary and $p \subseteq P^*(M/N)$. On the other hand, if $S+N = M$ then $N/N \cap S \subseteq M/S$ is $\wp$-coprimary and $p \subseteq P^*(N)$. The inclusion $P^*(M/N) \subseteq P^*(M)$ is almost evident.

2. Coprimary decompositions. A finite set $\{M_i | i \in I\}$ of coprimary (resp. s-coprimary) submodules of $M$ is called a coprimary (resp. s-coprimary) decomposition of $M$ if $M = \sum_{i \in I} M_i$, $M \not= \sum_{i \in I'} M_i$ for every proper subset $I'$ of $I$, and $\varphi(M_i) \not= \varphi(M_j)$ for every $i \not= j$. If $\{N_j | j \in J\}$ is a finite set of coprimary (resp. s-coprimary) submodules of $M$ with $M = \sum_{j \in J} N_j$, then Prop. 2 (3) (resp. Prop. 3 (3)) secures the existence of a coprimary (resp. s-coprimary) decomposition of $M$.

Proposition 5. Let $\{M_i | i = 1, \cdots, k\}$ be an s-coprimary decomposition of $M$, and $\wp_i = \varphi(M_i)$ $(i = 1, \cdots, k)$. Let $a$ be an ideal of $R$.

(1) If $aM = M$ then $a \nsubseteq \wp_i$ $(i = 1, \cdots, k)$, and conversely.

(2) $M^a = \sum_{i \in a} M_i$. If $a$ does not contain all $\wp_i$'s then $M^a = bM$ with a finite product $b$ of ideals of $R$ not contained in $a$. 

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Proof. (1) If \( a \) is contained in some \( p_i \), then \( a^\infty M_i = 0 \) for some positive integer \( h \). Accordingly, we have \( a^\infty M = M \), whence it follows \( a M = M \). The converse is obvious.

(2) Without loss of generality, we may assume that \( p_1, \ldots, p_i \subseteq a \) and \( p_{i+1}, \ldots, p_k \not\subseteq a \). In case \( l = k \), our assertion is evident by (1). Henceforth, we assume that \( 0 < l < k \). There exists a positive integer \( h \) such that \( p_j^h M_i = 0 \) \((j = l + 1, \ldots, k)\). Since every \( p_i \) is prime by Prop. 3 (1), \( b = (p_{i+1} \cdots p_k)^h \not\subseteq p_i \) \((i = 1, \ldots, l)\). There holds then \( M^a = \sum_{i=1}^l M_i^a = \sum_{i=1}^l M_i p_i = \sum_{i=1}^l M_i = b M \supseteq M^a \), namely, \( M = \sum_{i=1}^l M_i = M \).

Theorem 1. Let \( \{M_i | i = 1, \ldots, k \} \) be an \( s \)-coprimary decomposition of \( M \), and \( p_i = p(M_i) \) \((i = 1, \ldots, k)\). Then there holds the following:

1. \( P^*(M) = \{p, \ldots, p_k\} \).

2. A prime divisor \( p \) of \( l(M) \) is contained in \( P^*(M) \) if and only if \( p M^p \not\subseteq M^p \). Every minimal prime divisor of \( l(M) \) is contained in \( P^*(M) \), and if \( p_i \) is minimal in \( P^*(M) \) then \( M^{p_i} = M_i \).

Proof. (1) Evidently, \( p(M) \) is nilpotent modulo \( l(M) \). Next, we claim that if \( M \) is \( s \)-coprimary then \( k = 1 \). Since \( p = p(M) = \cap_{i=1}^k p_i \) is prime by Prop. 3 (1), without loss of generality, we may assume that \( p_1, \ldots, p_m \subseteq p \) \((m > 1)\) and \( p_{m+1}, \ldots, p_k \not\subseteq p \). Then, by Prop. 5, \( M = M^p = \sum_{i=1}^m M_i \), whence it follows \( m = k \). Combining this with \( p = \cap_{i=1}^k p_i \), we obtain \( k = 1 \).

Now, we shall proceed into the proof of (1). Obviously, \( M = \sum_{i=1}^k M_i \) is \( p \)-coprimary as a non-zero homomorphic image of \( M_0 \), and so \( P^*(M) \supseteq \{p_i, \ldots, p_k\} \). Conversely, assume that \( M/N \) is \( p \)-coprimary. Then, \( M/N = \sum_{j=1}^m (M_j + N)/N \), where \( (M_j + N)/N \) is either 0 or \( p \)-coprimary by Prop. 3 (2). Accordingly, by Prop. 5 (3), \( M/N \) has an \( s \)-coprimary decomposition \( \{M_j/N | j = 1, \ldots, l\} \) such that \( p(M_j/N) | j = 1, \ldots, l \subseteq \{p_i, \ldots, p_k\} \).

Then, as was mentioned above, we obtain \( l = 1 \) and \( p \subseteq \{p_i, \ldots, p_k\} \).

(2) If \( p \) is contained in \( P^*(M) = \{p_i, \ldots, p_k\} \), then we may assume that \( p_i, \ldots, p_m \subseteq p = p_m \) and \( p_{m+1}, \ldots, p_k \not\subseteq p \). Then, \( M^p = M_1 + \cdots + M_m \not\subseteq M^p \) by Prop. 5. Next, we shall prove the converse. Since \( p \supseteq \cap_{i=1}^m p_i \), we may assume that \( p_i, \ldots, p_m \subseteq p \) \((m > 1)\) and \( p_{m+1}, \ldots, p_k \not\subseteq p \). If \( p \) is a minimal prime divisor of \( l(M) \) then it is obviously in \( P^*(M) \).) Since \( M^p \not\subseteq P^*(M) \), we obtain \( \sum_{i=1}^m M_i \not\subseteq p(\sum_{i=1}^m M_i) \) by Prop. 5 (2), and hence \( p \subseteq p_i \), namely, \( p = p_i \), for some \( i < m \) (Prop. 5 (1)). The final assertion is evident by Prop. 5 (2).

Now, let \( \{M_i | i = 1, \ldots, k\} \) be an \( s \)-coprimary decomposition of \( M \). A subset \( P^* \) of \( \{p_i = p(M_i) | i = 1, \ldots, k\} \) is called an isolated subset of \( \{p_i | i = 1, \ldots, k\} \) if every \( p_i \) contained in one of the members of \( P^* \) belongs to \( P^* \). For an isolated subset \( P^* \) of \( \{p_i | i = 1, \ldots, k\} \) we set \( M^{P^*} = \sum_{p_i \in P^*} M_i \).
which coincides with $\sum_{p \in p^*} M^p$ by Prop. 5 (2) and is called a coisolated component of $M$. By Th. 1, we readily obtain the following:

**Theorem 2.** Suppose that $M$ has an $s$-coprimary decomposition. Then, the set of coisolated components of $M$ does not depend on the choice of $s$-coprimary decompositions of $M$.

Finally, we shall examine cases in which every $s$-coprimary decomposition is direct.

**Theorem 3.** Suppose $R$ contains 1 and $M$ is unital. Let $\{M_i \mid i = 1, \ldots, k\}$ be a finite set of $s$-coprimary submodules of $M$ such that $M = \sum_{i=1}^k M_i$ and $(R \neq 0) \implies p_i = p(M_i)$ $(i = 1, \ldots, k)$. If $p_i$'s are pairwise comaximal, then $M = \bigoplus_{i=1}^k M_i$ and this is the unique $s$-coprimary decomposition of $M$.

**Proof.** Since $p_i$'s are comaximal, so are $l(M_i)$'s, and so $l(M_i) + l(\sum_{j \neq i} M_j) = R$. Hence, $M_i = (l(M_i) + l(\sum_{j \neq i} M_j))(M_i \cap \sum_{j \neq i} M_j) = 0$, which means $M = \bigoplus_{i=1}^k M_i$. Obviously, the last is an $s$-coprimary decomposition of $M$ and $P^*(M) = \{p_1, \ldots, p_k\}$ by Th. 1. Further, every $p_i$ is minimal in $P^*(M)$ and $M_i = M^{p_i}$ by Th. 1 (2), which means the uniqueness of the $s$-coprimary decompositions.

**Corollary.** Let $R$ be a left Artinian ring with 1. If $M$ is a completely reducible module with a finite number of homogeneous components, then the idealistic decomposition of $M$ is the unique $s$-coprimary decomposition of $M$.

**Proof.** If $N$ is an arbitrary irreducible submodule of $M$ then $l(N) = p(N)$ is a maximal ideal of $R$ and $N$ is isomorphic to a minimal left ideal of $R/l(N)$. We have seen therefore that if $N'$ is another irreducible submodule of $M$ non-isomorphic to $N$ then $p(N') \neq p(N)$. Further, to be easily seen, the homogeneous component of $M$ containing $N$ is $p(N)$-s-coprimary. Now, our assertion is a consequence of Th. 3.

3. **Coprimary decomposition theory and $AR^*$-modules.** When every non-zero submodule of $M$ has a coprimary (resp. $s$-coprimary) decomposition, $M$ is said to have the coprimary (resp. $s$-coprimary) decomposition theory. In case $M$ has the coprimary (resp. $s$-coprimary) decomposition theory, every non-zero factor submodule of $M$ has a coprimary (resp. $s$-coprimary) decomposition by Prop. 2 (resp. Prop. 3), and if $N$ is a primary submodule of $M$ then $M/N$ is coprimary. Conversely, in case $M$ has the primary decomposition theory, if $M/N$ is coprimary then $N$ is primary. (Cf. [3].)

If $M$ satisfies one of the following equivalent conditions (I) and (II),
it is called an $AR^*$-module:

(1) For each submodule $N$ of $M$ and each ideal $a$ of $R$, there exists a positive integer $h$ such that $N + a^{-h}0 \supseteq a^{-1}N (= \{ x \in M | ax \subseteq N \})$.

(2) For each submodule $N$ of $M$ and each ideal $a$ of $R$, there exists a positive integer $h$ such that $aN + (a^{-h}0 \cap N) = N$.

One may remark here that if $M$ is an $AR^*$-module, then so is every non-zero factor submodule of $M$. Finally, $M$ is said to be $p^*$-worthy if $P^*(M^*)$ is finite and non-empty for every non-zero factor submodule $M^*$ of $M$.

**Proposition 6.** If $M$ has the $s$-coprimary decomposition theory, then there holds the following:

1. $M$ is an $s$-module, that is, $p(M^*)$ is nilpotent modulo $l(M^*)$ for every non-zero factor submodule $M^*$ of $M$.

2. For every submodule $N$ of $M$, if $N^0 \supseteq (N^0)^0 \supseteq \cdots \supseteq (\cdots (N^0)^0 \cdots)^0_n$ then $n \leq s(N)$ with a positive integer $s(N)$ depending solely on $N$.

3. $M$ is $p^*$-worthy.

4. $M$ is an $AR^*$-module.

**Proof.** (1)-(3) are easy by Props. 3 and 5 and Th. 1.

(4) It suffices to consider non-zero $N$. Let $\{ N_i \mid i = 1, \cdots, k \}$ be an $s$-coprimary decomposition of $N$. We may assume then $a \subseteq p(N^0), \cdots, p(N_k)$ and $a \not\subseteq p(N_{i+1}), \cdots, p(N_k)$. There exists a positive integer $h$ such that $a^hN_i = 0 (i = 1, \cdots, l)$. Since $N_1 + \cdots + N_l \subseteq a^{-h}0 \cap N$ and $N_{l+1} + \cdots + N_k \subseteq aN$, it follows $aN + (a^{-h}0 \cap N) = N$.

**Proposition 7.** Let $M$ be an $AR^*$-module and an $s$-module.

1. If $N$ is a $P^*$-submodule of $M$ then $N$ is $s$-coprimary.

2. If $M$ is Artinian, then $M$ has the $s$-coprimary decomposition theory.

**Proof.** (1) Let $N'$ be an arbitrary proper submodule of $N$. Since $P^*(N/N') = \{ p \}$, there exists a proper submodule $N''$ of $N$ containing $N'$ such that $N/N''$ is $p$-$s$-coprimary. Now, let $W$ be an arbitrary proper submodule of $N$, and choose a proper submodule $W'$ of $N$ containing $W$ such that $N/W'$ is $p$-$s$-coprimary. Since $l(N/N') \subseteq l(N/N'') \subseteq p$, Prop. 3 (1) yields $l(N/N')N + W' \subseteq N$, which means that $l(N/N')N$ is small in $N$. By the condition (I), there exists a positive integer $h$ such that $l(N/N')N + ((l(N/N'))^{-h}0 \cap N) = N$. It follows then $(l(N/N'))^{-h}0 \cap N = N$, namely, $l(N/N')^hN = 0$. This means that $l(N/N') \subseteq p(N)$, that is, $N$ is $s$-coprimary.

(2) Since every non-zero submodule of $M$ is a finite sum of sum-
irreducible submodules, it remains only to show that if a non-zero submodule \( N \) of \( M \) is not \( s \)-coprimary then \( N \) is not sum-irreducible. There exists a proper submodule \( N' \) of \( N \) such that \( a = l(N/N') \nsubseteq p(N) \), or \( a^s N \neq 0 \) for every positive integer \( n \). By the condition (II), there exists a positive integer \( h \) such that \( aN + (a^{-h}0 \cap N) = N \). It is obvious that \( aN \subseteq N' \subseteq N \) and \( a^{-h}0 \cap N \subseteq N \).

Combining Prop. 6 with Prop. 7, we obtain at once

**Theorem 4.** Let \( M \) be an Artinian module. In order that \( M \) have the \( s \)-coprimary decomposition theory, it is necessary and sufficient that \( M \) be an \( AR^* \)-module and an \( s \)-module.

In [2], D. Kirby has proved that every unital Artinian \( s \)-module over a commutative ring with 1 has the \( s \)-coprimary decomposition theory. However, the following example will show that it is not the case for non-commutative rings.

**Example.** Let \( R = \begin{pmatrix} F & 0 & 0 \\ F & F & 0 \\ F & F & F \end{pmatrix} \), where \( F \) is a field, and \( M \) the left \( R \)-module \( R \). To be easily seen, \( a = \begin{pmatrix} 0 & 0 & 0 \\ F & 0 & 0 \\ F & F & F \end{pmatrix} \) is an ideal of \( R \) and \( a^s = a \cdot \begin{pmatrix} 0 & 0 & 0 \\ F & 0 & 0 \\ F & F & F \end{pmatrix} \).

Moreover, \( a^{-2}0 \cap a = 0 \), and we have \( a \cdot a + (a^{-2}0 \cap a) \neq a \), which means that \( M \) is not an \( AR^* \)-module.

**References**


**Okayama University**

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