On a generalized p-vector curvature and its applications

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ON A GENERALIZED $p$-VECTOR CURVATURE AND ITS APPLICATIONS

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Introduction

Let $(M, g)$ be an $n$-dimensional Riemannian manifold with Riemannian metric $g$, and $R$ be its curvature tensor ($n \geq 3$). In this paper, we shall consider a special curvature structure of order $p$ given by

$$\phi_r = \frac{1}{r!} g^r \wedge \omega \quad (0 \leq r \leq n-2),$$

where $\omega$ is a curvature structure of order 2 and $p=r+2$. This was suggested to us by the work of S. Tachibana. Especially, if $\omega = R$, $\phi_r$ is nothing else but E. Cartan's notion of "$p$-vector curvature", which was formulated in the present form by R. S. Kulkarni.

In Theorem 1, we shall find a main property of this curvature structure. As simple application of this theorem, we shall give in Theorem 2 a sufficient condition for a Riemannian manifold with non-vanishing constant $2p^{th}$ sectional curvature to be of constant curvature in usual sense. In the last section, we shall study somewhat in detail the mean curvature $\rho$ for $p$-plane, which was introduced by Tachibana [7] in connection with the work in K. Yano and S. Bochner [10]. As second application of Theorem 1, we shall prove in Theorem 3 that this curvature $\rho$ generally determines the metric $g$ itself of $(M, g)$.

We shall assume, throughout this paper, that all manifolds are connected and all objects are of differentiability class $C^\infty$. For the terminology and notation, we generally follow [4].

1. Preliminaries on curvature structures

In this section, let us recall some basic facts on the ring of double forms for later use (for the details, see [4]).

Let $(M, g)$ be an $n$-dimensional smooth Riemannian manifold and let $\mathcal{F}(M)$ be the ring of smooth functions on $M$. Let $\Lambda^p(M)$ and $\Lambda^*(M)$ denote the bundles of $p$-vectors and of $p$-forms on $M$, respectively. For simplicity, we denote the space of sections of a bundle by the same notation as the bundle space. We consider the spaces

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\[ D^{p,q} = \Lambda^p(M) \otimes \Lambda^q(M), \quad 0 \leq p, q \leq n, \quad D = \sum_{p,q} D^{p,q}, \]

where the tensor product is taken over \( \mathcal{F}(M) \). An element \( \omega \) of \( D^{p,q} \) is an \( \mathcal{F}(M) \)-linear map \( \omega : \Lambda^p(M) \times \Lambda^q(M) \rightarrow \mathcal{F}(M) \) and the value of \( \omega \) on decomposable elements \( u = x_1 \wedge x_2 \wedge \cdots \wedge x_p \) and \( v = y_1 \wedge y_2 \wedge \cdots \wedge y_q \) is denoted by

\[ \omega(u \otimes v) = \omega(x_1, x_2, \cdots, x_p \otimes y_1, y_2, \cdots, y_q), \]

where \( x_1, \cdots, x_p, y_1, \cdots, y_q \) are vector fields on \( M \). \( D \) forms an associative ring with respect to the natural "exterior product" as follows: for \( \omega \in D^{p,q} \) and \( \theta \in D^{r,t} \), we define

\[ (\omega \wedge \theta)(x_1 \cdots x_{p+r} \otimes y_1 \cdots y_{q+t}) = \sum_{\tau \in S_{p+r}} \sum_{\mu \in S_{q+t}} e_\tau e_\mu \omega(x_{\tau(1)} \cdots x_{\tau(p+r)} \otimes y_{\mu(1)} \cdots y_{\mu(q+t)}) \theta(x_{\tau(p+1)} \cdots x_{\tau(p+r)} \otimes y_{\mu(q+1)} \cdots y_{\mu(q+t)}) \]

for any vector fields \( x_1, \cdots, x_{p+r}, y_1, \cdots, y_{q+t} \). Here, \( Sh(p, r) \) denotes the set of all \( (p, r) \)-shuffles

\[ Sh(p, r) = \{ \tau \in S_{p+r} ; \tau_1 < \cdots < \tau_p \quad and \quad \tau_{p+1} < \cdots < \tau_{p+r} \}, \]

where \( S_{p+r} \) is the symmetric group of degree \( p+r \). Then, we have

\[ \omega \wedge \theta = (-1)^{p+r} \theta \wedge \omega \]

for any \( \omega \in D^{p,q} \) and \( \theta \in D^{r,t} \). A symmetric element of \( D^{p,p} \) is called the curvature structure of order \( p \) and the set of such elements is denoted by \( \mathcal{C}^p \). \( \mathcal{C} = \sum \mathcal{C}^p \) is a commutative subring of \( D \) called the ring of curvature structures on \( M \).

The first Bianchi sum \( \mathcal{G} \) maps \( D^{p,q} \) into \( D^{p+1,q-1} \) and is defined as follows. Let \( \omega \in D^{p,q} \). If \( q = 0 \), we set \( \mathcal{G} \omega = 0 \). If \( q \geq 1 \), then we set

\[ \mathcal{G} \omega(x_1 \cdots x_{p+1} \otimes y_1 \cdots y_{q-1}) = \sum_{j=1}^{p+1} (-1)^j \omega(x_1 \cdots \hat{x}_j \cdots x_{p+1} \otimes x_j y_1 \cdots y_{q-1}) \]

for any vector fields \( x_1, \cdots, x_{p+1}, y_1, \cdots, y_{q-1} \), where as usual \( \wedge \) denotes omission. Then, for any \( \omega \in D^{p,q} \) and \( \theta \in D^{r,t} \) we have

\[ \mathcal{G}(\omega \wedge \theta) = \mathcal{G} \omega \wedge \theta + (-1)^{p+q} \omega \wedge \mathcal{G} \theta. \]

We define \( \mathcal{G}_1 = \mathcal{C}^p \cap \ker \mathcal{G} \) and set \( \mathcal{C}_1 = \sum \mathcal{G}_1 \). Then, owing to the above formula, \( \mathcal{C}_1 \) is a subring of \( \mathcal{C} \).

The contraction \( c \) maps \( D^{p,q} \) into \( D^{p-1,q-1} \) and is defined as follows. If \( \omega \in D^{p,q} \) and \( p = 0 \) or \( q = 0 \), we set \( c \omega = 0 \). If both \( p, q \geq 1 \), then for any vector fields \( x_1, \cdots, x_{p-1}, y_1, \cdots, y_{q-1} \), we set
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\[ c_{\omega}(x_1 \cdots x_{p-1} \otimes y_1 \cdots y_{q-1}) = \sum_{k=1}^{n} \omega(e_k x_1 \cdots x_{p-1} \otimes e_k y_1 \cdots y_{q-1}), \]

where \( \{e_1, \cdots, e_n\} \) is a locally defined orthonormal frame field with respect to the metric \( g \). Then, we have

\[ \mathcal{E} \cdot c = c \cdot \mathcal{E} \]

on \( \mathcal{D} \) and

\[ c(g \wedge \omega) = g \wedge c\omega + (n-p-q)\omega \]

for any \( \omega \in \mathcal{D}^{p,q} \).

Let \( \omega^p \) denote the exterior product of \( \omega \in \mathcal{E} \) with itself \( p \) times. Then, by the formula (1.1) we find

\[ \omega^p(x_1 \times x_2 \cdots x_p \otimes y_1 \times y_2 \cdots y_p) = p! \det \| \omega(x_i \otimes y_j) \| \]

for any \( \omega \in \mathcal{E}^1 \). Particularly, the norm \( \| \cdot \| \) of a \( p \)-vector induced by the metric \( g \) can be written as

\[ \| x_1 \wedge x_2 \wedge \cdots \wedge x_p \|^p = \frac{1}{p!} g^p(x_1 \times x_2 \cdots x_p \otimes x_1 \times x_2 \cdots x_p) \]

for the decomposable \( p \)-vector \( x_1 \wedge x_2 \wedge \cdots \wedge x_p \).

Let \( G_p \) denote the Grassmann bundle of \( p \)-planes on \( M \), and \( \pi: G_p \to M \) be its projection. For \( \omega \in \mathcal{E}^p \), we define the corresponding curvature function \( K_\omega : G_p \to \mathbb{R} \) as follows: for any \( \sigma \in G_p \)

\[ K_\omega(\sigma) = \frac{\omega(x_1 \times x_2 \times \cdots \times x_p)}{\| x_1 \wedge \cdots \wedge x_p \|^2}, \]

where \( \{x_1, \cdots, x_p\} \) is a base of \( \sigma \). The value \( K_\omega(\sigma) \) depends only on \( \sigma \). We say a point \( m \in M \) is isotropic with respect to \( \omega \) if \( K_\omega \) is identically constant on the fibre \( \pi^{-1}(m) \); otherwise, we call \( m \) non-isotropic.

The curvature function \( K_\omega \) generally determines \( \omega \), that is, if \( K_\omega = K_\theta \) on \( \pi^{-1}(m) \), then we have \( \omega = \theta \) at \( m \in M \), for any \( \omega, \theta \in \mathcal{E}^p \). In particular, from (1.3) we have

**Lemma.** \( K_\omega \mid \pi^{-1}(m) \equiv \text{const.} \kappa \) if and only if \( \omega = \frac{\kappa}{p!} g^p \) at \( m \), for any \( \omega \in \mathcal{E}^p \).

Finally, let \( R_{xy} \) be the curvature operator defined by

\[ R_{xy} = [\nabla_x, \nabla_y] - \nabla_{[x,y]} \]

for any vector fields \( x \) and \( y \), where \( \nabla \) denotes the covariant differentia-
tion with respect to the metric $g$. The curvature tensor $R$ of type $(0, 4)$ is defined by the formula

$$R(xy \otimes uv) = \langle R_{xy} u, v \rangle$$

for any vector fields $x$, $y$, $u$, and $v$. It is well-known that $R \in \mathcal{S}_4^1$. Also, $-cR \in \mathcal{S}_1^1$ and $-c^2 R \in \mathcal{F}(M)$ are the Ricci tensor $Ric$ and the scalar curvature $Sc$ of $(M, g)$, respectively.

2. Generalized $p$-vector curvature structures

In this section, let us consider the generalized $p$-vector curvature structure

$$\phi_r = \frac{1}{r!} g^r \wedge \omega \quad (0 \leq r \leq n - 2),$$

where $\omega$ is an element of $\mathcal{S}_1^1$ and $p = r + 2$. It is easy to see that for any $p$-plane $\sigma$ we have

$$K_\omega(\sigma) = \sum_{1 \leq i < j \leq p} \omega(e_i \wedge e_j)$$

from (1.1) and (1.4), where $\{e_1, \ldots, e_p\}$ is an orthonormal base of $\sigma$. Thus, the value $K_\omega(\sigma)$ differs by constant factor from the average value of $K_\omega$ over all 2-planes spanned by $e_i$ and $e_j$. Similarly, for any $(n-1)$-plane $\sigma$ we have

$$\frac{1}{2} c^r \omega = K_{n-2}(\sigma) + K_{n-2}(v),$$

where $v$ is the normal vector of $\sigma$ in the tangent space $T_{\omega}(M)$.

One of the principal properties of the curvature structure $\phi_r$ is the following theorem, whose proof is essentially due to Kulkarni [4].

**Theorem 1.** Suppose that $K_{\omega} |_{\pi^{-1}(m)} = \text{const.} \ a$ for some point $m \in M$ and for some fixed integer $r$ such that $0 \leq r \leq n-4$. Then we have

$$\omega = \frac{\kappa}{2m(n-1)} g^{2r} \quad \text{at } m,$$

where $\kappa = 2an(n-1)/(r+1)(r+2)$. The converse is also true.

**Proof.** If $r = 0$, Theorem 1 is trivial. Hence, we suppose $r \geq 1$. The assumption $K_\omega |_{\pi^{-1}(m)} = \text{const.} \ a$ implies
(2.3) \[ \phi_r = \frac{\kappa}{2n(n-1)(r!)} g^{r+1} \] at \( m \)

by Lemma, from which we obtain easily

\[ K_s | \pi^{-1}(m) = \frac{\kappa(s+1)(s+2)}{2n(n-1)} \]

for any \( s \) satisfying \( r \leq s \leq n-2 \). Especially, we get

\[ K_{n-1} | \pi^{-1}(m) = \frac{\kappa(n-2)}{2n} \]

from which we find \( K_{\omega} | \pi^{-1}(m) = \text{const.} \), by (2.2). Hence, we have by Lemma

(2.4) \[ c_\omega = \frac{\kappa}{n} g \] at \( m \).

On the other hand, from the identity (1.2) we have inductively

\[ c(g^r \wedge \omega) = g^r \wedge c_\omega + r(n-r-3) g^{r-1} \wedge \omega \] \( \quad (r \geq 1). \)

Accordingly, we get by (2.4)

(2.5) \[ c \phi_r = \frac{\kappa}{n(r!)} g^{r+1} + (n-r-3) \phi_{r-1} \] at \( m \).

Since \( r \leq n-4 \), by substituting (2.3) into (2.5) and then using the identity

\[ c g^t = t(n-t+1) g^{t-1} \] for any \( t \geq 1 \),

we obtain

\[ \phi_{r-1} = \frac{\kappa}{2n(n-1) (r-1)!} g^{r+1} \] at \( m \).

It is easy to check that, continuing this way, we have finally

\[ \phi_0 = \frac{\kappa}{2n(n-1)} g^2 \] at \( m \).

It will be easily seen that the converse is true. q. e. d.

Suppose \( a = 0 \) in Theorem 1. Then we have immediately a certain cancellation law in the ring \( \mathfrak{C}_1 \) of curvature structures as follows (cf. Lemma 1 and Lemma 2 in Tachibana [7]):

**Corollary.** Suppose that \( \omega \in \mathfrak{C}_1 \). If \( g^r \wedge \omega = 0 \) at \( m \in M \) for some
r such that $0 \leq r \leq n-4$, then we have $\omega = 0$ at $m$.

3. Application to the Riemannian manifold with constant $2p$th sectional curvature

The $2p$th sectional curvature $\gamma_{2p}$ of Thorpe [8] is given by the formula

$$\gamma_{2p}(\sigma) = \frac{(-1)^{p}}{2^p \cdot (2p)!} \sum_{r, \mu \in \mathcal{S}_{2p}} \varepsilon_{r} \varepsilon_{\mu} R(e_{1} e_{2} \otimes e_{\mu_{1}} e_{\mu_{2}}) \cdots R(e_{2p-1} e_{2p} \otimes e_{\mu_{2p-1}} e_{\mu_{2p}})$$

for any $2p$-plane $\sigma \subseteq G_{2p}$, where $\{e_{1}, \ldots, e_{2p}\}$ is an orthonormal base of $\sigma$. In the case $p = 1$, $\gamma_{2}$ is the usual sectional curvature of $(M, g)$. Since we have from the formula (1.1)

$$\omega^{p}(x_{1} \cdots x_{2p} \otimes y_{1} \cdots y_{2p})$$

$$= \frac{1}{2^p} \sum_{r, \mu \in \mathcal{S}_{2p}} \varepsilon_{r} \varepsilon_{\mu} \omega(x_{1} x_{2} \otimes y_{\mu_{1}} y_{\mu_{2}}) \cdots \omega(x_{2p-1} x_{2p} \otimes y_{\mu_{2p-1}} y_{\mu_{2p}})$$

for any $\omega \in \mathfrak{g}^{2,2}$ and any vector fields $x_{1}, \ldots, x_{2p}, y_{1}, \ldots, y_{2p}$, it follows that the formula (3.1) reduces to the expression

$$\gamma_{2p}(\sigma) = (-2)^{p} \cdot ((2p)!)^{-1} R^{p}(e_{1} \cdots e_{2p} \otimes e_{1} \cdots e_{2p}),$$

that is to say, $\gamma_{2p}$ is the curvature function $K_{p}$ corresponding to the curvature structure

$$\omega = (-2)^{p} \cdot ((2p)!)^{-1} R^{p}.$$ 

Since $R^{p} \in \mathfrak{g}^{2,2}$, we have from Lemma

$$\gamma_{2p} \equiv \text{const. } \kappa_{2p} \quad \text{iff} \quad R^{p} = (-2)^{-p} \kappa_{2p} \ g^{2p},$$

for any $p \geq 1$.

Now, the condition $\gamma_{2p} \equiv \text{const. } (p \geq 2)$ does not always imply $\gamma_{2} \equiv \text{const.}$ (e. g. see A. Stehney [6, §2]). However, we have

Theorem 2. Let $(M, g)$ be an $n$-dimensional Riemannian manifold with non-vanishing constant $2p$th sectional curvature. If $0 < 2p \leq n-4$ and its $2(p+1)^{th}$ sectional curvature is also identically constant, then $(M, g)$ is of constant curvature in usual sense.

Proof. The assumption $\gamma_{2p} \equiv \text{const. } \kappa_{2p} (\neq 0)$ in Theorem 2 implies

$$R^{p} = (-2)^{-p} \kappa_{2p} \ g^{2p}$$

by (3.2). Furthermore, suppose $\gamma_{2(p+1)} \equiv \text{const. } \kappa_{2(p+1)}$. Then we have
similarly
\begin{equation}
R^{p+1} = (-2)^{-p} \kappa_{(p+1)} \frac{e^{2p}}{2 \kappa_{2p}} \cdot g^{2(p+1)}.
\end{equation}
Substituting (3.3) into the left hand side of (3.4) and applying Corollary to Theorem 1, we obtain \( R = - \{ \kappa_{(p+1)} / 2 \kappa_{2p} \} \cdot g^2 \). Hence, we find \( \gamma \equiv \kappa_{(p+1)} / \kappa_{2p} \), that is, \((M, g)\) is of constant curvature. q. e. d.

4. Application to the mean curvature for \( p \)-plane

Let \( p \) be an integer such that \( 1 < p < n \), and we put
\[ \omega = 2R - \frac{1}{p-1} g \wedge cR. \]
We consider, throughout this section, the generalized \( p \)-vector curvature structure \( \phi_\tau \), defined by this \( \omega \in C^2 \):
\[ \phi_\tau = \frac{1}{r!} g^r \wedge \omega \quad (r = p-2). \]

The mean curvature \( \rho \) for \( p \)-plane of Tachibana [7] is given by the formula
\begin{equation}
(4.1) \quad \rho(\sigma) = \frac{1}{p(n-p)} \sum_{i=1}^{p} \sum_{j=p+1}^{n} \gamma_e(e_i, e_j)
\end{equation}
for any \( \sigma \in G_p \), where \( \{ e_1, \ldots, e_n \} \) is an orthonormal base of the tangent space \( T_{\gamma(e)}(M) \) such that \( e_1, \ldots, e_p \) span \( \sigma \), and \( \gamma_e(e_i, e_j) \) denotes the sectional curvature of the 2-plane spanned by \( e_i \) and \( e_j \). On the other hand, we get
\[
K e(e_i, e_j) = \sum_{i=1}^{p} R(e_i e_j \otimes e_i e_j) - \frac{1}{p-1} \sum_{1 \leq i < j \leq p} (g \wedge cR)(e_i e_j \otimes e_i e_j)
= \sum_{i=1}^{p} R(e_i e_j \otimes e_i e_j) - \sum_{i=1}^{p} \sum_{k=1}^{n} R(e_i e_k \otimes e_k e_i)
= \sum_{i=1}^{p} \sum_{j=p+1}^{n} \gamma_e(e_i, e_j)
\]
by the formula (2.1). Hence, \( \rho \) is a curvature function corresponding to the curvature structure \( \{ p(n-p) \}^{-1} \phi_\tau \in C^2 \), that is,
\begin{equation}
(4.2) \quad \rho = \frac{1}{p(n-p)} K e : G_p \longrightarrow \mathbb{R}
\end{equation}
From (4.1) and Theorem 1, we have the following proposition,
which has been obtained by Tachibana (cf. Theorem in [7]).

**Proposition.** Let $1 < p < n$. Each point of $M$ is isotropic with respect to the mean curvature $\rho$ for $p$-plane if and only if

(i) $(M, g)$ is Einsteinian for $p=n-1$,

(ii) $(M, g)$ is of constant curvature, for $1 < p < n-1$ and $2p \neq n$,

(iii) $(M, g)$ is conformally flat, for $1 < p < n-1$ and $2p = n$.

**Remark 1.** It is interesting to compare this proposition in the case (iii) with the following (cf. Theorem 3.2 in Kulkarni [3]) : $(M, g)$ is conformally flat if and only if at every point of $M$ we have

$$\gamma_2(e_1, e_2) + \gamma_2(e_3, e_4) = \gamma_1(e_1, e_2) + \gamma_1(e_3, e_4)$$

for every quadruple of orthogonal vectors $(e_1, e_2, e_3, e_4)$.

Now, let us assume $1 < p < n-1$ and show the mean curvature $\rho$ for $p$-plane generally determines the metric $g$. Let $(\bar{M}, \bar{g})$ be another Riemannian manifold and $f : (M, g) \rightarrow (\bar{M}, \bar{g})$ be a diffeomorphism. We indicate the corresponding quantities with respect to the metric $\bar{g}$ or the induced metric $g^* = f^* \bar{g}$ by bar overhead or asterisking, respectively. Suppose that $f$ is $K_\rho$-preserving, that is, for every $\sigma \in G_p$ we have

$$K_\rho(f_\sigma) = K_\rho(\sigma).$$

Furthermore, if

(\*) the set of non-isotropic points w. r. t. $\phi$, is dense in $M$,

then $f$ is conformal, that is, we have

$$g^* = e^{2\phi} g \quad (\psi \in \mathcal{F}(M))$$

by the well-known theorem of Kulkarni (see General Theorem 5.1 in [4]). Under these circumstances, we shall prove $f$ is an isometry, that is, $\phi = 0$.

First of all, we remark that the assumption (\*) means

(\*)' the set of non-isotropic points w. r. t. $\omega$ is dense in $M$,

by Theorem 1. Also, under the conformal change (4.4) of metric we have

$$R^* = e^{2\phi} (R + g \wedge \kappa(\psi)),$$

$\kappa(\psi)$ being an element of $\mathcal{F}$, which depends on $\psi$. From (4.5) we obtain

$$e^{\phi} R^* = cR + (n - 2)\kappa(\psi) + \text{Trace} \kappa(\psi) g.$$
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Substituting (4.4), (4.5) and (4.6) into $\phi_r^*$. we have

$$
(4.7) \quad \phi_r^* = \frac{1}{r!} e^{2\kappa+\psi} g^* \wedge \left\{ \omega + \frac{2p-n}{p-1} g^* \wedge \kappa(\psi) - \frac{1}{p-1} \text{Trace} \kappa(\psi) g^* \right\}.
$$

On the other hand, the condition (4.3) can be written as

$$
K_r^* = \epsilon^{p\psi} K^*_r,
$$

which implies

$$
(4.8) \quad \phi_r^* = \epsilon^{p\psi} \phi_r,
$$

because we have $\phi_r, \phi_r^* \in \mathcal{G}^p$. By (4.4), (4.8) and Corollary to Theorem 1 we obtain

$$
(4.9) \quad \omega^* = \epsilon^{p\psi} \omega.
$$

Eliminate $\phi_r^*$ from two equations (4.7) and (4.8). Then we have similarly

$$
(4.10) \quad (p-1)(\epsilon^{p\psi} - 1) \omega = (2p-n) g^* \wedge \kappa(\psi) - \text{Trace} \kappa(\psi) g^*.
$$

**Case (i) $n = 2p$.** Suppose that $M' = \{ m \in M ; \psi(m) \neq 0 \}$ has non-empty interior. Then, each point of $M'$ is isotropic w.r.t. $\omega$ by the equation (4.10) and Lemma. But this contradicts the assumption (**"**). Hence, we have $\psi \equiv 0$.

**Case (ii) $n \neq 2p$.** In the case, it will be easily seen that the assumption (**"**) means

$$
(**") \quad \text{the set of non-isotropic points w.r.t. $R$ is dense in $M$.}
$$

By operating the contraction $c$ to the equation (4.10), we have

$$
(\epsilon^{p\psi} - 1) (2p-n) cR = c^2 R g
$$

$$
= (2p-n)(n-2) \kappa(\psi) + (2p-3n+2) \text{Trace} \kappa(\psi) g.
$$

Furthermore, operating $c$ to (4.11) we get

$$
(\epsilon^{p\psi} - 1) c^2 R = 2(n-1) \text{Trace} \kappa(\psi).
$$

Substitute this into the left hand side of (4.11). Then owing to $n \neq 2p$ we have

$$
(\epsilon^{p\psi} - 1) cR = (n-2) \kappa(\psi) + \text{Trace} \kappa(\psi) g,
$$

which implies

$$
(4.12) \quad c^* R^* = \epsilon^{p\psi} cR
$$

by (4.6). Substitute (4.4) and (4.12) into (4.9). Then we find $R^* = \epsilon^{p\psi} R$. 

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Thus, \( f \) is \( K_\nu \)-preserving. Since we assume \((*)''\) and \( n > 3 \), \( f \) is an isometry by Theorem 7.1 in [4]. Thus, we have proved the following theorem:

**Theorem 3.** Let \((M, g)\) and \((\overline{M}, \overline{g})\) be two Riemannian manifolds of dimension \( n \). Let \( f : (M, g) \rightarrow (\overline{M}, \overline{g}) \) be a diffeomorphism which preserves the mean curvature for \( p \)-plane, where \( 1 < p < n-1 \). Suppose that the set of non-isotropic points with respect to the mean curvature for \( p \)-plane is dense in \( M \). Then \( f \) is an isometry.

**Remark 2.** Theorem 3 is not true when \( p = n - 1 \). In fact, if \( p = n - 1 \), then the formula (4.1) reduces to the expression

\[
\rho(\sigma) = \frac{1}{n-1} K_{Rho}(e_n).
\]

The present author found a counterexample for corresponding local statement for the Ricci curvature \( K_{Rho} \) (cf. [5]).

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