A note on commutative separable algebras

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A NOTE ON COMMUTATIVE SEPARABLE ALGEBRAS

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In this note, we prove that separability descends by faithful flatness and hence is a local property. We also prove that separability is a punctual property over a semi-local ring.

Throughout this paper, rings and algebras are commutative with identity and ring homomorphisms carry the identity to the identity. In what follows, $A$ denotes a ring with identity 1 and $B$ an $A$-algebra. $B$ is a separable $A$-algebra if and only if there exists an element $e$ in $B \otimes_A B$ such that $(b \otimes 1) e = (1 \otimes b)e$ for all $b \in B$ and $\psi(e) = 1$, where $\psi$ is the multiplication map from $B \otimes_A B$ to $B$ ([3, p. 40]). It is easily seen that $e$ is idempotent and unique. This element is called the separability idempotent of $B$ over $A$, and is invariant under the switch map $B \otimes_A B \to B \otimes_A B$ given by $b_i \otimes b_j \mapsto b_j \otimes b_i$.

Now, it is well known that if $B$ is separable over $A$ then for any $A$-algebra $C$, the $C$-algebra $B \otimes_A C$ is again separable. Moreover, by [4, Prop. 2.2 (c)], it is known that if $C$ is a faithfully flat $A$-algebra and $B \otimes_A C$ is separable over $C$ then $B$ is separable over $A$, provided that $B$ is finitely generated as an $A$-algebra. Our main result is that the hypothesis on $B$ is not necessary.

Theorem 1. Let $B$ be an $A$-algebra and $C$ a faithfully flat $A$-algebra. If the $C$-algebra $B \otimes_A C$ is separable, then $B$ is separable over $A$.

Proof. Let $\epsilon_0$ (resp. $\epsilon_1$) : $C \to V = C \otimes_A C$ denote the $A$-algebra homomorphism defined by $c \mapsto 1 \otimes c$ (resp. $c \mapsto c \otimes 1$). Then, the homomorphisms

$$1 \otimes \epsilon_i : U = B \otimes_A C \longrightarrow W = B \otimes_A C \otimes_A C \quad (i = 0, 1)$$

give rise to the homomorphisms

$$(1 \otimes \epsilon_i) \otimes (1 \otimes \epsilon_i) : U \otimes_U U \longrightarrow W \otimes_U W \quad (i = 0, 1).$$

Moreover, we have the homomorphisms

$$U \overset{\mu}{\longrightarrow} U \otimes_{\text{Im}(\epsilon_i)} V \overset{\nu_i}{\longrightarrow} \text{Im} (1 \otimes \epsilon_i) \cdot V = W \quad (i = 0, 1)$$

where $\mu(u) = u \otimes 1$, and $\nu_i(u \otimes v) = (1 \otimes \epsilon_i)(u) v$. Now, in general,
if $U$ is separable over $T$ and $V$ is any $T$-algebra, then the separability idempotent for $U$ over $T$ goes to the separability idempotent for any homomorphic image of $U \otimes \tau V$ over $V$. Applying this to our case, we see that the separability idempotent $e'$ of $U$ over $C$ (which is an element of $U \otimes_c U$) must be sent to 0 under the difference $d = (1 \otimes \epsilon_0) \otimes (1 \otimes \epsilon_0) - (1 \otimes \epsilon_1) \otimes (1 \otimes \epsilon_1)$. Since $C$ is faithfully flat over $A$, it follows from [2, Lemma 3.8] that the sequence

$$
0 \longrightarrow B \otimes_A B \overset{\rho}{\longrightarrow} B \otimes_A B \otimes_A C \overset{1 \otimes 1 \otimes \epsilon_0 - 1 \otimes 1 \otimes \epsilon_1}{\longrightarrow} B \otimes_A B \otimes_A C \otimes_A C
$$

where $\rho(m) = m \otimes 1$, is exact. From this, one will easily see that the sequence

$$
0 \longrightarrow B \otimes_A B \longrightarrow U \otimes_c U \longrightarrow W \otimes \tau W
$$

where $\sigma(b_1 \otimes b_2) = (b_1 \otimes 1) \otimes (b_2 \otimes 1)$, is also exact. Hence there exists an element $e$ in $B \otimes_A B$ so that $\sigma(e) = e'$. Obviously, $p(e) = 1$, and $(b \otimes 1)e = (1 \otimes b)e$ for all $b \in B$. Thus, $B$ is separable over $A$, completing the proof.

An application of Th. 1 is the following corollary which shows that separability is a local property.

**Corollary 2.** Let $B$ be an $A$-algebra and let \{ $f_1, \cdots, f_s$ \} be a family of elements of $A$ which generates the unit ideal of $A$. Then $B$ is separable over $A$ if and only if for all $i$, $B_{f_i}$ is separable over $A_{f_i}$, where $A_{f_i}$ is the ring of fractions of $B$ having denominators equal to some power of $f$, and $B_{f_i} = B \otimes_A A_{f_i}$.

**Proof.** By the result of [1, Ch. II, Prop. 5.1.3], $C = \prod_{i=1}^s A_{f_i}$ is a faithfully flat $A$-algebra. From this, the assertion follows immediately.

As a second application of Th. 1, we have the following corollary which shows that separability is a punctual property provided that the base ring has only a finite number of maximal ideals.

**Corollary 3.** Let $A$ be a semi-local ring and $B$ an $A$-algebra. Then the following conditions are equivalent.

i) $B$ is separable over $A$.

ii) $B_\wp$ is separable over $A_\wp$ for each prime ideal $\wp$ of $A$.

iii) $B_\m$ is separable over $A_\m$ for each maximal ideal $\m$ of $A$.
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Proof. Only iii) ⇒ i) needs proof. Let $\mathcal{O}$ denote the set of maximal ideals of $A$. Since $\mathcal{O}$ is finite, $\prod_{m \in \mathcal{O}} A_m$ is faithfully flat over $A$ by [1, Ch. II, Prop. 3.3.10].

Remark 1. By virtue of Corollary 2, we easily see that an $A$-algebra $B$ is separable if and only if for every prime ideal $\mathfrak{p}$ of $A$, there exists an element $t$ in $A - \mathfrak{p}$ (the complement of $\mathfrak{p}$ in $A$) such that $B_t$ is separable over $A$. Moreover, the result of Corollary 3 is a partial generalization of [4, Prop. 2.5].

Remark 2. By Theorem 1, we see that for $A$-algebras $B$, $C$, if $B \otimes_A C$ is separable over $C$ then $B$ is separable over $A$, provided that \{A, C\} is one of (1), (2) and (3):

(1) $A :=$ a Noetherian ring, $C =$ the $I$-adic completion of $A$ where $I$ is an ideal of $A$ contained in the Jacobson radical of $A$ ([1, p. 206]).
(2) $A :=$ a local ring, $C =$ the Henselization of $A$ ([3, p. 73]).
(3) $A =$ a coherent ring (e. g., a Noetherian ring), $C =$ $A[[X_1, \ldots, X_n]]$, a formal power series ring over $A$ ([1, p. 49]).

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REFERENCES


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