On modified chain conditions

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ON MODIFIED CHAIN CONDITIONS

To Professor Yoshikazu Nakai on his sixtieth birthday

HIROAKI KOMATSU and HISAO TOMINAGA

Throughout the present paper, $A$ will represent a ring without (possibly with) identity, $N$ the prime radical of $A$, and $M$ a left $A$-module. Given a left ideal $I$ of $A$ and an $A$-submodule $M'$ of $M$, for each positive integer $i$ we set $I^{-i}M' = \{ u \in M \mid I^{i}u \subseteq M' \}$. Following F. S. Cater [1], we say that $M$ is almost Artinian (resp. almost Noetherian) if for each infinite descending (resp. ascending) chain $M_{1} \supseteq M_{2} \supseteq \cdots$ (resp. $M_{1} \subseteq M_{2} \subseteq \cdots$) of $A$-submodules of $M$ there exist positive integers $m$, $q$ such that $A^{q}M_{m} \subseteq M_{i}$ (resp. $M_{i} \subseteq A^{-q}M_{m}$) for all $i$, or equivalently there exists a positive integer $p$ such that $A^{p}M_{p} \subseteq M_{i}$ (resp. $M_{i} \subseteq A^{-p}M_{p}$) for all $i$. Every left $A$-module which is Artinian (resp. Noetherian) in the usual sense is clearly almost Artinian (resp. almost Noetherian). If $A$ is almost Artinian (resp. almost Noetherian), we say that $A$ is an almost left Artinian (resp. almost left Noetherian) ring.

If $M$ is a trivial left $A$-module, i.e. $AM = 0$, then clearly $M$ is both almost Artinian and almost Noetherian. Further, every nilpotent ring is both almost left Artinian and almost left Noetherian. It is easy to construct a nilpotent ring which is neither left Artinian nor left Noetherian, e.g. $\begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix}$ is such a ring. On the other hand, $\begin{pmatrix} Q & 0 \\ Q & 0 \end{pmatrix}$ is a non-nilpotent ring which is almost left Artinian but not left Artinian, and $\begin{pmatrix} Z & 0 \\ Q & 0 \end{pmatrix}$ is a non-nilpotent ring which is almost left Noetherian but neither left Noetherian nor almost left Artinian.

In §1, several preliminary results in [1] will be reproved with notable briefness. In §2, we shall improve Theorems A, B of [1] (Theorems 1 and 2). The principal theorem of §3 states that if $A$ is almost left Noetherian then $A$ satisfies the ascending chain condition for semiprime ideals, every nil subring of $A$ is nilpotent and the nilpotency indices of nil subrings are bounded (Theorem 3). In §4, we shall give some new conditions for a ring to be almost left Artinian (Theorem 4).

1. We begin with improving Propositions 4 and 9 of [1] all together.

**Proposition 1.** (1) The following are equivalent:

131
1) \( \lambda M \) is almost Artinian.

2) For each infinite descending chain \( M_1 \supseteq M_2 \supseteq \cdots \) of \( \lambda \)-submodules of \( M \) there exists a positive integer \( q \) such that \( A^q M_i = A^q M_i \) for all \( i \geq q \).

3) In each non-empty family \( \mathcal{M} \) of \( \lambda \)-submodules of \( M \) such that \( M' \in \mathcal{M} \) implies \( \lambda M' \in \mathcal{M} \), there exists a minimal member.

4) For each non-empty family \( \mathcal{M} \) of \( \lambda \)-submodules of \( M \), there exists a positive integer \( q \) and a member \( M' \) of \( \mathcal{M} \) such that \( \lambda A^q M' \subseteq M'' \) for every \( M'' \in \mathcal{M} \) with \( M' \subseteq M'' \).

(2) The following are equivalent:

1) \( \lambda M \) is almost Noetherian.

2) For each infinite descending chain \( M_1 \subseteq M_2 \subseteq \cdots \) of \( \lambda \)-submodules of \( M \) there exists a positive integer \( q \) such that \( A^q M_i = A^q M_i \) for all \( i \geq q \).

3) In each non-empty family \( \mathcal{M} \) of \( \lambda \)-submodules of \( M \) such that \( M' \in \mathcal{M} \) implies \( \lambda A^{-1} M' \in \mathcal{M} \), there exists a maximal member.

4) For each non-empty family \( \mathcal{M} \) of \( \lambda \)-submodules of \( M \), there exists a positive integer \( q \) and a member \( M' \) of \( \mathcal{M} \) such that \( M'' \subseteq \lambda A^{-1} M' \) for every \( M'' \in \mathcal{M} \) with \( M' \subseteq M'' \).

\textbf{Proof.}  (1) As is easily seen, 4) \( \implies 3) \implies 2) \implies 1) \). Now, suppose 4) does not hold for some \( \mathcal{M} \). Then we can find successively \( M_i \in \mathcal{M} \) \( (i = 1, 2, \ldots) \) such that \( M_{i+1} \subseteq M_i \) but \( \lambda A^i M_i \not\subseteq M_{i+1} \). We have thus seen that 1) implies 4).

(2) Obviously, 4) \( \implies 3) \implies 2) \implies 1) \). Suppose now that 4) does not hold for some \( \mathcal{M} \). Then we can find successively \( M_i \in \mathcal{M} \) \( (i = 1, 2, \ldots) \) such that \( M_i \subseteq M_{i+1} \) but \( M_{i+1} \not\subseteq \lambda A^{-1} M_i \). Thus we have seen that 1) implies 4).

Now, Proposition 1 makes short the proof of [1, Proposition 7].

\textbf{Proposition 2} ([1, Proposition 7]). (1) Let \( \lambda M' \) be an \( \lambda \)-submodule of \( M \). Then \( \lambda M \) is almost Artinian if and only if both \( \lambda M' \) and \( \lambda M/M' \) are almost Artinian.

(2) Let \( \lambda M' \) be an \( \lambda \)-submodule of \( M \). Then \( \lambda M \) is almost Noetherian if and only if both \( \lambda M' \) and \( \lambda M/M' \) are almost Noetherian.

\textbf{Proof.}  (1) It suffices to prove the if part. Let \( M_1 \supseteq M_2 \supseteq \cdots \) be an arbitrary descending chain of \( \lambda \)-submodules of \( M \). By Proposition 1 (1), there exists a positive integer \( p \) such that \( A^p M_i + M' = A^p M_i + M' \) and \( A^p (M_i \cap M') = A^p (M_i \cap M') \) for all \( i \geq p \). Since \( A^n M_i \subseteq A^n M_i + (A^p M_i \cap M') \subseteq A^n M_i + (M_i \cap M') \), it follows that \( A^n M_i \subseteq A^n M_i + \)
ON MODIFIED CHAIN CONDITIONS

\[ A^p(M \cap M') = A^p M_1 + A^p(M_i \cap M') \subseteq M_i. \]

(2) It is enough to prove the if part. Let \( M_1 \subseteq M_2 \subseteq \ldots \) be an arbitrary ascending chain of \( A \)-submodules of \( M \). There exists a positive integer \( p \) such that \( A^p M_1 + M' \subseteq M_p + M' \) and \( A^p(M_i \cap M') \subseteq M_p \cap M' \) for all \( i \). Since \( A^p M_1 \subseteq M_p + (M_i \cap M') \), it follows \( A^p M_1 \subseteq A^p M_p + A^p(M_i \cap M') \subseteq M_p. \)

A left \( A \)-module \( M \) is said to be \( s \)-unital if \( u \in Au \) for each \( u \in M \), or equivalently if \( M' = AM' \) for each \( A \)-submodule \( M' \) of \( M \). If \( A \) is \( s \)-unital, we term \( A \) a left \( s \)-unital ring. Any ring \( A \) with a left identity is a left \( s \)-unital ring. Obviously, for \( s \)-unital left \( A \)-modules, the concept of "almost Artinian" (resp. "almost Noetherian") coincides with that of "Artinian" (resp. "Noetherian"). Now, suppose that \( A/\text{Ann} (M) \) is left \( s \)-unital. Then by [6, Theorem 1], \( A/AM \) is seen to be \( s \)-unital, and therefore by Proposition 2 (1) (resp. (2)), \( A/AM \) is almost Artinian (resp. almost Noetherian) when and only when \( A/AM \) is Artinian (resp. Noetherian). In particular, if \( A/I(A) \) is left \( s \)-unital, then \( A \) is almost left Artinian (resp. almost left Noetherian) when and only when \( A^2 \) is a left Artinian (resp. Noetherian) ring.

Lemma 1. (1) If a unital left \( A \)-module \( M \) is almost Artinian, then the socle of \( AM \) is essential in \( AM \).

(2) If a left \( A \)-module \( M \) is the sum of \( s \)-unital \( A \)-submodules \( M_\lambda \) (\( \lambda \in \Lambda \)), then \( M \) is \( s \)-unital. In particular, every completely reducible left \( A \)-module is \( s \)-unital.

Proof. (1) Immediate from the condition 3) of Proposition 1 (1).

(2) Let \( u \) be an arbitrary element of \( M \). Then \( u = u_1 + \cdots + u_k \) with some \( u_i \subseteq M_\lambda \). If \( k = 1 \) then \( au = u \) with some \( a \in A \), by hypothesis. Now, assume \( k > 1 \), and choose \( b \in A \) such that \( bu_k = u_k \). Then \( u - bu = (u_1 - bu_2) + \cdots + (u_{k-1} - bu_{k-1}) \). By induction method, there exists \( c \in A \) such that \( c(u - bu) = u - bu \). We conclude then \( u = (b + c - cb) u \).

The next is [1, Lemma 2]. However, for the sake of convenience, we shall give a somewhat economical proof.

Lemma 2. Let \( A \) be an almost left Artinian ring.

(1) Every non-nilpotent left ideal contains a minimal non-nilpotent left ideal.

(2) Every nil left ideal of \( A \) is nilpotent.
Proof. (1) is obvious by the condition 3) of Proposition 1 (1). In order to prove (2), suppose contrarily that there exists a nil left ideal \( I \) which is not nilpotent. By (1), we may assume that \( I \) is a minimal non-nilpotent left ideal. Consider the family of all left subideals \( I' \) of \( I \) with \( II' \not= 0 \). Then, again by the condition 3) of Proposition 1 (1), the family contains a minimal member \( I^* \). Since \( II^* = I^* \), there exists \( a^* \in I^* \) such that \( Ia^* = I^* \). Hence, \( aa^* = a^* (\neq 0) \) with some \( a \in I \). Obviously, \( a \) is not nilpotent. But this contradicts the hypothesis that \( I \) is nil.

Now, by making use of Lemmas 1 and 2, we reprove [1, Theorem 1].

Proposition 3. If \( A \) is almost left Artinian, then \( A \) is semiprimary, namely \( N \) is nilpotent and \( A/N \) is Artinian (semisimple).

Proof. Since \( N \) is nilpotent by Lemma 2 (2), it suffices to prove that if \( A \) is semiprime and almost left Artinian then \( A \) is Artinian semi-

simple. By Lemma 1 (1), the left socle \( S \) of \( A \) is essential in \( _A\!A \). Since \( _A\!S \) is completely reducible and Artinian (Lemma 1 (2)) and every minimal left ideal of \( A \) is generated by an idempotent, it is known that \( S \) itself is generated by an idempotent. Hence, \( S \) coincides with \( A \), whence we can conclude the assertion.

2. First, we state the following that includes Theorems A and B of [1].

Theorem 1. Let \( I, I_1, \ldots, I_k \) be left ideals of \( A \).

(1) If \( _A\!A/I \) is completely reducible and \( IM = 0 \), then the following are equivalent:
1) \( _A\!M \) is almost Artinian.
2) \( _A\!AM \) is Artinian.
3) \( _A\!AM \) is finitely generated.
4) \( _A\!AM \) is Noetherian.
5) \( _A\!M \) is almost Noetherian.

(2) If \( _A\!A/I_j (j = 1, \ldots, k) \) are completely reducible and \( I_1 \cdots I_k M = 0 \), then the following are equivalent:
1) \( _A\!M \) is almost Artinian.
2) \( _A\!(AM/I_j M), (_A\!M/I_{j-1} I_k M), \ldots, (_A\!M/I_2 \cdots I_k M) \) are finitely generated.
3) \( _A\!M \) is almost Noetherian.
In particular, if \( A \) is semiprimary then a left \( A \)-module is almost Artinian
if and only if it is almost Noetherian.

Proof. (1) It is easy to see that $\mathcal{A}M$ is completely reducible. Hence, the equivalence of 2), 3) and 4) is obvious. Since $\mathcal{A}(A/{\text{Ann}}(M))$ is $s$-unital by Lemma 1 (2), the equivalences of 1) and 2) and of 4) and 5) are evident by the remark mentioned just before Lemma 1.

(2) Observe the descending chain

$$M \supseteq I_k M \supseteq I_{k-1} M \supseteq \cdots \supseteq I_2 \cdots I_k M \supseteq I_1 \cdots I_k M = 0.$$ 

Then the assertion can be proved by (1) and Proposition 2 (1).

Now, let $\mathcal{A}_M$ be the set of almost Artinian $A$-submodules of $M$, and $\Gamma_M$ the set of $A$-submodules $U$ of $M$ such that $\mathcal{A}M/U$ is almost Noetherian. Obviously, $\mathcal{A}_M$ and $\Gamma_M$ contain $0$ and $M$, respectively. Moreover, by Proposition 2 (1) (resp. (2)), if $M'$ and $M''$ are in $\mathcal{A}_M$ (resp. $\Gamma_M$) then $M' + M''$ and $A^{-1}M'$ (resp. $M' \cap M''$ and $AM'$) are in $\mathcal{A}_M$ (resp. $\Gamma_M$).

We set $\mathcal{A}(M) = \sum_{i \in \mathcal{A}_M} U$ and $\Gamma(M) = \cap_{i \in \Gamma_M} U$. Needless to say, if $\mathcal{A}M$ is almost Artinian (resp. almost Noetherian) then $\mathcal{A}(M) = M$ (resp. $\Gamma(M) = 0$), but not conversely. If $\mathcal{A}M$ is almost Noetherian, then by Proposition 1 (2) we see that $\mathcal{A}(M)$ is the greatest member of $\mathcal{A}_M$ and is characterized as the least one among the $A$-submodules $U$ of $M$ with $\mathcal{A}(M/U) = 0$; in particular $\mathcal{A}M$ is almost Artinian if and only if $\mathcal{A}(M) = M$. On the other hand, if $\mathcal{A}M$ is almost Artinian, then by Proposition 1 (1) we see that $\Gamma(M)$ is the least member of $\Gamma_M$ and is characterized as the greatest one among the $A$-submodules $U$ of $M$ with $\Gamma(U) = U$; in particular $\mathcal{A}M$ is almost Noetherian if and only if $\Gamma(M) = 0$.

In the proof of the following partial extension of Theorem 1 (1), we shall use freely the facts mentioned above.

Theorem 2. Let $I$ be a left ideal of $A$ such that $\mathcal{A}A/I$ is completely reducible.

(1) If $\mathcal{A}M$ is almost Noetherian and $I^{-1}M' \neq M'$ for every proper $A$-submodule $M'$ of $M$, then $\mathcal{A}M$ is almost Artinian.

(2) If $\mathcal{A}M$ is almost Artinian and $IM' \neq M'$ for every non-zero $A$-submodule $M'$ of $M$, then $\mathcal{A}M$ is almost Noetherian.

Proof. (1) Suppose $\mathcal{A}(M) \neq M$, and choose an $A$-submodule $M'' \supseteq \mathcal{A}(M)$ such that $IM'' \subseteq \mathcal{A}(M)$. Since $\mathcal{A}(M/A(M)) = 0$, we see that $\mathcal{A}(M''/\mathcal{A}(M)) \neq 0$. Then, $\mathcal{A}(M''/\mathcal{A}(M))$ is completely reducible and Noetherian (Lemma 1), and therefore Artinian. This is a contradiction. Thus $\mathcal{A}M$ is almost Artinian.
(2) Obviously, $\mathcal{A} \Gamma (M) = \Gamma (M)$. Now, suppose $\Gamma (M) \neq 0$. Then $\mathcal{A} \Gamma (M)/\Gamma (M)$ is completely reducible and Artinian (Lemma 1), and therefore Noetherian. This contradiction means that $\mathcal{A} M$ is almost Noetherian.

3. In this section, we shall prove the following:

**Theorem 3.** Let $A$ be an almost left Noetherian ring.

(1) $A$ satisfies the ascending chain condition for semiprime ideals.

(2) Every nil subring of $A$ is nilpotent and the nilpotency indices of nil subrings are bounded.

In preparation for the proof, we establish the next lemma.

**Lemma 3.** Let $A$ be an almost left Noetherian ring. If $r(A) = 0$ (in particular, if $A$ is semiprime), then $A$ is a left Goldie ring.

**Proof.** Let $L_1 \subseteq L_2 \subseteq \cdots$ be an infinite ascending chain of left annihilators, where $L_i = l(S_i)$. Then there exists a positive integer $p$ such that $A^p L_i \subseteq L_p$ for all $i$. Since $A^p L_i S_p = 0$, it follows $L_i S_p = 0$, namely $L_i \subseteq L_p$. Next, assume that $A$ contains an infinite direct sum of non-zero left ideals $I_1 \oplus I_2 \oplus \cdots$. There exists a positive integer $q$ such that

$$A^q (I_1 \oplus \cdots \oplus I_i) \subseteq I_1 \oplus \cdots \oplus I_i$$

for all $i$. Then $A^q I_i = 0$, and therefore $I_i = 0$ for all $i > q$. But, this is a contradiction.

**Proof of Theorem 3.** (1) The proof is straightforward.

(2) There exists a positive integer $q$ such that $A^q r(A^i) \subseteq r(A^i)$ for all $i$. Since $A^{2q} r(A^i) \subseteq A^q r(A^i) = 0$, there holds $r(A^i) \subseteq r(A^{2q})$. This means that the right annihilator of $A/r(A^{2q})$ is zero. Hence, $A/r(A^{2q})$ is a left Goldie ring by Lemma 3. According to [2, Corollary 1.7], there exists a positive integer $n$ such that $K^n \subseteq r(A^{2q})$ for all nil subrings $K$ of $A$. It is immediate that $K^{2q+n} = 0$.

Combining Theorem 3 (2) with Proposition 3 and the latter part of Theorem 1 (2), we readily obtain

**Corollary 1.** If $A$ is almost left Artinian, then every nil subring of $A$ is nilpotent and the nilpotency indices of nil subrings are bounded.

4. In advance of stating the main theorem of this section, we shall
prove the following

Lemma 4. (1) If $A$ is almost left Artinian, then $A$ is a $\pi$-regular ring of bounded index.

(2) Let $A$ be an almost left Noetherian, $\pi$-regular ring. If $A/N$ is left s-unital, then $A/N$ is Artinian.

Proof. (1) By Proposition 3 and [5, Lemma 2].

(2) By Lemma 3, $A/N$ is a left Goldie ring. Then, as was claimed in the proof of [6, Theorem 3], $A/N$ contains the identity. Moreover, it is easy to see that every regular element of $A/N$ is a unit. Hence, $A/N$ coincides with its left quotient ring that is Artinian semisimple.

A left ideal $I$ of $A$ is said to be almost maximal if $A/I$ is a sum of minimal $A$-submodules. If a prime ideal $P$ is an almost maximal left ideal, then $\downarrow A/P$ is completely reducible. (In [1], a prime ideal is called an almost prime ideal.)

We are now ready to complete the proof of our main theorem, which includes Theorems 5, 6 and 11 of [1].

Theorem 4. The following are equivalent:

1) $A$ is almost left Artinian.

2) $N$ is nilpotent and $\langle AN^{i-1}/N^{i} \rangle$ is Artinian for all $i \geq 0$.

3) $A$ is almost left Noetherian and $A/N$ is left Artinian.

4) $A$ is almost left Noetherian and $\pi$-regular, and $A/N$ is left s-unital.

5) $A$ is almost left Noetherian and every proper prime ideal of $A$ is an almost maximal left ideal.

6) $N$ is nilpotent, $\langle AN^{i-1}/N^{i} \rangle$ is finitely generated for all $i \geq 0$, $A$ satisfies the ascending chain condition for semiprime ideals, and every proper prime ideal of $A$ is an almost maximal left ideal.

Proof. 1) $\iff$ 2) $\iff$ 3). Under any of the conditions 1) — 3), $N$ is nilpotent: $N^{n} = 0$, and $\downarrow A/N$ is completely reducible (Theorem 3 (2) and Proposition 3). Observe the descending chain $A \supseteq N \supseteq N^{2} \supseteq \cdots \supseteq N^{n} = 0$. By Theorem 1 (1), $\langle N^{i-1}/N^{i} \rangle$ is almost Artinian if and only if $\langle AN^{i-1}/N^{i} \rangle$ is Artinian, or equivalently $\langle (N^{i-1}/N^{i}) \rangle$ is almost Noetherian. Hence, by Proposition 2 all the conditions 1) — 3) are equivalent.

1) $\implies$ 4) $\implies$ 3). By Propositions 2 (2), 3 and Lemma 4.

5) $\implies$ 6). By Theorem 3 (1), $A$ satisfies the ascending chain condition for semiprime ideals. Hence, by [4, Theorem 3], $N = \cap_{i=1}^{t} P_{i}$ with
some prime ideals $P_i$. Since $\alpha(\cap \{P_i \mid \exists \lambda \in P_i \})/(\cap \{P_i \mid \exists \lambda \in P_i \}) \cong P_i + \cap \{P_i \mid \exists \lambda \in P_i \}$ and $\alpha A/P_i$ is completely reducible, we see that $\alpha A/N$ is Artinian (Lemma 1). Now, (6) is obvious by Theorem 1 (1) and Theorem 3 (2).

(6) $\Rightarrow$ (1). Again by [4, Theorem 3], $N = \cap \lambda \in P_i$ with some prime ideals $P_i$, and $\alpha A/P_i (= \alpha (A/N)/(P_i/N))$ is a completely reducible module of finite length. Hence, $A/N$ is Artinian semisimple. Then, by Theorem 1 (2), $\alpha N$ is almost Artinian, and therefore $A$ is almost left Artinian by Proposition 2 (1).

The next is an easy combination of [3, Theorem 9] and Theorem 4.

Corollary 2. If $A$ is almost left Artinian then the full matrix ring $(A)_n$ is almost left Artinian, and $eAe$ is left Artinian for every idempotent $e$ of $A$.

Corollary 3 (cf. [1, Theorems 3 and 12]). (1) A (left and right) duo ring $A$ is almost left Artinian if and only if $A$ is the direct sum of an Artinian ring with identity and a nilpotent ring.

(2) A left duo ring $A$ is almost left Artinian if and only if $A$ is almost left Noetherian and every proper prime ideal of $A$ is maximal.

Proof. (2) is immediate from Theorem 4. It remains only to prove the only if part of (1). Let $e$ be an idempotent lifted from the identity of $A/N$ (Proposition 3). Since $A$ is a duo ring, $eA$ coincides with $eA$. Hence $A$ is the direct sum of the Artinian ring $eAe = A$ (Corollary 2) and the nilpotent ideal $I(e)$ contained in $N$ (Proposition 3).

Remark. Let $A$ be an almost left Artinian ring. If $AN = N$ then, as was claimed in [1, p. 17], $A$ is left Noetherian by Proposition 3 and Theorem 1 (1). However, this is a consequence of Hopkins' theorem, too. In fact, by [7, Theorem 1], $A$ has then a left identity.

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