On von Neumann regular rings. V

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Introduction. For several years, von Neumann regular rings and related rings are extensively studied (cf. for example, the bibliographies of [3] and [5], while for rings without identity, consult [6, [15], [16]). This paper is motivated by a question of Goodearl [5, Problem 10] concerning unit-regular rings and the question raised in [21]. Semi-simple Artinian rings and strongly regular rings are well-known examples of unit-regular rings, but arbitrary von Neumann regular rings need not be unit-regular. The class of unit-regular rings is closed under homomorphic images, direct products and direct limits. Such rings have many interesting properties (cf. [5]). Certain regular rings, \( V \)-rings and associated rings are here proved to be unit-regular. For example, left and right \( V \)-rings whose essential left ideals are ideals are unit-regular. (This is related to [5, Problem 10] and extends [5, Corollary 4.2].) Such rings, if indecomposable, are simple Artinian. But we first continue the study of ALD rings (redefined below) introduced in [21]. Further properties of ALD rings are developed, and certain results in [21] will be improved. A positive answer is given to the question raised in [21, Remark (p. 340)] which is related to [3, Query (a)].

Throughout, \( A \) represents an associative ring with identity and \( A \)-modules are unitary. We follow the notations and definitions in [6], [12] to [21]. Since for a given proper essential left ideal \( E \) of \( A \), any maximal left subideal is either essential in or a direct summand of \( E \) (in the latter case, it is a complement left subideal), the definition of ALD rings in [21] may be reformulated as follows: \( A \) is called an \( ALD (almost left duo) ring \) if, for any proper essential left ideal \( E \) of \( A \), every complement left subideal is an ideal of \( E \) and \( E \) is an ideal of \( A \).

In [21, Proposition 2.1], it is proved that a simple left module over an ALD ring is injective iff it is \( p \)-injective. Consequently, ALD regular rings are left \( V \)-rings. It is then natural to ask whether ALD left \( V \)-rings are von Neumann regular [21, Remark]. This is answered in the first section. Results in [5], [10] and [21] are generalized. Next, the decompositions of certain \( p \)-injective and \( p \)-\( V \)-rings will yield unit regular rings and new characteristic properties of semi-simple Artinian rings.

1. Von Neumann regular rings: We first prove an important lemma for ALD rings.

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Lemma 1.1. If $A$ is a semi-prime ALD ring, then $A$ is either semi-simple Artinian or reduced.

Proof. Assume that $A$ is not semi-simple Artinian. Then there exists a maximal left ideal $M$ of $A$ which is essential in $A$. Suppose there exists non-zero $a \in A$ such that $a^2 = 0$. If $l(a) \subseteq M$, then $a \in M$. If $l(a) \nsubseteq M$, then $M + l(a) = A$ and $1 = b + c$, where $b \in M$, $c \in l(a)$, which implies $a = ba \in M$ (an ideal of $A$). Thus $a \in M$ in any case. By Zorn's lemma, there exists a complement left subideal $K$ of $M$ such that $Aa \oplus K$ is essential in $A$. Then $KM \subseteq K$ implies $Ka \subseteq K \cap Aa = 0$, and hence $Aa \oplus K \subseteq l(a)$. Since $l(a)$ is then an ideal, we have $(Aa)^2 = A(aA)a \subseteq Al(a)a = 0$, contradicting the semi-primeness of $A$. This proves the lemma.

With suitable modifications, the proof of [14, Lemma 3], [4, Theorem 2.38] and Lemma 1.1 yield

Proposition 1.2. Let $A$ be a semi-prime ALD ring.
(1) The maximal left quotient ring of $A$ coincides with the right one. (This partly extends Utumi's result (cf. [5, Theorem 3.8]).)
(2) If $A$ is left or right continuous, then $A$ is either semi-simple Artinian or a left and right continuous strongly regular ring.

Remark 1. Since flat modules play an important role in ring theory, the following may be noted: For any $p$-injective left ideal $I$ of $A$, $A/I$ is a flat left $A$-module.

Following [12], a left $A$-module $M$ is called semi-simple if $f(M)$, the Jacobson radical of $M$, is zero. The next result, which is related to [3, Query (a)], improves [21, Proposition 2.1 (3), (4), Corollary 2.2, Theorems 2.4 and 2.5], while answering at the same time the question raised in [21, Remark].

Theorem 1.3. The following conditions are equivalent for an ALD ring $A$:
1) $A$ is von Neumann regular.
2) $A$ is a left V-ring.
3) $A$ is a right V-ring.
4) $A$ is fully idempotent.
5) $E = E^*$ for every essential left subideal $E$ of any proper principal left ideal of $A$. 

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6) Every cyclic semi-simple left \( A \)-module is flat.
7) Every maximal essential right ideal of \( A \) is \( p \)-injective.
8) Every simple right \( A \)-module is flat.
9) \( A \) is a semi-prime ring such that \( A/P \) is regular for every completely prime ideal \( P \) of \( A \).
10) \( A \) is a semi-prime left or right \( p \)-injective ring.

Proof. 1) implies 2) and 3) by Lemma 1.1, while 2) implies 5) by [12, Theorem 2.1].
3) implies 1) by [20, Proposition 9].
1) implies 4) and 6) through 10) evidently.
4) \( \implies 1) \): Note that a reduced ring is fully left idempotent iff fully right idempotent. Then [20, Proposition 9] and [21, Proposition 2.1 (4)] prove the implication.
5) \( \implies 2) \): Since the Jacobson radical of \( A \) is zero, every minimal left ideal of \( A \) is injective [21, Lemma 1.1]. Suppose there exists a simple left \( A \)-module \( V \) which is not injective. Then for any proper principal left ideal \( P \) of \( A \) and any non-zero left \( A \)-homomorphism \( f : P \rightarrow V \), \( \text{Ker} f \) is a maximal left subideal of \( P \) which is necessarily essential in \( A \), whence \( K = K^* \). Then there exists a maximal left ideal \( M \) of \( A \) such that \( K \subseteq M \) but \( P \not\subseteq M \). Since \( M \cap P = K \) and \( M + P = A \), then \( A/K = P/K \oplus M/K \), which shows that \( f \) may be extended to \( g : A \rightarrow V \). Therefore \( A \) is \( p \)-injective, which implies that \( A \) is injective by [21, Proposition 2.1 (1)]. This contradiction proves that 5) implies 2).
6) \( \implies 1) \): Since \( J(A/J(A)) = 0 \), \( A/J(A) \) is a flat left \( A \)-module. For any \( b \in J(A) \), \( b = bc \) for some \( c \in J(A) \) and there exists \( d \in A \) such that \( (1 - c)d = 1 \). Then \( b = b(1 - c)d = 0 \). Thus \( J(A) = 0 \). Since every simple left \( A \)-module is flat, \( A \) is regular by [19, Theorem 1.4] and Lemma 1.1.
7) implies 8) by Remark 1.
8) implies 1) by [19, Theorem 1.4 and Lemma 2.1] and Lemma 1.1.
9) implies 1) by [5, Theorem 1.21] and Lemma 1.1.
10) implies 1) by [7, Theorem 1], [16, Theorem 5] and Lemma 1.1.

Remark 2. (1) [5, Corollary 1.18] and Theorem 1.3 (4) imply that if every prime factor ring of \( A \) is ALD, then \( A \) is regular iff \( A \) is fully idempotent.
(2) An ALD left \( V \)-ring is either semi-simple Artinian or strongly regular and therefore unit-regular (cf. [19, Question]).

Theorem 1.4. The following are equivalent:
1) A is a prime ALD ring.
2) A is either simple Artinian or a left duo, left Ore domain.

Proof. 1) $\implies$ 2): Suppose A is not simple Artinian. By Lemma 1.1, A is reduced and hence an integral domain. If C is a non-zero complement left ideal of A then there exists a left ideal D such that $C \oplus D$ is essential in _A. Since A is an integral domain, it is easy to see that $D = 0$. This proves $C = A$. Noting that any left ideal of A is essential in a complement left ideal, we readily see that A is a left duo, left Ore domain.

2) $\implies$ 1): Obvious.

Call A an ELT ring if every essential left ideal is an ideal of A. It is known that prime ELT left self-injective rings are simple Artinian [8]. We may add the following to [21, Theorem 2.8].

**Theorem 1.5.** The following conditions are equivalent:
1) A is simple Artinian.
2) A is a prime ALD left V-ring.
3) A is a prime ALD ring with a divisible simple left module.
4) A is a prime ELT left and right V-ring.

Proof. Obviously, 1) $\implies$ 2) $\implies$ 3) and 1) $\implies$ 4).

3) $\implies$ 1): If A is not simple Artinian, then A is a left duo, left Ore domain by Theorem 1.4. Let U be a simple left A-module which is divisible. Then U is p-injective, since A is an integral domain. If $U = A/M$, then M is a maximal essential left ideal of A. For any non-zero $b \in M$, we consider the left A-homomorphism $f: Ab \to A/M$ defined by $f(ab) = a + M$ for all $a \in A$. Then there exists $c \in A$ such that $1 + M = f(b) = bc + M$. Since $bc \in M$, we obtain $1 \in M$, which contradicts $M \neq A$.

4) $\implies$ 1): By a remark in [5, Problem 52 (p. 350)], [20, Proposition 9] and Proposition 2. 3 below.

**Corollary 1.6.** Let A be a fully idempotent ring whose prime factor rings are ALD. Then A is a unit-regular left and right V-ring. In that case, A is left self-injective iff A is right self-injective.

Proof. Apply [5, Theorem 6.10 and Proposition 6.18] to Remark 2 (1) and Theorem 1.5. The last part follows from [5, Corollary 6.22].

We now consider strongly regular rings. The next result contains
improvements of some of the equivalent conditions in [10, Theorem].

**Theorem 1.7.** The following conditions are equivalent:

1) $A$ is strongly regular.

2) $A$ is a left and right duo ring such that $L \cap R = LR$ for every essential left ideal $L$ and every essential right ideal $R$ of $A$.

3) $A$ is either a left or right duo ring such that $L \cap R = RL$ for every essential left ideal $L$ and every essential right ideal $R$ of $A$.

4) $A$ is a left duo ring such that $L_1 \cap L_2 = L_1L_2$ for all essential left ideals $L_1, L_2$ of $A$.

5) $P \cap L = PL$ for every principal left ideal $P$ and every essential left ideal $L$ of $A$.

6) $A$ is an ELT fully left idempotent ring such that every proper prime ideal is completely prime.

7) Every maximal left ideal of $A$ is an ideal and every simple right $A$-module is flat.

8) Every maximal left ideal $I$ of $A$ is an ideal and $A/I_A$ is flat.

**Proof.** By [1, Remark (1) (p.248)], $A$ is fully idempotent iff $I^2 = I$ for any essential ideal $I$ of $A$. It therefore follows that if $A$ is a left duo ring such that $I^2 = I$ for every essential ideal $I$ of $A$, then $A$ is strongly regular. It is then easy to see that 1), 2) and 4) are equivalent.

Obviously, 1) implies 3), 5) through 8).

3) $\Rightarrow$ 1): For any $b \in A$, there exists a left ideal $K$ and a right ideal $R$ such that $L = Ab \oplus K$ is essential in $A$ and $E = bA \oplus R$ is essential in $A$. Then $b \in EL$ yields $b \in bAb$.

5) $\Rightarrow$ 1): Obviously, $A$ is a left duo ring. For any $b \in A$, there exists a left ideal $K$ such that $L = Ab \oplus K$ is essential in $A$. Then $b \in AbL$ yields $b \in Ab^2$.

6) $\Rightarrow$ 1): If $P$ is a proper prime ideal of $A$, then $A/P$ is an ELT, fully left idempotent integral domain, and therefore a division ring. Hence $A$ is strongly regular by [5, Corollary 1.18 and Theorem 3.2].

7) $\Rightarrow$ 8): If $I$ is a maximal left ideal, then $A/I$ is a division ring, and therefore $A/I_A$ is flat by 7).

8) $\Rightarrow$ 1): It suffices to show that $Ab + l(b) = A$ for every $b \in A$. Suppose $Ac + l(c) \neq A$ for some $c \in A$. If $L$ is a maximal left ideal containing $Ac + l(c)$, then $A/L_A$ is flat and hence $Ay \cap L = Ly$ for any $y \in A$, in particular $c = dc$ with some $d \in L$. Then, $1 = (1 - d) + d \in l(c) + L = L$, which is a contradiction.
2. ELT rings. A well-known theorem of Jain-Mohamed-Singh [9, Theorem 2.3] states that $A$ is an ELT left self-injective ring if and only if every left ideal of $A$ is quasi-injective. We begin this section with a lemma which contains direct consequences of definitions.

Lemma 2.1. Let $A = B \oplus C$, where $B, C$ are ideals of $A$.

1. If $A$ is an ELT ring then both $B$ and $C$ are ELT rings.
2. Every left $C$-module is a left $A$-module, and a left $C$-module which is a $p$-injective left $A$-module is a $p$-injective left $C$-module.
3. If $A$ is a left $p$-injective ring then both $B$ and $C$ are left $p$-injective rings.

The next decomposition theorem is motivated by [20, Question (p. 128)].

Following [15], we say that $A$ is a left $p$-$V$-ring if every simple left $A$-module is $p$-injective. Throughout, $S$ denotes the left socle of $A$.

Theorem 2.2. The following conditions are equivalent for a ring $A$:

1. $A$ is a direct sum of a semi-simple Artinian ring and a strongly regular ring with zero socle.
2. $A$ is an ELT left $p$-$V$-ring with finitely generated left socle.
3. $A$ is a semi-prime ELT ring whose simple right modules are flat and such that $\mathcal{S}$ is finitely generated.
4. $A$ is a semi-prime ELT left $p$-injective ring such that $\mathcal{S}$ is finitely generated.

Proof. It is easy to see that 1) implies 2) through 4).

2) $\implies$ 1): Since a left $p$-$V$-ring is fully left idempotent, $A$ is a semi-prime ring such that $\mathcal{S}$ is finitely generated. Then it is known that $S = Ae$ for some central idempotent $e$. Let $T = A(1-e)$. Then $A = S \oplus T$, $S$ is semi-simple Artinian, and $T$ is an ELT left $p$-$V$-ring with zero socle (Lemma 2.1 (1), (2)). Since $T/I_T$ is flat for any ideal $I$ of $T$, the condition (8) of Theorem 1.7 is satisfied. Hence, $T$ is strongly regular.

3) $\implies$ 1): Again $A = S \oplus T$, where $S$ is a semi-simple Artinian ring and $T$ is an ELT ring with zero socle such that every simple right $T$-module is flat. Then $T$ is strongly regular by Theorem 1.7 7).

4) $\implies$ 1): We obtain $A = S \oplus T$, where $S$ is a semi-simple Artinian ring and $T$ is a semi-prime ELT left $p$-injective ring with zero socle (Lemma 2.1 (1), (2), (3)). Since $T$ is semi-prime ELT, by applying [19, Lemma 2.1], we see that $T$ is left non-singular, and $J(T) = 0$ [9, p. 213]. Let $M$ be an arbitrary maximal left ideal of $T$, and let $t \in T$ such that $t^2 = 0$. If $I_r(t) \subseteq M$ then $M + I_r(t) = T$ implies $t \in M$ (an ideal of $T$).
Thus \( t \in M \) in any case, and \( t \in f(T) = 0 \), which proves that \( T \) is reduced. Now, \( T \) is strongly regular by [7, Theorem 1] and [16, Theorem 5].

In view of Theorem 2.2, we raise the following

**Question 1.** Suppose that \( A \) is a semi-prime ELT ring satisfying any one of the following conditions: 1) \( A \) is left \( p \)-injective; 2) every simple right \( A \)-module is flat. Then, is \( A \) von Neumann regular?

It is now known that a prime regular ring need not be primitive [2]. However, the proof of Theorem 2.2 and [19, Theorem 1.4] imply the following:

**Proposition 2.3.** A prime ELT ring is primitive with non-zero socle if \( A \) satisfies any one of the following conditions: 1) \( A \) is left \( p \)-injective; 2) \( A \) is fully left or right idempotent; 3) every cyclic semi-simple left \( A \)-module is flat (cf. [3, Problem 3]).

**Remark 3.** (1) If \( A \) is a prime ring with non-zero left singular ideal, then every non-zero left ideal of \( A \) contains a non-zero nilpotent element.

(2) \( A \) is a primitive left self-injective regular ring with non-zero socle iff \( A \) is a prime left self-injective ring with a maximal right annihilator.

In [16, Theorem 6], semi-simple Artinian rings are characterized in terms of ELT left Goldie rings. We here give a few characteristic properties in terms of ELT rings with maximum condition on annihilators.

**Theorem 2.4.** The following conditions are equivalent:
1) \( A \) is semi-simple Artinian.
2) \( A \) is an ELT left \( V \)-ring with maximum condition on left annihilators.
3) \( A \) is an ELT left \( V \)-ring with maximum condition on right annihilators.
4) \( A \) is an ELT left \( p \)-\( V \)-ring without infinite sets of orthogonal idempotents.
5) \( A \) is an ELT left and right \( V \)-ring whose proper factor rings satisfy the maximum condition on left annihilators.

**Proof.** Apply [17, Proposition 3], [19, Lemma 1.2], Theorem 2.2.
and [13, Corollary 1].

We now return to unit-regular rings and consider the following question of Goodearl [5, Problem 10 (p. 344)]: Are regular left and right $V$-rings unit-regular? A particular answer is contained in the next theorem.

**Theorem 2.5.** An ELT left and right $V$-ring is unit-regular.

**Proof.** Since every factor ring of an ELT ring is ELT, it is sufficient to apply [5, Theorem 6.10], [20, Proposition 9] and Theorem 1.5.4.

The next is a combination of [8, Theorem 2.3], [20, Proposition 9], Theorems 2.2 and 2.5.

**Corollary 2.6.** $A$ is unit-regular in each of the following cases:

1) $A$ is an ELT continuous right $V$-ring.
2) $A$ is a right $V$-ring whose essential left ideals are quasi-injective.
3) $A$ is an ELT right $V$-ring whose minimal left ideals are injective.
4) $A$ is an ELT left $p$-$V$-ring such that $A$ is finitely generated.
5) $A$ is a semi-prime ELT left $p$-injective ring such that $A$ is finitely generated.

As usual, $M_n(A)$ denotes the ring of all $n \times n$ matrices over $A$. For direct finiteness, consult [5, Chapter 5]. Applying [5, Proposition 6.11, Corollaries 4.7, 6.4, 6.12, 6.16, and Theorems 4.14, 6.6, 6.21] to Theorems 1.5.4) and 2.5, we get

**Proposition 2.7.** Let $A$ be an ELT left and right $V$-ring.

1) The maximal left quotient ring of $A$ coincides with the right one.
2) Every non-zero ideal of $A$ contains a non-zero central idempotent.
3) If $F$ is a finitely generated left $A$-module, then every injective or surjective endomorphism of $F$ is an automorphism, and $F$ is directly finite.
4) Let $P$ be a finitely generated projective left $A$-module. Then (a) $\text{End}_A(P)$ is a unit-regular left and right $V$-ring; (b) if $M, N$ are left $A$-modules such that $P \oplus M \cong P \oplus N$, then $M \cong N$.
5) If $M, N$ are finitely generated projective left $A$-modules and $n$ is a positive integer such that the direct sum of $n$ copies of $M$ is isomorphic to the direct sum of $n$ copies of $N$, then $M \cong N$.
6) If $B$ is an ELT left and right $V$-ring and $n$ is a positive integer such that $M_n(A) \cong M_n(B)$, then $A \cong B$.

Proposition 2.7 (2) implies the following...
Corollary 2.8. An indecomposable ELT left and right V-ring is simple Artinian.

Finally, [5, Corollary 6.3 and Theorem 6.10], theorem 1.5 4), theorem 2.5 and Proposition 2.7 (1) yield the following answer to [3, Query (c)].

Corollary 2.9. Let $A$ be a regular ring whose maximal left quotient ring $Q$ is an ELT, right V-ring. Then $Q$ is right self-injective and $A$ is a unit-regular left and right V-ring.

In [11], it is shown that certain results on injective and $p$-injective modules have analogues in the theory of semi-groups. We conclude with the following:

Question 2. Are there semi-group analogues of Theorems 1.3 and 2.4?

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