Ergodic theorems for d-parameter semigroups of Dunford-Schwartz operators

Ryotaro Sato*

*Okayama University

ERGODIC THEOREMS FOR $d$-PARAMETER SEMIGROUPS OF DUNFORD-SCHWARTZ OPERATORS

RYOTARO SATO

1. Introduction. Let $\Gamma = \{T(t_1, \ldots, t_d); t_1, \ldots, t_d > 0\}$ be a strongly continuous $d$-parameter semigroup of Dunford-Schwartz operators on $L_1(\Omega) = L_1(\Omega, \mathcal{F}, \mu)$, where $(\Omega, \mathcal{F}, \mu)$ is a $\sigma$-finite measure space. In this paper $\Gamma$ will be extended to a semigroup of linear operators on the class $L_1(\Omega) + L_\infty(\Omega)$ of all functions $f$ of the form $f = g + h$, with $g \in L_1(\Omega)$ and $h \in L_\infty(\Omega)$, so that $\|T(t_1, \ldots, t_d)f\|_p \leq 1$ for every $1 \leq p \leq \infty$ and also so that $\lim_{n} T(t_1, \ldots, t_d)f_n = T(t_1, \ldots, t_d)f$ almost everywhere on $\Omega$ whenever $f_n \in L_\infty(\Omega)$, $\sup_n \|f_n\|_\infty < \infty$ and $\lim_{n} f_n = f$ almost everywhere on $\Omega$. Then for every $f \in L_1(\Omega) + L_\infty(\Omega)$ the averages

$$A(\alpha_1, \ldots, \alpha_d)f = \frac{1}{\alpha_1 \cdots \alpha_d} \int_0^{\alpha_1} \cdots \int_0^{\alpha_d} T(t_1, \ldots, t_d)f \, dt_1 \cdots dt_d$$

are well-defined, and now it would be interesting to ask the following questions: For what functions $f$ does the almost everywhere convergence of the averages $A(\alpha_1, \ldots, \alpha_d)f$ hold as $\alpha_1 \to 0$, $\cdots$, $\alpha_d \to 0$ independently? For what functions $f$ does the almost everywhere convergence of the averages $A(\alpha_1, \ldots, \alpha_d)f$ hold as $\alpha_1 \to \infty$, $\cdots$, $\alpha_d \to \infty$ independently?

It will be proved below that if $f \in L_1(\Omega) + L_\infty(\Omega)$ satisfies

$$\left\{ \frac{|f|}{t} \left[ \log \frac{|f|}{t} \right]^{d-1} \right\} \, d\mu < \infty$$

for every $t > 0$, then the averages $A(\alpha_1, \ldots, \alpha_d)f$ converge almost everywhere on $\Omega$ as $\alpha_1 \to 0$, $\cdots$, $\alpha_d \to 0$ independently, and also the averages $A(\alpha_1, \cdots, \alpha_d)f$ converge almost everywhere on $\Omega$ as $\alpha_1 \to \infty$, $\cdots$, $\alpha_d \to \infty$ independently. This may be considered to be an extension of Terrell's local ergodic theorem [10] and Dunford-Schwartz's ergodic theorem [2].

The method of proof chiefly depends upon a weak type maximal inequality similar to Fava's [4].

2. Preliminaries. Let $(\Omega, \mathcal{F}, \mu)$ be a $\sigma$-finite measure space and let $L_p(\Omega) = L_p(\Omega, \mathcal{F}, \mu)$, $1 \leq p \leq \infty$, be the usual Banach spaces of real or complex functions on $(\Omega, \mathcal{F}, \mu)$. A Dunford-Schwartz operator $T$ on $L_1(\Omega)$
is a linear contraction on $L_1(\Omega)$ (i.e. $\|T\|_1 \leq 1$) such that for every $f \in L_1(\Omega) \cap L_\infty(\Omega)$
\[ \|Tf\|_\infty \leq \|f\|_\infty. \]
It is well-known that a Dunford-Schwartz operator $T$ on $L_1(\Omega)$ satisfies
\[ \|Tf\|_p \leq \|f\|_p \]
for all $f \in L_1(\Omega) \cap L_p(\Omega)$, with $1 < p < \infty$. By this, $T$ can be uniquely extended to a linear contraction on each $L_p(\Omega)$, with $1 < p < \infty$. Furthermore it can be extended to a linear contraction on $L_\infty(\Omega)$ as follows. If $0 \leq f \in L_\infty(\Omega)$, choose $f_n \in L_1(\Omega)$ so that $0 \leq f_n \leq f_{n+1} \leq f$ and $\lim f_n = f$ almost everywhere (a.e.) on $\Omega$. Then for $n > m$ we have
\[ |Tf_n - Tf_m| \leq \tau(f_n - f_m) \leq (\lim_k \tau f_k) - \tau f_m \text{ a.e. on } \Omega \]
where $\tau$ denotes the linear modulus of $T$ in the sense of Chacon-Krenkel [1]. (Thus $\tau$ is a positive Dunford-Schwartz operator on $L_1(\Omega)$ such that
\[ \tau g = \sup \{|Th| : h \in L_1(\Omega), |h| \leq g \text{ a.e. on } \Omega}\]
for any $0 \leq g \in L_1(\Omega)$.) On the other hand, if $0 \leq u \in L_1(\Omega) \cap L_\infty(\Omega)$ and $0 < u$ a.e. on $\Omega$, then it may be readily seen that $0 \leq \tau^* u \in L_1(\Omega)$ and $\|\tau^* u\|_1 \leq \|u\|_1$, where $\tau^*$ denotes the adjoint operator of $\tau$, acting on $L_\infty(\Omega) = L_1(\Omega)^*$. Thus, putting
\[ g_n = (\lim_k \tau f_k) - \tau f_n \text{ a.e. on } \Omega, \]
we have, by Lebesgue's dominated convergence theorem,
\[ \int g_n u \, d\mu = \int (\lim_k \tau f_k) \tau^* u \, d\mu - \int f_n \tau^* u \, d\mu \to 0 \]
as $m \to \infty$. Since $u > 0$ a.e. on $\Omega$ and $g_n \geq g_{n+1} \geq 0$ a.e. on $\Omega$, this proves that $\lim g_n = 0$ a.e. on $\Omega$, and hence for almost all $\omega \in \Omega$ the sequence $Tf_n(\omega)$, $n = 1, 2, \cdots$, is a Cauchy sequence. Therefore it is possible to define
\[ Tf(\omega) = \lim_{n \to \infty} Tf_n(\omega) \text{ a.e. on } \Omega. \]

It is now a routine matter to check that this definition of $Tf$ does not depend upon the particular choice of such a sequence $(f_n)$ in $L_1(\Omega)$, and so by linearity $T$ can be extended to a linear operator on $L_\infty(\Omega)$. From the definition of $T$ on $L_\infty(\Omega)$, it follows that $\|T\|_\infty \leq 1$ and that if $f_n \in L_\infty(\Omega)$, $n = 1, 2, \cdots$, is a sequence satisfying $\sup_n \|f_n\|_\infty < \infty$ and $\lim f_n = f$ a.e. on $\Omega$ for some $f \in L_\infty(\Omega)$, then

http://escholarship.lib.okayama-u.ac.jp/mjou/vol23/iss1/8
ERGODIC THEOREMS

\[ Tf = \lim_{n} T f_n \, \text{a.e. on } \Omega. \]

The above discussion ensures that we may and will assume, throughout this paper, that a Dunford-Schwartz operator \( T \) is a linear operator on the class \( L_1(\Omega) + L_\infty(\Omega) \) such that \( \|T\|_p \leq 1 \) on each \( L_p(\Omega) \) with \( 1 \leq p \leq \infty \) and also such that

\[ Tf = \lim_{n} T f_n \, \text{a.e. on } \Omega \]

whenever \( f_n \in L_\infty(\Omega) \), \( \sup\{\|f_n\|_\infty : n \geq 1\} < \infty \) and \( f = \lim f_n \, \text{a.e. on } \Omega. \)

Let us now consider a \( d \)-parameter semigroup \( \Gamma = \{T(t_1, \cdots, t_d) : t_1, \cdots, t_d > 0\} \) of Dunford-Schwartz operators on \( L_1(\Omega) + L_\infty(\Omega), \ d \geq 1 \) being a fixed integer. Thus each \( T(t_1, \cdots, t_d) \) is a Dunford-Schwartz operator on \( L_1(\Omega) + L_\infty(\Omega) \), and \( \Gamma \) satisfies

\[ T(t_1, \cdots, t_d)T(s_1, \cdots, s_d) = T(t_1 + s_1, \cdots, t_d + s_d). \]

Throughout this paper we shall assume that \( \Gamma \) is strongly continuous with respect to the norm topology of \( L_1(\Omega) \), i.e. for each \( f \in L_1(\Omega) \) the function \( T(t_1, \cdots, t_d)f \) of \( (t_1, \cdots, t_d) \in R^d_+ \), where \( R^d_+ = \{(t_1, \cdots, t_d) : t_1, \cdots, t_d > 0\} \), is continuous with respect to the norm topology of \( L_1(\Omega) \). It then follows from an approximation argument that \( \Gamma \) is strongly continuous with respect to the norm topology of each \( L_p(\Omega) \) with \( 1 \leq p < \infty \), and that for each \( f \in L_p(\Omega) \) with \( 1 \leq p < \infty \) there exists a scalar function \( g(t_1, \cdots, t_d, \omega) \), defined on \( R^d_+ \times \Omega \) and measurable with respect to the product of the Lebesgue measurable subsets of \( R^d_+ \) and \( \mathcal{F} \), such that for each fixed \( (t_1, \cdots, t_d) \in R^d_+ \), \( g(t_1, \cdots, t_d, \omega) \) as a function of \( \omega \in \Omega \) belongs to the equivalence class of \( T(t_1, \cdots, t_d)f \in L_p(\Omega) \). Therefore, in the sequel, \( g(t_1, \cdots, t_d, \omega) \) will be denoted by \( T(t_1, \cdots, t_d)f(\omega) \). It then follows from Fubini's theorem that there exists a \( \mu \)-null set \( E(f) \), dependent on \( f \) but independent of \( (t_1, \cdots, t_d) \), such that for each fixed \( \omega \in E(f) \), \( T(t_1, \cdots, t_d)f(\omega) \) as a function of \( (t_1, \cdots, t_d) \in R^d_+ \) is Lebesgue integrable over every finite interval \( (\alpha_1, \beta_1) \times \cdots \times (\alpha_d, \beta_d) \subset R^d_+ \) with respect to the Lebesgue measure, and the integral

\[ \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_d}^{\beta_d} T(t_1, \cdots, t_d)f(\omega) \, dt_1 \cdots dt_d \quad (\omega \notin E(f)) \]

as a function of \( \omega \in \Omega \) belongs to the equivalence class of the Bochner integral

\[ \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_d}^{\beta_d} T(t_1, \cdots, t_d)f \, dt_1 \cdots dt_d \quad (\in L_p(\Omega)). \]

Next we will observe that a similar situation holds for \( f \in L_\infty(\Omega) \). In fact, let \( (f_n) \) be a sequence in \( L_1(\Omega) \) such that \( |f_n| \leq |f| \) and \( \lim f_n = f \) a.e.
on $\Omega$. Then for every $(t_1, \cdots, t_d) \in \mathbb{R}^d$

$$T(t_1, \cdots, t_d)f = \lim_n T(t_1, \cdots, t_d)f_n \quad \text{a.e. on } \Omega,$$

and hence by Fubini's theorem we may define

$$g(t_1, \cdots, t_d, \omega) = \lim_n T(t_1, \cdots, t_d)f_n(\omega)$$

for almost all $(t_1, \cdots, t_d, \omega) \in \mathbb{R}^d \times \Omega$ with respect to the product of the Lebesgue measure and $\mu$. Since, for each fixed $(t_1, \cdots, t_d) \in \mathbb{R}^d$, $g(t_1, \cdots, t_d, \omega)$ as a function of $\omega \in \Omega$ belongs to the equivalence class of $T(t_1, \cdots, t_d)f \in L_\omega(\Omega)$, $g(t_1, \cdots, t_d, \omega)$ will be again denoted by $T(t_1, \cdots, t_d)f(\omega)$. It then follows from Fubini's theorem that there exists a $\mu$-null set $E(f)$, dependent on $f$ but independent of $(t_1, \cdots, t_d)$, such that for each fixed $\omega \notin E(f)$, $T(t_1, \cdots, t_d)f(\omega)$ as a function of $(t_1, \cdots, t_d) \in \mathbb{R}^d$ is Lebesgue integrable over every finite interval $(\alpha_1, \beta_1) \times \cdots \times (\alpha_d, \beta_d) \subset \mathbb{R}^d$, and the integral

$$\int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_d}^{\beta_d} T(t_1, \cdots, t_d)f(\omega) \, dt_1 \cdots dt_d \quad (\omega \notin E(f))$$

as a function of $\omega \in \Omega$ belongs to $L_\omega(\Omega)$ and satisfies

$$\left( \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_d}^{\beta_d} T(t_1, \cdots, t_d)f(\omega) \, dt_1 \cdots dt_d, \ u(\omega) \right) = \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_d}^{\beta_d} f \cdot T(t_1, \cdots, t_d)^*u \, dt_1 \cdots dt_d$$

for all $u \in L_1(\Omega) \cap L_\omega(\Omega)$ (where we let $\langle f, \ u \rangle = \int_\Omega fu \, d\mu$) and hence for all $u \in L_1(\Omega)$, because the adjoint semigroup $\Gamma^* = \{ T(t_1, \cdots, t_d)^* : t_1, \cdots, t_d > 0 \}$ may be regarded as a semigroup of Dunford-Schwartz operators on $L_1(\Omega) + L_\omega(\Omega)$ which is strongly continuous with respect to the norm topology of $L_1(\Omega)$.

Now let $f$ be in the class $L_1(\Omega) + L_\omega(\Omega)$ and write $f = g + h$ with $g \in L_1(\Omega)$ and $h \in L_\omega(\Omega)$. Then we may define the integral

$$\int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_d}^{\beta_d} T(t_1, \cdots, t_d)f \, dt_1 \cdots dt_d \quad (\in L_1(\Omega) + L_\omega(\Omega))$$

over the finite interval $(\alpha_1, \beta_1) \times \cdots \times (\alpha_d, \beta_d) \subset \mathbb{R}^d$ to be the function

$$\left( \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_d}^{\beta_d} T(t_1, \cdots, t_d)f \, dt_1 \cdots dt_d \right)(\omega) = \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_d}^{\beta_d} T(t_1, \cdots, t_d)g(\omega) \, dt_1 \cdots dt_d + \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_d}^{\beta_d} T(t_1, \cdots, t_d)h(\omega) \, dt_1 \cdots dt_d \quad \text{a.e. on } \Omega.$$
It is clear that this definition of the integral does not depend upon the particular choice of such a decomposition \( f = g + h \), and we have the relation
\[
\left< \int_{t_1}^{s_1} \cdots \int_{t_d}^{s_d} T(t_1, \ldots, t_d) f \, dt_1 \cdots dt_d, \ u \right>
= \int_{t_1}^{s_1} \cdots \int_{t_d}^{s_d} \left< T(t_1, \ldots, t_d) f, \ u \right> \, dt_1 \cdots dt_d
\]
for all \( u \in L_1(\Omega) \cap L_\infty(\Omega) \).

3. Maximal operators and inequalities. We will call an operator \( M \), which maps functions in \( L_1(\Omega) + L_\infty(\Omega) \) into measurable functions on \( (\Omega, \mathcal{F}) \), a maximal operator if it satisfies:

(a) \(| M(f + g) | \leq | Mf | + | Mg |\) and \(| M(cf) | = | c | \, | Mf |\) a.e. on \( \Omega \)
   where \( c \) is a constant;

(b) There exists a constant \( A > 0 \) such that for every \( f \in L_1(\Omega) + L_\infty(\Omega) \)
   and all \( \lambda > 0 \)
\[
||Mf||_\infty \leq A \|f\|_\infty \quad \text{and} \quad \mu \{|Mf| > \lambda\} \leq \frac{A}{\lambda} \|f\|_1.
\]

Lemma 1. If \( M \) is a maximal operator on \( L_1(\Omega) + L_\infty(\Omega) \) then for every \( f \in L_1(\Omega) + L_\infty(\Omega) \) and all \( t > 0 \)
\[
\mu \{|Mf| \geq (A+1)t\} \leq \frac{A}{t} \int_{|f| > t} |f| \, d\mu.
\]

Proof. Putting \( f^t(\omega) = f(\omega)1_{\{|f| > t\}}(\omega) \) and \( f_1(\omega) = f(\omega) - f^t(\omega) \), we have \( \|f_1\|_\infty \leq t \) and
\[
|Mf| \leq |M(f^t)| + |M(f_1)| \leq |M(f^t)| + At.
\]
Thus \( \{|Mf| \geq (A+1)t\} \subseteq \{|M(f^t)| > t\} \) and consequently we have
\[
\mu \{|Mf| \geq (A+1)t\} \leq \mu \{|M(f^t)| > t\} \leq \frac{A}{t} \int_{|f| > t} |f| \, d\mu,
\]
which completes the proof.

Corollary. If \( M \) is a maximal operator on \( L_1(\Omega) + L_\infty(\Omega) \) then for every \( f \in L_1(\Omega) + L_\infty(\Omega) \)
\[
\int |Mf|^p \, d\mu \leq \frac{p A(A+1)^p}{p-1} \int |f|^p \, d\mu \quad (1 < p < \infty)
\]
and
\begin{equation}
\int |Mf| \, d\mu \leq (A+1) \left[ \mu(\mathcal{Q}) + A \int_{\{|f| > 1\}} |f| \log |f| \, d\mu \right].
\end{equation}

**Proof.** If $1 < p < \infty$ then, by Fubini's theorem,

\begin{align*}
\int |Mf|^p \, d\mu &= p \int_0^\infty r^{p-1} \mu\{|Mf| > r\} \, dr \\
&\leq p \int_0^\infty dr \left[ r^{p-1} \frac{A(A+1)}{r} \int_{\{|f| > \frac{r}{A+1}\}} |f| \, d\mu \right] \\
&= p A(A+1) \int_0^\infty d\mu(\omega) \left[ |f(\omega)|^{(A+1)/|f(\omega)|} r^{p-2} \right] \, dr \\
&= \frac{p A(A+1)^p}{p - 1} \int_0^\infty |f(\omega)|^p \, d\mu(\omega),
\end{align*}

and if $p = 1$ then, again by Fubini's theorem,

\begin{align*}
\int |Mf| \, d\mu &= \int_0^\infty \mu\{|Mf| > r\} \, dr \\
&\leq (A+1) \mu(\mathcal{Q}) + \int_{A+1}^\infty \mu\{|Mf| > r\} \, dr \\
&\leq (A+1) \mu(\mathcal{Q}) + \int_{A+1}^\infty dr \left[ \frac{A(A+1)}{r} \int_{\{|f| > \frac{r}{A+1}\}} |f| \, d\mu \right] \\
&= (A+1) \mu(\mathcal{Q}) + A(A+1) \int_{\{|f| > 1\}} d\mu(\omega) \left[ |f(\omega)|^{1/r} \frac{1}{r} \right] dr \\
&= (A+1) \mu(\mathcal{Q}) + A(A+1) \int_{\{|f| > 1\}} |f(\omega)| \log |f(\omega)| \, d\mu(\omega).
\end{align*}

Hence the proof is completed. (This argument is standard.)

For each $n \geq 0$, let $R_n(\mathcal{Q})$ be the class of all functions $f$ in $L_1(\mathcal{Q}) + L_\infty(\mathcal{Q})$ such that

\begin{equation}
\int_{\{|f| > 1\}} \left| f \right| \left[ \log \left| \frac{f}{t} \right| \right]^n \, d\mu < \infty
\end{equation}

for all $t > 0$, and let $L(\mathcal{Q})[\log^+ L(\mathcal{Q})]^n$ be the class of all functions $f$ in $L_1(\mathcal{Q}) + L_\infty(\mathcal{Q})$ such that

\begin{equation}
\int_{\{|f| > 1\}} |f| \left[ \log |f| \right]^n \, d\mu < \infty.
\end{equation}

The classes $R_n(\mathcal{Q})$, $n \geq 0$, originally introduced by Fava [4] in order to
obtain a weak type inequality for a product of maximal operators, have the
following properties:

(i) For each \( n \geq 0 \), \( R_n(\Omega) \subset L(\Omega) [\log^{-1} L(\Omega)]^n \) and both classes coincide
if and only if \( \mu(\Omega) < \infty \).

(ii) \( L_1(\Omega) \subset R_0(\Omega) \) and both classes coincide if and only if \( \mu(\Omega) < \infty \).

(iii) For each \( n \geq 0 \), \( R_{n+1}(\Omega) \subset R_n(\Omega) \) and both classes coincide if and
only if there exists a constant \( \delta > 0 \) such that \( E \in \mathcal{F} \) and \( \mu(E) > 0 \) implies
\( \mu(E) > \delta \).

(iv) For each \( n \geq 0 \), \( R_n(\Omega) \) is a linear manifold of \( L_1(\Omega) + L_\infty(\Omega) \).

(v) For each \( n \geq 0 \), \( R_n(\Omega) \) includes the linear manifold generated by
\( \bigcup_{1 \leq p \leq n} L_p(\Omega) \), and both manifolds coincide if and only if \( \mu(\Omega) < \infty \) and there
exists a constant \( \delta > 0 \) such that \( E \in \mathcal{F} \) and \( \mu(E) > 0 \) implies \( \mu(E) > \delta \).

Some of the above properties are found in [4] and the others may be
directly proved, and hence we omit the details.

The following maximal theorem is a key lemma to prove individual
ergodic theorems for \( d \)-parameter semigroups \( \Gamma = \{ T(t_1, \ldots, t_d) : t_1, \ldots, t_d > 0 \} \) of Dunford-Schwartz operators on \( L_1(\Omega) + L_\infty(\Omega) \).

**Theorem 1.** Let \( M \) be a maximal operator on \( L_1(\Omega) + L_\infty(\Omega) \), and let
\( A > 0 \) be the constant relating to \( M \) as in the definition of a maximal operator.
Then for each \( n \geq 0 \) there corresponds a constant \( B_n = B(n, A) > 0 \) so that
for every \( f \in R_{n+1}(\Omega) \) and all \( t > 0 \)

\[
\int_{\{ |Mf| > t \}} \left[ \log \frac{|Mf|}{t} \right]^n d\mu \leq \int_{\{ \mu_{n}>t \}} \frac{B_n}{t} \left[ \log \frac{B_n}{t} \right]^{n+1} d\mu.
\]

Consequently \( f \in R_{n+1}(\Omega) \) implies \( Mf \in R_n(\Omega) \).

**Proof.** Fix any \( a > 1 \). Then for \( f \in R_{n+1}(\Omega) \) and \( t > 0 \), putting \( g = f/t \), we have by Fubini's theorem

\[
\int_{\{ |Mf| > at \}} \left[ \log \frac{|Mf|}{t} \right]^n d\mu = \int_{\{ \mu f > a \}} |Mg| \left( [\log r]^n + n [\log r]^{n-1} \right) d\mu
\]

\[
= \int_{a}^{\infty} \mu \{ |Mg| > a \} \left( [\log r]^n + n [\log r]^{n-1} \right) dr
\]

\[
= \int_{a}^{\infty} \mu \{ |Mg| > a \} \left( [\log r]^n + n [\log r]^{n-1} \right) dr
\]

\[
= I + II.
\]

Since Lemma 1 implies that
\[
\mu \left( |Mg| > r \right) \leq \frac{A(A+1)}{r} \int_{\{g > r\}} |g| \ d\mu \quad (r > 0),
\]

it follows that

\[
I \leq \frac{A}{a} \int_{\{g > a\}} (A+1) |g| \ d\mu \times \int_{1}^{a} \left( [\log r]^{n} + n [\log r]^{n-1} \right) \ dr
\]

\[
\leq I(A) \int_{\{g > a\}} (A+1) |g| \left( [\log (A+1) |g|] \right)^{n+1} \ d\mu,
\]

where

\[
I(A) = \frac{A}{a} [\log a]^{-1} \int_{1}^{a} \left( [\log r]^{n} + n [\log r]^{n-1} \right) \ dr.
\]

Further, since \( a > 1 \) and \( n \geq 0 \), it follows that

\[
II \leq \int_{a}^{\infty} \left( \frac{A(A+1)}{r} \int_{\{g > r\}} |g| \ d\mu \left( [\log r]^{n} + n [\log r]^{n-1} \right) \right) \ dr
\]

\[
= \int_{a}^{\infty} d\mu(\omega) \left[ A(A+1) |g(\omega)| \int_{a}^{\infty} \frac{1}{r} \left( [\log r]^{n} + n [\log r]^{n-1} \right) \ dr \right]
\]

\[
\leq \int_{a}^{\infty} d\mu(\omega) \left[ A(A+1) |g(\omega)| \left( [\log (A+1) |g(\omega)|]^{n+1} \right.ight.
\]

\[
+ [\log (A+1) |g(\omega)|]^{n+1} \left. \right) \right]
\]

\[
\leq A(1 + [\log a]^{-1}) \int_{a}^{\infty} (A+1) |g(\omega)| \left( [\log (A+1) |g(\omega)|]^{n+1} \right. \ d\mu(\omega).
\]

Thus, letting \( B_{n} = \left[ I(A) + A(1 + [\log a]^{-1}) + 1 \right] (A+1) \), we get

\[
\int_{\{|Mg| > a\}} \frac{|Mg|}{t} \left( [\log |Mg|]^{n} \right) \ d\mu \leq \int_{\{|B_{n}| > a\}} \frac{B_{n}|f|}{t} \left( [\log B_{n}|f|]^{n+1} \right) \ d\mu,
\]

which completes the proof.

**Theorem 2.** Let \( \Gamma = \{ T(t_{1}, \cdots, t_{d}) ; \ t_{1}, \cdots, t_{d} > 0 \} \) be a \( d \)-parameter semigroup of Dunford-Schwartz operators on \( L_{1}(\Omega) + L_{\infty}(\Omega) \) which is assumed to be strongly continuous with respect to the norm topology of \( L_{1}(\Omega) \). For \( f \in L_{1}(\Omega) + L_{\infty}(\Omega) \), define

\[
f^{*}(\omega) = \sup_{\alpha_{1}, \cdots, \alpha_{d} > 0} \frac{1}{\alpha_{1} \cdots \alpha_{d}} \left| \int_{0}^{t_{d}} \cdots \int_{0}^{t_{1}} T(t_{1}, \cdots, t_{d}) f(\alpha_{1} \cdots \alpha_{d}) dt_{1} \cdots dt_{d} \right|.
\]

Then for each \( k \geq d - 1 \) there corresponds a constant \( C_{k}(d) > 0 \) so that
(i) if \( k \geq d \) then for every \( f \in R_k(\Omega) \) and all \( t > 0 \)
\[
\int_{\{f^* > t\}} \frac{f^*}{t} \left[ \log \frac{f^*}{t} \right]^{k-d} \, d\mu \\
\leq \int_{\{C_k(d)f^* > t\}} \frac{C_k(d)}{t} \left[ \log \frac{C_k(d)}{t} \right]^{k} \, d\mu \quad (\leq \infty),
\]

(ii) if \( k = d-1 \) then for every \( f \in R_{d-1}(\Omega) \) and all \( t > 0 \)
\[
\mu \{f^* > t\} \leq \int_{\{C_{d-1}(d)f^* > t\}} \frac{C_{d-1}(d)}{t} \left[ \log \frac{C_{d-1}(d)}{t} \right]^{d-1} \, d\mu \quad (\leq \infty).
\]

Consequently \( f \in R_k(\Omega) \) with \( k \geq d \) implies \( f^* \in R_{k-d}(\Omega) \).

**Proof.** We proceed by induction on \( d \).

First suppose that \( d = 1 \). It is then known by [7] that there exists a one-parameter semigroup \( \{\tau_t(t_1) : t_1 > 0\} \) of positive Dunford-Schwartz operators on \( L_1(\Omega) + L_\infty(\Omega) \), strongly continuous with respect to the norm topology of \( L_1(\Omega) \), such that for every \( f \in L_1(\Omega) + L_\infty(\Omega) \) and all \( t_1 > 0 \)
\[
|T(t_1)f| \leq \tau_1(t_1)|f| \quad \text{a.e. on } \Omega.
\]

Thus, for \( f \in L_1(\Omega) + L_\infty(\Omega) \), if we set
\[
M^- f(\omega) = \sup_{\alpha > 0} \frac{1}{\alpha} \int_{\alpha}^{\infty} \tau_1(t_1)|f|(\omega) \, dt_1,
\]
then we have
\[
f^* \leq M^- f \quad \text{a.e. on } \Omega.
\]

Since \( M^- \) is a maximal operator on \( L_1(\Omega) + L_\infty(\Omega) \) with \( A = 1 \) (cf. [4] or [5]), we observe by Lemma 1 and Theorem 1 that the theorem holds for \( d = 1 \).

Next let us assume that the theorem holds for \( d = i - 1 \). To show that the theorem holds for \( d = i \), we define for each \( n \geq 1 \) an \( i \)-parameter semigroup \( F_n = \{T_n(t_1, \cdots, t_i) : t_1, \cdots, t_i \geq 0\} \) of Dunford-Schwartz operators on \( L_1(\Omega) + L_\infty(\Omega) \) by the relation
\[
T_n(t_1, \cdots, t_i) = \begin{cases} 
I & \text{if } t_1 = t_2 = \cdots = t_i = 0 \\
T(t_1 + u_i/n, \cdots, t_i + u_i/n) & \text{otherwise}
\end{cases}
\]

where \( u_i = (t_1 + \cdots + t_i) - t_k \) for \( 1 \leq k \leq i \). Put, for \( f \in L_1(\Omega) + L_\infty(\Omega) \),
\[
M_n f(\omega) = \sup_{\alpha_1, \cdots, \alpha_i > 0} \frac{1}{\alpha_1 \cdots \alpha_i} | \int_0^{\alpha_1} \cdots \int_0^{\alpha_i} T_n(t_1, \cdots, t_i) f(\omega) \, dt_1 \cdots dt_i |.
\]

Then clearly we have
\[ f^*(\omega) \leq \lim \inf_n M_n f(\omega) \text{ a.e. on } \Omega, \]

and hence by Fatou's lemma it is sufficient to observe that the inequalities of the theorem hold, replacing \( f^* \) by \( M_n f \).

For this purpose we next define, for each \( n \geq 1 \), an \((i-1)\)-parameter semigroup \( J_n = \{ S_n(t_2, \cdots, t_i) ; t_2, \cdots, t_i > 0 \} \) of Dunford-Schwartz operators on \( L_1(\Omega) + L_\infty(\Omega) \) by the following relation

\[ S_n(t_2, \cdots, t_i) = T_n(0, t_2, \cdots, t_i). \]

Let us denote by \( \{ \tau_n(t_1) ; t_1 \geq 0 \} \) a one-parameter semigroup of positive Dunford-Schwartz operators on \( L_1(\Omega) + L_\infty(\Omega) \), strongly continuous with respect to the norm topology of \( L_1(\Omega) \), such that for every \( f \in L_1(\Omega) + L_\infty(\Omega) \) and all \( t_1 > 0 \)

\[ |T_n(t_1)f| \leq \tau_n(t_1)|f| \quad \text{a.e. on } \Omega, \]

where we let \( T_n(t_1) = T_n(t_1, 0, \cdots, 0) \) for \( t_1 > 0 \). Then for \( f \in L_1(\Omega) + L_\infty(\Omega) \) and \( \alpha_1, \cdots, \alpha_i > 0 \) we have

\[ \frac{1}{\alpha_1 \cdots \alpha_i} \left| \int_0^{\alpha_1} \int_0^{\alpha_2} \cdots \int_0^{\alpha_i} T_n(t_1, \cdots, t_i) f dt_1 \cdots dt_i \right| \]

\[ = \frac{1}{\alpha_1} \int_0^{\alpha_1} T_n(t_1) \left[ \int_0^{\alpha_2} \cdots \int_0^{\alpha_i} S_n(t_2, \cdots, t_i) f dt_2 \cdots dt_i \right] dt_1 \]

\[ \leq \frac{1}{\alpha_1} \int_0^{\alpha_1} \tau_n(t_1) \left[ \int_0^{\alpha_2} \cdots \int_0^{\alpha_i} S_n(t_2, \cdots, t_i) f dt_2 \cdots dt_i \right] dt_1 \]

Therefore if \( f \in R_n(\Omega) \) and \( k \geq i-1 \) then the function \( g_n \) defined by

\[ g_n(\omega) = \sup_{\alpha_1, \cdots, \alpha_i > 0} \frac{1}{\alpha_1 \cdots \alpha_i} \left| \int_0^{\alpha_1} \int_0^{\alpha_2} \cdots \int_0^{\alpha_i} S_n(t_2, \cdots, t_i) f(\omega) dt_2 \cdots dt_i \right| \]

is, by induction hypothesis, in \( R_{i-1}(\Omega) \), and for every \( t > 0 \)

\[ \int_{|f| > t} \frac{g_n}{t} \left[ \log \frac{g_n}{t} \right]^{k-1} d\mu \]

\[ \leq \int_{|f| > t} \frac{C_k(i-1)|f|}{t} \left[ \log \frac{C_k(i-1)|f|}{t} \right]^{k} d\mu < \infty, \]

and thus if we set

\[ M_n^- g_n(\omega) = \sup_{\alpha > 0} \frac{1}{\alpha} \int_0^{\alpha} \tau_n(t_1) g_n(\omega) dt_1, \]

then \( M_n f \leq M_n^- g_n \) a.e. on \( \Omega \), and furthermore we have:

1. if \( k \geq i \) and \( f \in R_n(\omega) \) then for every \( t > 0 \)
ERGODIC THEOREMS

\[ \int_{\{G_n > t\}} \frac{M_n}{t} \left[ \log \frac{M_n}{t} \right]^{k-1} d\mu \leq \int_{\{G_n > t\}} \frac{C_{k-t+1}(1)g_n}{t} \left[ \log \frac{C_{k-t+1}(1)g_n}{t} \right]^{k-1} d\mu, \]

(ii) if \( k = i - 1 \) and \( f \in R_{i-1}(\Omega) \) then for every \( t > 0 \)

\[ \mu \{ M_n > t \} \leq \int_{\{G_n > t\}} \frac{C_{0}(1)g_n}{t} d\mu. \]

Therefore, replacing \( f^* \) by \( M_n f \), the inequalities of the theorem hold with \( C_k(i) = C_k(i-1)C_{k-t+1}(1) \), and so the theorem holds for \( d = i \).

The proof is completed.

Remark. It may be readily seen from the above-given argument that if \( 1 < p < \infty \) and \( f \in L_p(\Omega) \) then the function \( f^* \) of Theorem 2 is in \( L_p(\Omega) \) and also satisfies

\[ \int \left| f^* \right|^p d\mu \leq \left[ \frac{p2^p}{p-1} \right]^d \int |f|^p d\mu. \]

4. Ergodic theorems.

**Theorem 3.** Let \( T = \{ T(t_1, \cdots, t_d) ; t_1, \cdots, t_d > 0 \} \) be a \( d \)-parameter semigroup of Dunford-Schwartz operators on \( L_1(\Omega) + L_\infty(\Omega) \) which is assumed to be strongly continuous with respect to the norm topology of \( L_1(\Omega) \). If \( 1 \leq p < \infty \) and \( f \in L_p(\Omega) \), then \( T(t_1, \cdots, t_d)f \) converges in the norm topology of \( L_p(\Omega) \) as \( t_1 \to 0, \cdots, t_d \to 0 \) independently.

**Proof.** Put, for \( t > 0 \),

\[ S(t) = T(t, \cdots, t). \]

Since \( J = \{ S(t) ; t > 0 \} \) is a one-parameter semigroup of Dunford-Schwartz operators on \( L_1(\Omega) + L_\infty(\Omega) \) which is strongly continuous with respect to the norm topology of \( L_1(\Omega) \), it follows from [6] together with an approximation argument that if \( 1 \leq p < \infty \) and \( f \in L_p(\Omega) \) then \( S(t)f \) converges in the norm topology of \( L_p(\Omega) \) as \( t \to 0 \). Write

\[ f_0 = \lim_{t \to 0} S(t)f \ (\in L_p(\Omega)), \]

then we have \( S(t)f_0 = S(t)f \) for all \( t > 0 \), and thus if \( t_1, \cdots, t_d > t > 0 \) then we have
\[ T(t_1, \ldots, t_d)f = T(t_1 - t, \ldots, t_d - t)S(t)f \]
\[ = T(t_1 - t, \ldots, t_d - t)S(t)f_0 = T(t_1, \ldots, t_d)f_0. \]

Therefore for each fixed \( a > 0 \), it follows that
\[
\|T(t_1, \ldots, t_d)f - f_0\|_p = \|T(t_1, \ldots, t_d)f_0 - f_0\|_p \\
\leq \|T(t_1, \ldots, t_d)S(a)f_0 - S(a)f_0\|_p + \|S(a)f_0 - f_0\|_p \\
+ \|T(t_1, \ldots, t_d)[f_0 - S(a)f_0]\|_p.
\]

Since \( \|S(a)f_0 - f_0\|_p \rightarrow 0 \) as \( a \rightarrow 0 \), given an \( \varepsilon > 0 \) there exists an \( a > 0 \) so that \( \|S(a)f_0 - f_0\|_p < \varepsilon \). Then we get
\[
\|T(t_1, \ldots, t_d)f - f_0\|_p < \|T(t_1, \ldots, t_d)S(a)f_0 - S(a)f_0\|_p + 2\varepsilon.
\]

On the other hand, since \( \Gamma = \{T(t_1, \ldots, t_d); t_1, \ldots, t_d > 0\} \) is strongly continuous with respect to the norm topology of \( L_p(\Omega) \) for \( 1 \leq p < \infty \), it follows that
\[
\|T(t_1, \ldots, t_d)S(a)f_0 - S(a)f_0\|_p \rightarrow 0
\]
as \( t_1 \rightarrow 0, \ldots, t_d \rightarrow 0 \) independently. Therefore \( T(t_1, \ldots, t_d)f \) converges to \( f_0 \) in the norm topology of \( L_p(\Omega) \) as \( t_1 \rightarrow 0, \ldots, t_d \rightarrow 0 \) independently.

The proof is complete.

**Theorem 4.** Let \( \Gamma = \{T(t_1, \ldots, t_d); t_1, \ldots, t_d > 0\} \) be a \( d \)-parameter semigroup of Dunford-Schwartz operators on \( L_1(\Omega) + L_\infty(\Omega) \) which is assumed to be strongly continuous with respect to the norm topology of \( L_1(\Omega) \). If \( f \in R_{d-1}(\Omega) \) then the averages
\[
A(\alpha_1, \ldots, \alpha_d)f(\omega) = \frac{1}{\alpha_1 \cdots \alpha_d} \int_{t_1}^{t_1 + \alpha_1} \cdots \int_{t_d}^{t_d + \alpha_d} T(t_1, \ldots, t_d)f(\omega) \, dt_1 \cdots dt_d
\]
converge almost everywhere on \( \Omega \) as \( \alpha_1 \rightarrow 0, \ldots, \alpha_d \rightarrow 0 \) independently.

**Proof.** Theorem 3 ensures us to define an operator \( T_0 \) on \( L_1(\Omega) \) by the relation
\[
T_0 f = \lim_{t_1, \ldots, t_d \rightarrow 0} T(t_1, \ldots, t_d)f \quad (f \in L_1(\Omega))
\]
where the limit is in the norm topology of \( L_1(\Omega) \) and where \( t_1, \ldots, t_d \) tend to zero independently. Then we have \( \|T_0\|_1 \leq 1 \) and furthermore \( \|T_0f\|_\infty \leq \|f\|_\infty \) for every \( f \in L_1(\Omega) \cap L_\infty(\Omega) \). Thus, as in Section 2, we may and will assume that \( T_0 \) is a Dunford-Schwartz operator on \( L_1(\Omega) \cap L_\infty(\Omega) \).

It will be proved that if \( f \in R_{d-1}(\Omega) \) then
\[
A(\alpha_1, \ldots, \alpha_d)f(\omega) \rightarrow T_0f(\omega) \quad \text{a.e. on } \Omega
\]
as \( \alpha_1 \rightarrow 0, \ldots, \alpha_d \rightarrow 0 \) independently.
To do this, first suppose that $1 < p < \infty$ and $f \in L_p(\Omega)$. Let, for each $n \geq 1$,

$$f_n = (n^n)^{1/n} \cdots \left( \int_0^{1/n} T(t_1, \cdots, t_d) f \, dt_1 \cdots dt_d \right) \in L_p(\Omega).$$

Then we see that

$$\lim_n \|f_n - T_0 f\|_p = 0.$$ 

Furthermore it may be readily seen that for almost all $(t_1, \cdots, t_d, \omega) \in \mathbb{R}_+^d \times \Omega$ with respect to the product of the Lebesgue measure and $\mu$ we have

$$T(t_1, \cdots, t_d) f_n(\omega) = (n^n)^{1/n} \cdots \left( \int_0^{1/n} T(t_1 + s_1, \cdots, t_d + s_d) f(\omega) \, ds_1 \cdots ds_d \right)$$

where of course $T(t_1, \cdots, t_d) f_n(\omega)$ denotes a scalar representation of $T(t_1, \cdots, t_d) f_n$, $(t_1, \cdots, t_d) \in \mathbb{R}_+^d$. Thus for almost all $\omega \in \Omega$, $T(t_1, \cdots, t_d) f_n(\omega)$ as a function of $(t_1, \cdots, t_d) \in \mathbb{R}_+^d$ is continuous, and clearly

$$A(\alpha_1, \cdots, \alpha_d) f_n(\omega) \longrightarrow f_0(\omega) \quad \text{a.e. on } \Omega$$

as $\alpha_1 \longrightarrow 0, \cdots, \alpha_d \longrightarrow 0$ independently. Since $T_0 f_n = f_n$, it then follows that

$$\limsup_{\alpha_1, \cdots, \alpha_d \to 0} \left| A(\alpha_1, \cdots, \alpha_d) f(\omega) - T_0 f(\omega) \right|$$

$$\leq \limsup_{\alpha_1, \cdots, \alpha_d \to 0} \left| A(\alpha_1, \cdots, \alpha_d)(f - f_n)(\omega) - T_0(f - f_n)(\omega) \right|$$

$$\leq \sup_{t_1, \cdots, t_d > 0} \left| A(\alpha_1, \cdots, \alpha_d)(f - f_n)(\omega) \right| + \left| T_0(f - f_n)(\omega) \right|$$

$$\leq (f - f_n)^*(\omega) + \left| T_0(f - f_n)(\omega) \right| \quad \text{a.e. on } \Omega.$$

Since $\lim_n \|(f - f_n)^*\|_p = 0$ by the remark in the preceding section and $\lim_n \|T_0(f - f_n)\|_p = \lim_n \|f - f\|_p = 0$, this implies that for $f \in L_p(\Omega)$ with $1 < p < \infty$, $A(\alpha_1, \cdots, \alpha_d) f(\omega)$ converges to $T_0 f(\omega)$ a.e. on $\Omega$ as $\alpha_1 \longrightarrow 0, \cdots, \alpha_d \longrightarrow 0$ independently.

Next suppose that $f \in L_d(\Omega)$, and then take $f_n \in L_d(\Omega)$, where $1 < p < \infty$, so that $|f - f_n| \leq |f|$ and $\lim_n f_n = f$ a.e. on $\Omega$. Then

$$\limsup_{\alpha_1, \cdots, \alpha_d \to 0} \left| A(\alpha_1, \cdots, \alpha_d) f(\omega) - T_0 f(\omega) \right|$$

$$\leq \limsup_{\alpha_1, \cdots, \alpha_d \to 0} \left| A(\alpha_1, \cdots, \alpha_d)(f - f_n)(\omega) - T_0(f - f_n)(\omega) \right|$$

$$\leq (f - f_n)^*(\omega) + \left| T_0(f - f_n)(\omega) \right| \quad \text{a.e. on } \Omega,$$

and by Theorem 2, for every $t > 0$
\[
\mu \left\{ (f-f_n)^* > t \right\} \leq \int_{\{C_d = 1(d) \left| \frac{f-f_n}{t} \right| \left[ \log \frac{C_d = 1(d) \left| \frac{f-f_n}{t} \right|}{\left| f-f_n \right|} \right] \right\} \leq \int_{\{C_d = 1(d) \left| \frac{f-f_n}{t} \right| \left[ \log \frac{C_d = 1(d) \left| \frac{f-f_n}{t} \right|}{\left| f-f_n \right|} \right] \right\}} d\mu,
\]

where the right-hand side of the last inequality tends to zero as \( n \to \infty \), by virtue of Lebesgue's dominated convergence theorem. On the other hand, as in Lemma 1, we have for every \( t > 0 \)
\[
\mu \left\{ \left| T_0(f-f_n) \right| > t \right\} \leq \frac{2}{t} \int_{\{C_d = 1(d) \left| \frac{f-f_n}{t} \right| \left[ \log \frac{C_d = 1(d) \left| \frac{f-f_n}{t} \right|}{\left| f-f_n \right|} \right] \}} d\mu,
\]
and the right-hand side of this inequality tends to zero as \( n \to \infty \), by Lebesgue's convergence theorem, too. Therefore we observe that the theorem holds for \( f \in R_{d-1}(\Omega) \), and the proof is completed.

**Remark.** It is known (cf. [9]) that if \( d = 1 \) then Theorem 4 holds for every \( f \in L_1(\Omega) + L_\infty(\Omega) \). But, as is well-known (cf. [4] or [10]), if \( d \geq 2 \) then the theorem may fail to hold for some \( f \in L_1(\Omega) \).

**Lemma 2.** Let \( \Gamma = \{T(t_1, \ldots, t_d) ; t_1, \ldots, t_d > 0 \} \) be a \( d \)-parameter semigroup of Dunford-Schwartz operators on \( L_1(\Omega) + L_\infty(\Omega) \) which is assumed to be strongly continuous with respect to the norm topology of \( L_1(\Omega) \). If \( 1 < p < \infty \) and \( f \in L_p(\Omega) \), then the averages \( A(\alpha_1, \ldots, \alpha_d) f \) converge in the norm topology of \( L_p(\Omega) \) as \( \alpha_1 \to \infty, \ldots, \alpha_d \to \infty \) independently.

**Proof.** \( \{A(\alpha_1, \ldots, \alpha_d) ; \alpha_1, \ldots, \alpha_d > 0 \} \) may and will be regarded as a net of bounded linear operators on \( L_p(\Omega) \). Then it follows that this net is \( \Gamma \)-ergodic in the sense of [8], and since \( L_p(\Omega) \) with \( 1 < p < \infty \) is a reflexive Banach space, it follows from [8] that \( A(\alpha_1, \ldots, \alpha_d) f \) converges in the strong operator topology as \( \alpha_1 \to \infty, \ldots, \alpha_d \to \infty \) independently. This completes the proof.

**Theorem 5.** Let \( \Gamma = \{T(t_1, \ldots, t_d) ; t_1, \ldots, t_d > 0 \} \) be a \( d \)-parameter semigroup of Dunford-Schwartz operators on \( L_1(\Omega) + L_\infty(\Omega) \) which is assumed to be strongly continuous with respect to the norm topology of \( L_1(\Omega) \). If \( f \in R_{d-1}(\Omega) \) then the averages \( A(\alpha_1, \ldots, \alpha_d) f(\omega) \) converge almost everywhere on \( \Omega \) as \( \alpha_1 \to \infty, \ldots, \alpha_d \to \infty \) independently.

**Proof.** Let \( 1 < p < \infty \). Lemma 2 enables us to define an operator \( T_\infty(\cdot, \ldots, \cdot) \) on \( L_p(\Omega) \) by the relation
\[
T_\infty f = \lim_{\alpha_1, \ldots, \alpha_d \to \infty} A(\alpha_1, \ldots, \alpha_d) f \quad (f \in L_p(\Omega))
\]
where the limit is in the norm topology of \( L_p(\Omega) \) and where \( \alpha_1, \ldots, \alpha_d \) tend
to infinity independently. Then we have \( \|T_n\|_p \leq 1 \), and for \( f \in L_1(\Omega) \cap L_p(\Omega) \) there exists a sequence \( (f_n) \) in the set \( \{A(\alpha_1, \cdots, \alpha_d) f : \alpha_1, \cdots, \alpha_d > 0\} \) such that

\[
T_n f = \lim_{n} f_n \quad \text{a.e. on } \Omega.
\]

Since \( \|f_n\|_1 \leq \|f\|_1 \) for each \( n \geq 1 \), it follows from Fatou's lemma that

\[
\|T_n f\|_1 \leq \liminf_{n} \|f_n\|_1 \leq \|f\|_1.
\]

Hence \( T_n \) can be uniquely extended to a linear contraction on \( L_1(\Omega) \), which satisfies \( \|T_n f\|_\infty \leq \|f\|_\infty \) for every \( f \in L_1(\Omega) \cap L_\infty(\Omega) \). Therefore, as in Section 2, we may and will assume that \( T_n \) is a Dunford-Schwartz operator on \( L_1(\Omega) + L_\infty(\Omega) \).

It will be proved that if \( f \in R_{\pi}(\Omega) \) then

\[
A(\alpha_1, \cdots, \alpha_d) f(\omega) \longrightarrow T_n f(\omega) \quad \text{a.e. on } \Omega
\]

as \( \alpha_1 \longrightarrow \infty, \cdots, \alpha_d \longrightarrow \infty \) independently.

To do this, however, in view of the proof of Theorem 4, it is enough to check that the theorem holds for every \( f \in L_p(\Omega) \). And to check this it is also enough to notice that the theorem holds for every \( f \) in a dense linear manifold of \( L_p(\Omega) \).

For this purpose, let \( M \) denote the linear manifold generated by the functions \( f \) of the form \( f = h + [g - T(s_1, \cdots, s_d) g] \), where \( h, g \in L_\infty(\Omega), T(t_1, \cdots, t_d) h = h \) for all \( t_1, \cdots, t_d > 0 \), and \( g \in L_\infty(\Omega) \). Lemma 2 implies that \( M \) is dense in \( L_p(\Omega) \) with respect to the norm topology of \( L_p(\Omega) \), and for such a function \( f \) it follows easily that \( T_n f = h \) and that

\[
A(\alpha_1, \cdots, \alpha_d) f(\omega) \longrightarrow h(\omega) \quad \text{a.e. on } \Omega
\]

as \( \alpha_1 \longrightarrow \infty, \cdots, \alpha_d \longrightarrow \infty \) independently. This completes the proof.

Let \( \Gamma_j = \{T_j(t) ; t > 0\}, 1 \leq j \leq d \), be one-parameter semigroups of Dunford-Schwartz operators on \( L_1(\Omega) + L_\infty(\Omega) \) which are assumed to be strongly continuous with respect to the norm topology of \( L_1(\Omega) \). (Here we do not assume that these one-parameter semigroups commute.) Then, since for each \( f \in L_p(\Omega) \), with \( 1 \leq p < \infty \), the function \( T_1(t_1) \cdots T_d(t_d) f \) of \( (t_1, \cdots, t_d) \in R_+^d \) is continuous with respect to the norm topology of \( L_p(\Omega) \), it follows, as in Section 2, that for every \( f \in L_1(\Omega) + L_\infty(\Omega) \) there exists a scalar function \( T_1(t_1) \cdots T_d(t_d) f(\omega) \), defined on \( R_+^d \times \Omega \) and measurable with respect to the product of the Lebesgue measurable subsets of \( R_+^d \), \( \mathcal{F} \), such that for each fixed \( (t_1, \cdots, t_d) \in R_+^d \), \( T_1(t_1) \cdots T_d(t_d) f(\omega) \) as a function of \( \omega \in \Omega \) belongs to the equivalence class of \( T_1(t_1) \cdots T_d(t_d) f \). Then we may define, for almost all \( \omega \in \Omega \),
A(\alpha_1, \ldots, \alpha_d) f(\omega) = \frac{1}{\alpha_1 \cdots \alpha_d} \int_0^{\alpha_1} \cdots \int_0^{\alpha_d} T_1(t_1) \cdots T_d(t_d) f(\omega) \, dt_1 \cdots dt_d

Then we have the following theorem which is similar to the above Theorem 5 and a generalization of Theorem 5 in Fava [4].

**Theorem 6.** Let \( \Gamma_j = \{ T_j(t) ; t > 0 \} \), \( 1 \leq j \leq d \), be one-parameter semigroups of Dunford-Schwartz operators on \( L_1(\Omega) + L_\infty(\Omega) \) which are assumed to be strongly continuous with respect to the norm topology of \( L_1(\Omega) \). If \( f \in R_{d-1}(\Omega) \) then the averages \( A(\alpha_1, \ldots, \alpha_d) f(\omega) \) converge almost everywhere on \( \Omega \) as \( \alpha_1 \to \infty, \ldots, \alpha_d \to \infty \) independently.

**Proof.** It is known (cf. [3], p. 694) that if \( 1 < p < \infty \) and \( f \in L_p(\Omega) \) then the averages \( A(\alpha_1, \ldots, \alpha_d) f(\omega) \) converge almost everywhere on \( \Omega \) and as well in the norm topology of \( L_p(\Omega) \) as \( \alpha_1 \to \infty, \ldots, \alpha_d \to \infty \) independently. Thus, by using Theorem 1 repeatedly, we may see, as in Theorem 5, that the desired result holds for \( f \in R_{d-1}(\Omega) \). We omit the details.

In conclusion we should like to remark that Yoshimoto [11] has obtained, using a maximal ergodic theorem due to Hasegawa-Sato-Tsurumi [5], vector valued ergodic theorems in the same direction for a one-parameter semigroup \( \{ T(t) ; t > 0 \} \) of linear operators on \( L_1(\Omega, X) + L_\infty(\Omega, X) \) which satisfies some norm and integrability conditions, \( X \) being a reflexive Banach space. Since the scalar field is a reflexive Banach space, Yoshimoto's results generalize ours when restricted to one-parameter semigroups. But we could not extend his results to \( d \)-parameter semigroups with \( d \geq 2 \), because the existence of a positive one-parameter semigroup is not known which dominates a given \( L_1(\Omega, X) \) contraction operator one-parameter semigroup.

**Added in proof.** Professor S.A. McGrath kindly informed me that he proved, in his recent paper [Local ergodic theorems for noncommuting semigroups, Proc. Amer. Math. Soc. 79 (1980), 212-216], the following local ergodic theorem:

Let \( \Gamma_j = \{ T_j(t) ; t > 0 \} \), \( 1 \leq j \leq d \), be as in Theorem 6. Assume, in addition, that \( \lim_{t \to 0} \| T_j(t) - I \|_1 = 0 \) for each \( j \). Then for any \( f \in R_{d-1}(\Omega) \), the averages \( A(\alpha_1, \ldots, \alpha_d) f(\omega) \) converge almost everywhere on \( \Omega \) as \( \alpha_1 \to 0, \ldots, \alpha_d \to 0 \) independently.

Modifying his argument and using the local ergodic theorem in [6], it is easily seen that McGrath's theorem holds, without the additional hypothesis that \( \lim_{t \to 0} \| T_j(t) - I \|_1 = 0 \) for each \( j \).
ERGODIC THEOREMS

REFERENCES


DEPARTMENT OF MATHEMATICS
OKAYAMA UNIVERSITY

(Received February 9, 1980)