A note on isomorphism invariants of a modular group algebra

Tôru Furukawa*
A NOTE ON ISOMORPHISM INVARIANTS OF
A MODULAR GROUP ALGEBRA

TÔRU FURUKAWA

1. Introduction. Let $F(G)$ be the group algebra of a group $G$ over the prime field $F = GF(p)$ and let $\{M_{i,p}(G)\}_{i \geq 1}$ be the Brauer-Jennings-Zassenhaus $M$-series of $G$ relative to the prime $p$: $M_{i,p}(G) = G$ and $M_{i,p}(G) = (G, M_{i-1,p}(G))M_{i/p,p}(G)$ for $i \geq 2$, where $(i/p)$ is the least integer not smaller than $i/p$ and $(G, M_{i-1,p}(G))$ is the subgroup generated by all commutators $(x, y) = x^{-1}yx$, $x \in G$, $y \in M_{i-1,p}(G)$. In [4], I. B. S. Passi and S. K. Sehgal showed that for each $i \geq 1$ the factor groups $M_{i,p}(G)/M_{i+1,p}(G)$ and $M_{i,p}(G)/M_{i+2,p}(G)$ are isomorphism invariants of $F(G)$. In this note we shall show that the factor groups $M_{i,p}(G)/M_{i+j,p}(G)$ are isomorphism invariants of $F(G)$ for all $i \geq 1$ and all $j$ with $1 \leq j \leq i + 1$, too.

2. Notations and preliminary results. Let $G$ be a group, $N$ a normal subgroup of $G$, and $R$ a commutative ring with identity. We adopt the following notations:

- $R(G)$ is the group ring of $G$ with coefficients in $R$.
- $\Delta_\mu(G, N) =$ the kernel of the natural homomorphism $R(G) \to R(G/N)$.
- $\Delta_\mu(G) =$ the augmentation ideal of $R(G)$.
- $\Delta_\mu^i(G) =$ the $i$-th power of $\Delta_\mu(G)$.
- $U(R(G)) =$ the unit group of $R(G)$.

It is easy to verify that if $S$ and $I$ are subrings of a ring such that $SI + IS \subseteq I$ then $S + I$ is a subring which contains $I$ as an ideal. Now, let $G^*$ be a subgroup of $G$, and $I$ an $R$-submodule of $R(G)$ satisfying $I^2 \subseteq I$ and $\Delta_\mu(G^*) + I \Delta_\mu(G^*) \subseteq I$. Since $R(G^*) = \Delta_\mu(G^*) + R$, we see that $R(G^*) + I$ forms a ring containing $I$ as an ideal. Let

$$\nu : U(R(G^*)) \to U(R(G^*) + I/I); \quad u \mapsto u + I$$

be the group homomorphism induced by the natural ring homomorphism $R(G^*) \to R(G^*) + I/I$. Denoting by $\nu^*$ the restriction of $\nu$ to $G^*$, we see that the kernel of $\nu^*$ coincides with $G^* \cap (1 + I)$ and the image of $\nu^*$ is $G^* + I/I$. Hence, we have an isomorphism $G^*/G^* \cap (1 + I) \cong G^* + I/I$. The next is an immediate consequence of this fact.
Lemma 1. Let $\theta : R(G) \to R(H)$ be an $R$-algebra isomorphism. Let $G^*$ and $H^*$ be subgroups of $G$ and $H$ respectively, and $I$ an $R$-submodule of $R(G)$ such that $I^2 \subseteq I$, $\Delta_R(G^*)I + I\Delta_R(G^*) \subseteq I$ and $\theta(G^* + I) = H^* + \theta(I)$, then $G^*/G^* \cap (1 + I) \cong H^*/H^* \cap (1 + \theta(I))$.

Let $F$ be the prime field $GF(p)$. Then, it is known that for each $i \geq 1$, $M_{i,p}(G)$ coincides with $D_{i,p}(G) = G \cap (1 + \Delta^i p(G))$, the $i$-th dimension subgroup of $G$ over $F$ (see, e.g. [1, 2, 3, 5 and 6]). Now, let $L_{i,p}(G) = \Delta^i p(G, M_{i,p}(G)) + \Delta^i p(G)$ for $i \geq 1$.

We borrow the following in [4].

Lemma 2. (1) $L_{i,p}(G) = \{ x - 1 + \alpha \mid x \in M_{i,p}(G), \alpha \in \Delta_i^{i+1}(G) \}$ for $i \geq 1$.

(2) Let $\theta : F(G) \to F(H)$ be a normalized isomorphism in the sense that the sum of the coefficients of $\theta(g)$ is 1 for all $g \in G$. Then $\theta(L_{i,p}(G)) = L_{i,p}(H)$ for all $i \geq 1$.

3. Main theorem. We are now in a position to prove our main theorem.

Theorem. Let $F$ be the prime field $GF(p)$, and $\{M_{i,p}(G)\}_{i \geq 1}$ the Brauer-Jennings-Zassenhaus $M$-series of $G$ relative to the prime $p$. If $F(G) \cong F(H)$, then $M_{i,p}(G)/M_{i+1,p}(G) \cong M_{i,p}(H)/M_{i+1,p}(H)$ for all $i \geq 1$ and all $j$ with $1 \leq i \leq j + 1$.

Proof. Throughout the proof, we shall omit their subscripts $p$ and $F$ from $M_{i,p}(\ )$, $L_{i,p}(\ )$ and $\Delta^i p(\ )$, which are denoted by $M(\ )$, $L(\ )$ and $\Delta(\ )$, respectively.

Let $I_{i,t} = L_i(G)$, and $I_{i,t+1} = \Delta^{i+1}(G)$ ($i \geq 1$). Since

$$I_{i,t+1} \supseteq I_{i+1,t+1} \equiv I_{i+1,t+2} \quad (i \geq 1),$$

we can find subspaces $I_{i,t+1}$ of $I_{i,t+1}$ containing $I_{i,t+2}$ such that $I_{i,t+1} = I_{i+1,t+1} + I_{i,t+2}$ and $I_{i+1,t+2} \supseteq I_{i+1,t+1} \cap I_{i,t+3}$. Obviously,

$$I_{i,t+2} \supseteq I_{i+1,t+2} \supseteq I_{i+1,i+3} \quad (i \geq 1),$$

and so we can repeat the same procedure to obtain subspaces $I_{i,t+3}$ of $I_{i,t+2}$ containing $I_{i+1,t+3}$ such that $I_{i,t+3} = I_{i+1,t+2} + I_{i,t+3}$ and $I_{i+1,t+3} \supseteq I_{i+1,t+2} \cap I_{i,t+3}$. In this way, for $j \geq 0$ we can construct inductively the series of subspaces of $F(G)$ such that

(1) $I_{i,t+j} \supseteq I_{i,i+j+1} \supseteq I_{i+1,i+1-j} \quad (i \geq 1; j \geq 0)$.
A NOTE ON ISOMORPHISM INVARIANTS OF A MODULAR GROUP ALGEBRA

(2) \( I_i, i + j = I_{i+1, i+j} + I_i, i+j+1\)  \( (i \geq 1; j \geq 1)\)

(3) \( I_{i+1, i+j} \supseteq I_{i, i+j} \cap I_i, i-j+1\)  \( (i \geq 1; j \geq 1)\).

From (1), we see that \( \{I_i, i\} \) is a decreasing series for \( i \geq 1\), and
moreover if \( 1 \leq i \leq k \) then \( I_{i, k+1} \supseteq I_{i+1, k+1}\). We have therefore

(4) \( I_{i, k+1} \equiv I_{i, k+1} \equiv \cdots \equiv I_{i, k+1} \equiv I_{k+1, i+1} \)  \( (k \geq 1)\).

Similarly, by (2) and (3), we can prove

(5) \( I_{i, k+1} = I_{i+1, k+1} + I_{i, k+2} \)  \( (1 \leq i \leq k)\).

(6) \( I_{i, k+1} \equiv I_{i+1, k+1} \cap I_{i, k+2} \)  \( (1 \leq i \leq k)\).

Combining (4) and (5), we obtain

(7) \( I_{i, k+1} = I_{i+1, k+1} + I_{i, k+2} \)  \( (1 \leq i \leq k)\).

Since \( I_{i+1, k+1} = \{x - 1 + \alpha \mid x \in M_{i+1}(G), \alpha \in I_{i+1, k+1}\} \) by Lemma 2 (1),
(7) together with (1) and (4) implies

(8) \( I_{i, i} = \{x - 1 + \alpha \mid x \in M_{i}(G), \alpha \in I_{i, i+j+1}\} \)  \( (i \geq 1; j \geq 0)\).

Now, we claim that

(9) \( I_{i, i} = \{x - 1 + \alpha \mid x \in M_i(G), \alpha \in I_{i, i+j+1}\} \)  \( (i \geq 1; j \geq 0)\).

According to (1), it suffices to show that the left-hand side of (9) is
contained in the right-hand side. We shall proceed by induction on \( j \),
keeping \( i \) fixed. The first step of induction, when \( j = 0 \), is assured by
(8). Suppose \( j \geq 1 \) and the statement holds for \( j - 1 \). Given \( \beta \in I_{i, i} \),
by the induction hypothesis, \( \beta = x - 1 + \alpha \) with some \( x \in M_i(G) \) and
\( \alpha \in I_{i, i+j+1} \). By (8), \( \alpha = y - 1 + \eta \) with some \( y \in M_{i+j}(G) \) and
\( \eta \in I_{i, i+j+1} \). Therefore,

\[ \beta = x - 1 + y - 1 + \eta \]

\[ = xy - 1 + \delta, \quad \text{where} \quad \delta = \eta - (x - 1)(y - 1). \]

By (4),

\[ (x - 1)(y - 1) \in \Delta(G) \Delta^{i+j+1}(G) \subseteq \Delta^{i+j+1}(G) = I_{i+j, i+j+1} \subseteq I_{i, i+j+1}, \]

which implies \( \delta \in I_{i, i+j+1} \). Since \( xy \in M_i(G) \), the induction is complete
and hence (9) is established.

Next, we claim that

(10) \( G \cap (1 + I_{i, k+1}) = M_{i+1}(G) \)  \( (1 \leq i \leq k)\).

By (4), the right-hand side of (10) is contained in the left-hand side.
To show the reverse inclusion, we proceed by induction on \( k \), the statement
being clear for \( k = 1 \). Suppose \( G \cap (1 + I_{i, k-1}) \subseteq M_{k-1}(G) \) \( (1 \leq i \leq k)\).
To complete the induction, we have to show that

\[ G \cap (1 + I_{i, k+2}) \subseteq M_{k+2}(G) \quad (1 \leq i \leq k + 1). \]

To see this, we use descending induction on \( t \), the above being obvious
for \( t = k + 1 \). Assume that \( G \cap (1 + I_{t,k+2}) \subseteq M_{k+2}(G) \) for some \( t \) with \( 2 \leq t \leq k + 1 \). Then, by our induction hypothesis \( G \cap (1 + I_{t-1,k+1}) \subseteq M_{k+1}(G) \). Let \( g \) be in \( G \cap (1 + I_{t-1,k+2}) \subseteq G \cap (1 + I_{t-1,k+1}) \) by (1). Then, \( g \in M_{k+1}(G) \), and hence \( g - 1 \in L_{k+1}(G) = I_{t-1,k+1} \subseteq I_{t,k+1} \) by (4). Noting here that \( I_{i,k+1} \cap I_{t-1,k+2} \subseteq I_{i,k+2} \) by (6), we obtain \( g - 1 \in I_{i,k+2} \). Now, according to the decreasing induction hypothesis, it follows \( g \in M_{k+2}(G) \). This completes the induction on \( k \), and hence (10) has been proved.

Now, assume that an isomorphism \( \theta : F(G) \to F(H) \) is given. Then, without loss of generality, we may assume that \( \theta \) is normalized, and therefore \( \theta(\Delta^i(G)) = \Delta^i(H) \) and \( \theta(I_i(G)) = L_i(H) \) (Lemma 2 (2)). Hence, applying the above argument to the subspaces \( \theta(I_{i,j}) \) of \( F(H) \), we do have the following:

\[
(9') \quad \theta(I_{i,j}) = \{ h - 1 + \beta \mid h \in M_i(H), \beta \in \theta(I_{i+j,i+1}) \} \quad (i \geq 1; j \geq 0).
\]

\[
(10') \quad H \cap (1 + \theta(I_{i+j,i})) = M_{i+j}(H) \quad (1 \leq i \leq k).
\]

(9) and (9') immediately imply

\[
\theta(M_i(G) + I_{i+j,i+1}) = M_i(H) + \theta(I_{i+j,i+1}) \quad (i \geq 1; j \geq 0).
\]

We are now ready to complete the proof of our theorem. Let \( i, j \) satisfy \( i \geq 1 \) and \( 1 \leq j \leq i + 1 \). Then, since \( I_{i+j-1,i+j} \subseteq I_{i+j} \subseteq I_{i+j+1} = \Delta^{i+1}(G) \) by (1) and (4), there holds that

\[
I_{i,j+1}^2 \subseteq \Delta^{i+1}(G) \subseteq \Delta^{i+j}(G) = I_{i+j-1,i+j} \subseteq I_{i+j}.
\]

Similarly,

\[
\Delta(M_i(G))I_{i,j} + I_{i,j} \Delta(M_i(G)) \subseteq I_{i+1}.
\]

Finally, by (11)

\[
\theta(M_i(G) + I_{i,j}) = M_i(H) + \theta(I_{i,j}).
\]

Thus, in virtue of Lemma 1, we get

\[
M_i(G)/M_i(G) \cap (1 + I_{i,j}) \equiv M_i(H)/M_i(H) \cap (1 + \theta(I_{i,j})).
\]

Since \( M_i(G) \cap (1 + I_{i,j}) = M_{i+j}(G) \) and \( M_i(H) \cap (1 + \theta(I_{i,j})) = M_{i+j}(H) \) by (10) and (10'), the theorem has been proved.

REFERENCES

A NOTE ON ISOMORPHISM INVARIANTS OF A MODULAR GROUP ALGEBRA


DEPARTMENT OF MATHEMATICS
OKAYAMA UNIVERSITY

(Received September 1, 1980)