On conformal diffeomorphisms with decomposable associated scalar field

Yoshihiro Tashiro*         Mitsuru Kora†

*Hiroshima University
†Himeji Technical High School
ON CONFORMAL DIFFEOMORPHISMS WITH DECOMPOSABLE ASSOCIATED SCALAR FIELD

YOSHIHIRO TASHIRO and MITSURU KORA

Introduction. In a recent paper [4], one of the present authors has shown that there is no global conformal diffeomorphism between complete product Riemannian manifolds $M$ and $M^*$ such that the product structures $F$ and $G$ of them are not commutative under it in a dense subset of $M$, and given an example of a global conformal diffeomorphism commuting the product structures. There remain some open problems concerning this, for instance, whether the condition "in a dense subset" of the theorem can be replaced with "in an open subset" or not.

Our purpose of the present paper is to show that the replacement is possible in the case where the scalar field $\rho$ associated with the conformal diffeomorphism is decomposable in an open subset of $M = M_1 \times M_2$ and depends on points of both $M_1$ and $M_2$, that is, $\rho$ is the sum

$$\rho = \rho_1 + \rho_2$$

of non-constant functions $\rho_1$ on $M_1$ and $\rho_2$ on $M_2$.

After preliminaries are given in §1, we shall prove in §2 that the parts $\rho_1$ and $\rho_2$ are special concircular scalar fields in $M_1$ and $M_2$ respectively. In §3 we shall give expressions of $\rho$ with respect to adapted coordinate systems in $M = M_1 \times M_2$. Then, in §4, we shall show that no global conformal diffeomorphism with such a solution of $\rho$ can be admitted between complete product Riemannian manifolds $M$ and $M^*$.

1. Preliminaries. Let $M = M_1 \times M_2$ and $M^* = M_1^* \times M_2^*$ be product Riemannian manifolds of dimension $n \geq 3$, and denote the structures by $(M, g, F)$ and $(M^*, g^*, G)$ where $g$ and $g^*$ are the metric tensors and $F$ and $G$ the product structures of $M$ and $M^*$ respectively. Throughout this paper, we shall assume that the manifolds are connected and the differentiability is of class $C^\infty$. The dimensions $n_1$ and $n_2$ of the parts $M_1$ and $M_2$ may be different from those of $M_1^*$ and $M_2^*$. Greek indices run on the range 1 to $n$, and Latin indices on the following ranges:

$$h, i, j = 1, 2, \ldots, n_1,$$

$$p, q, r = n_1 + 1, \ldots, n,$$

respectively. Summation convention is applied to repeated indices on their
own ranges.

We shall use a separate coordinate system \((x') = (x', x'')\) in \(M = M_1 \times M_2\) with respect to the product structure \(F\). The metric tensor \(g\) has components

\[
(g_{\nu \lambda}) = \begin{pmatrix}
g_{\mu \nu} & 0 \\
0 & g_{\eta \eta}\end{pmatrix},
\]

where \(g_{\mu \nu}\) depend on the coordinates \((x')\) of \(M_1\) only and \(g_{\eta \eta}\) on \((x'')\) of \(M_2\) only. The Christoffel symbol and the curvature tensor of \(M\) have pure components only, and the covariant differentiations \(\Gamma_\alpha\) along \(M_1\) and \(\Gamma_\beta\) along \(M_2\) commute with one another.

A conformal diffeomorphism \(f\) of \(M\) to \(M^*\) is characterized by the metric change

\[
f^*(g^*) = \frac{1}{\rho^2} g,
\]

and \(\rho\) is called the associated scalar field with \(f\). Hereafter, the image of a quantity of \(M^*\) to \(M\) by the induced map \(f^*\) of \(f\) will be denoted by the same letter as the original one, for example, we write \(g^*\) for \(f^*(g^*)\) and \(G\) for \(f^*(G)\). If \(FG = GF\) at a point \(P \in M\), then we say that the structures \(F\) and \(G\) are commutative at \(P\) under \(f\). The commutativity is equivalent to the purity of \(G\) with respect to \(F\). We put

\[
G_{\mu \lambda} = G_{\mu \eta} g_{\eta \lambda},
\]

which is symmetric in \(\lambda\) and \(\mu\).

Let \(Y\) be the gradient vector field \((\rho')\) of \(\rho\), and \(Y_1\) and \(Y_2\) the components \((\rho'^{1})\) and \((\rho'^{2})\) of \(Y\) belonging to \(M_1\) and \(M_2\) respectively. We denote by \(\phi\) the squared length of \(Y\):

\[
\phi = |Y|^2 = \rho \rho',
\]

and put the open subset \(U\) as

\[
U = \{ P \mid Y_1(P) \neq 0 \text{ and } Y_2(P) \neq 0 \}.
\]

It is proved in [4] that the product structures \(F\) and \(G\) are not commutative under \(f\) in \(U\), and we have the equations

\[
\begin{cases}
\Gamma_\mu \rho_\nu = \phi_1 g_{\mu \nu} + \frac{C}{\rho} G_{\mu \nu}, \\
\Gamma_\nu \rho_\mu = 0, \\
\Gamma_\eta \rho_\eta = \frac{C}{\rho} G_{\eta \eta},
\end{cases}
\]

(1.1)
in every connected component of $U$, where $C$ is a constant and the coefficients $\phi_1$ and $\phi_2$ are functions satisfying the relation

\[(1.2)\quad \phi_1 + \phi_2 = \frac{\phi}{\rho} = \frac{1}{\rho} \rho \cdot \rho'.\]

Moreover we have seen that we may put

\[(1.3)\quad \phi_1 - \phi_2 = k \rho,
\]

$k$ being a constant. The constants $C$ and $k$ might be different from one connected component of $U$ to another.

In general, a scalar field $\rho$ in a Riemannian manifold $M$ is said to be **conicircular** if it satisfies the equation

\[(1.4)\quad \Gamma_{\mu} \rho_{\lambda} = \phi g_{\mu \lambda},\]

and to be **special conicircular** if it satisfies the equation

\[(1.5)\quad \Gamma_{\mu} \rho_{\lambda} = (k \rho + b) g_{\mu \lambda},\]

$k$ and $b$ being constants. Properties of conicircular scalar fields play important roles, and we refer to [2] and [3] as to them. The trajectories of the gradient vector field $Y = (\rho')$ are geodesics, called $\rho$-curves, and, in a neighborhood of an ordinary point of $\rho$, there is a local coordinate system, called an adapted one, such that the first coordinate $x$ is the arclength of $\rho$-curves, $\rho$ is a function of $x$ only, and the metric form $ds^2$ of $M$ is given in the form

\[ds^2 = ds^2 + (\rho'(x))^2 \overline{ds}^2,\]

where prime indicates derivative in $x$ and $\overline{ds}^2$ is the metric form of an $(n-1)$-dimensional Riemannian manifold $\overline{M}$, see also [1]. Along the $\rho$-curves, the equations (1.4) and (1.5) reduce to the ordinary differential equations

\[\rho''(x) = \phi\]

and

\[\rho''(x) = k \rho + b.\]

2. **The decomposable associated scalar field.** Now we suppose that, in an open subset $U'$ of $U$, the associated scalar field $\rho$ is the sum

\[(2.1)\quad \rho = \rho_1 + \rho_2\]
of non-constant functions $\rho_1$ depending on $(x^i)$ only and $\rho_2$ depending on $(x^p)$ only. Then we have

$$\rho_i = \Gamma_i \rho_1, \quad \rho_p = \Gamma_p \rho_2.$$ 

and

$$\Gamma_q \Gamma_i \rho = 0$$

in the open subset $U'$. Since there is a hybrid component $G_{qi} \neq 0$, it follows from the equation (1.1, 2) that $C = 0$ and $\rho$ is decomposable in the connected component of $U$ containing $U'$. Consequently we may suppose that the open subset $U'$ is the connected component of $U$.

Then the equations (1.1, 1) and (1.1, 3) turn to

$$\begin{cases}
\Gamma_i \rho_i = \phi_1 g_{ji}, \\
\Gamma_p \rho_p = \phi_2 g_{qp}
\end{cases}$$

in $U'$. Therefore the function $\phi_1$ depends on $(x^i)$ only and $\phi_2$ on $(x^p)$ only. Substituting (2.1) into (1.3), we have the equation

$$\phi_1 - k \rho_1 = \phi_2 + k \rho_2.$$ 

Since the left hand side depends on $(x^i)$ only and the right hand side on $(x^p)$ only, both sides are equal to a constant, say $b$. Hence the functions $\phi_1$ and $\phi_2$ are given by

$$\phi_1 = k \rho_1 + b, \quad \phi_2 = -k \rho_2 + b,$$

and the equations (2.2) become

$$\begin{cases}
\Gamma_i \Gamma_j \rho_i = (k \rho_1 + b) g_{ji}, \\
\Gamma_p \rho_p = (k \rho_2 + b) g_{qp}.
\end{cases}$$

If we denote by $M_1(P)$ and $M_2(P)$ the parts of $M$ passing through a point $P \in M$, then these equations mean that the parts $\rho_1$ and $\rho_2$ of $\rho$ are non-constant special concircular scalar fields in the intersections $U' \cap M_1(P)$ and $U' \cap M_2(P)$ respectively. Since there are at most two isolated stationary points of a concircular scalar field, the closure of the intersection $U' \cap M_1(P)$ in $M_1(P)$ coincides with $M_1(P)$ provided $n_1 \geq 2$ and the closure of $U' \cap M_2(P)$ in $M_2(P)$ with $M_2(P)$ provided $n_2 \geq 2$. Therefore the open subset $U$ consists of one component $U'$, and the closure of $U$ coincides with the manifold $M$. If one of the parts, say $M_1$, is of dimension 1, then the part $\rho_1$ of $\rho$ is given by a hyperbolic or sine function in $U' \cap M_1(P)$, as will be seen later in the equations (3.6) and (3.9). Hence the stationary points of $\rho_1$ are isolated in $M_1$ by means of
differentiability of \( \rho_1 \), and the closure of \( U \) coincides with the manifold \( M \). Therefore, in any case, the equations (2.4) are valid over the whole manifold \( M = M_1 \times M_2 \). Thus we can state the following

**Lemma 1.** Let \( M \) and \( M^* \) be product Riemannian manifolds and \( \rho \) the scalar field associated with a non-homothetic conformal diffeomorphism \( f \) of \( M \) to \( M^* \). If \( \rho \) is decomposable in an open subset in \( M = M_1 \times M_2 \), and the parts \( \rho_1 \) and \( \rho_2 \) of \( \rho \) are not constants in the subset, then \( \rho \) is globally decomposable and the parts \( \rho_1 \) and \( \rho_2 \) are special concircular scalar fields in \( M_1 \) and \( M_2 \) satisfying the equations (2.4) respectively.

3. **Expressions of the associated scalar field \( \rho \).** We shall seek for expressions of \( \rho \) in all possible cases under the assumptions of Lemma 1.

In the case \( k = 0 \), the equations (2.4) become together the tensor equation

\[
\Gamma^*_{\mu \nu} = b g_{\mu\nu}
\]

and the gradient vector field \( Y \) is parallel if \( b = 0 \) and concurrent if \( b \neq 0 \). Along any geodesic curve with arc-length \( x \), the equation (3.1) reduces to the ordinary differential equation

\[
\rho'' = b.
\]

By choosing suitably the arc-length \( x \) of the \( \rho \)-curves, \( \rho \) is given by

\[
\rho = \begin{cases} 
ax & (b = 0), \\
\frac{1}{2} bx^2 + a & (b \neq 0),
\end{cases}
\]

\( a \) being a constant, and the metric form \( ds^2 \) of \( M \) is expressed as

\[
ds^2 = \begin{cases} 
\left(dx^2 + \frac{ds^2}{b} \right) & (b = 0), \\
\left(dx^2 + x^2 ds^2 \right) & (b \neq 0),
\end{cases}
\]

in the respective cases.

In the case \( k \neq 0 \), we may put \( k = c^2 \), \( c \) being a positive constant, without loss of generality. The equation (2.4, 1) reduces to the ordinary differential equation

\[
\rho''(x) = c^2 \rho_1 + b
\]

along any geodesic with arc-length \( x \) in \( M_1 \). By choosing suitably the arc-length \( x \) of the \( \rho \)-curves of \( \rho_1 \), the part \( \rho_1 \) is given by
\[ a_1 \text{ being a non-zero constant. In an adapted coordinate system in } M_i, \]
\[ the \text{ metric form } ds_i^2 \text{ of } M_i \text{ is expressed as} \]
\[ ds_i^2 = \begin{cases} 
(a) & dx^2 + (\exp 2cx) \overline{ds}_i^2, \\
(b) & dx^2 + (\cosh cx)^2 \overline{ds}_i^2, \\
(c) & dx^2 + (\sinh cx)^2 \overline{ds}_i^2 
\end{cases} \]
\[ in \text{ the respective cases of } (3.6), \text{ where } \overline{ds}_i^2 \text{ is the metric form of an} \]
\[ (n_i - 1)\text{-dimensional Riemannian manifold } \bar{M}_i. \]

On the other hand, the equation (2.4, 2) reduces to the ordinary differential equation
\[ (3.8) \quad \rho_i''(y) = -c^2 \rho_i + b \]
along any geodesic with arc-length \( y \) in \( M_i \). By choosing suitably the arc-length \( y \) of the \( \rho \)-curves of \( \rho_2 \), the part \( \rho_2 \) is given by
\[ (3.9) \quad \rho_2 = a_2 \cos cy + b/c^2, \]
\[ a_2 \text{ being a non-zero constant. In an adapted coordinate system in } M_2, \]
\[ the \text{ metric form } ds_2^2 \text{ of } M_2 \text{ is expressed as} \]
\[ ds_2^2 = dy^2 + (\sin cy)^2 \overline{ds}_2^2, \]
\[ where \overline{ds}_2^2 \text{ is the metric form of an } (n_2 - 1)\text{-dimensional manifold } \bar{M}_2. \]

By adding the expressions (3.6) and (3.9), we see that associated scalar field \( \rho \) is given by
\[ (3.11) \quad \rho = \begin{cases} 
(a) & a_1 \exp cx + a_2 \cos cy, \\
(b) & a_1 \sinh cx + a_2 \cos cy, \\
(c) & a_1 \cosh cx + a_2 \cos cy 
\end{cases} \]
in the respective cases.

4. Theorem. We recall the following lemma [4, Lemma 5] for later use:

Lemma 2. Let \( M \) and \( M^* \) be complete Riemannian manifolds and \( f \) a diffeomorphism of \( M \) onto \( M^* \). If the length of a differentiable curve \( \Gamma \) in \( M \) is bounded, then so is the length of the image \( \Gamma^* = f(\Gamma) \) in \( M^* \).
ON CONFORMAL DIFEOMORPHISMS

Now we suppose that the manifold $M$ and $M^*$ are complete and $f$ a global conformal diffeomorphism of $M$ onto $M^*$.

In the case $k = 0$ and $b = 0$, $M$ is globally the product of a straight line $I$ with an $(n-1)$-dimensional Riemannian manifold $\overline{M}$. Since the arc-length $x$ of $I$ is extendable to the infinity in a complete manifold, the associated scalar field $\rho$ given by the expression (3.3, 1) vanishes at the point of $I$ corresponding to $x = 0$. This contradicts to the positiveness of $\rho$.

In the case $k = 0$ and $b \neq 0$, the constants $a$ and $b$ in the expression (3.3, 2) should be positive because $\rho$ is positive for all value of $x$. The point $O$ corresponding to $x = 0$ is the stationary one of $\rho$, the $(n-1)$-dimensional Riemannian manifold $\overline{M}$ with metric form $d\overline{s}^2$ is of constant curvature 1, and $M$ itself a Euclidean space. The rays issuing from $O$ are $\rho$-curves. The arc-length $s^*$ of the image $\Gamma^*$ of a $\rho$-curve $\Gamma$ in $M$ under conformal diffeomorphism $f$ is related to the arc-length $x$ by

$$\frac{ds^*}{dx} = \frac{1}{\rho} = \frac{2}{bx^2 + a}.$$ 

Putting $s^* = 0$ corresponding to $x = 0$, we have

$$s^* = \frac{2}{\sqrt{ab}} \arctan \sqrt{\frac{b}{a}} x.$$

and see

$$s^* \longrightarrow \frac{\pi}{\sqrt{ab}} \quad (x \to \infty).$$

This means that the length of the image $\Gamma^*$ is bounded and it is a contradiction by means of Lemma 2.

In the case $k = c^2 \neq 0$, the arc-lengths $x$ and $y$ are extendable to the infinity. In the first case (a) and the second (b) of (3.11), $\rho$ has zero points. In the third case (c) with $|a_1| \leq |a_2|$, $\rho$ has zero points too. Thus in these cases there is no global conformal diffeomorphism of $M$ to $M^*$.

In the third case (c) with $|a_1| > |a_2|$, the constant $a_1$ should be positive. Let $P$ be a point corresponding to $y = \pi/2c$, and $\Gamma$ a $\rho$-curve lying in the part $M_1(P)$ passing through $P$. The arc-length $s^*$ of the image $\Gamma^*$ is related to $x$ by the equation

$$\frac{ds^*}{dx} = \frac{1}{a_1 \cosh cx}.$$
Y. TASHIRO and M. KORA

Integrating this equation, we have

\[ s^* - s_0^* = \frac{2}{a_1 c} \arctan (\exp cx) - \frac{\pi}{2a_1 c}, \]

where \( s_0^* \) is the value of \( s^* \) corresponding to the point \( P \) of \( \rho_1 \) on \( \Gamma \), and see

\[ s^* - s_0^* \xrightarrow{x \to \infty} \frac{\pi}{2a_1 c}. \]

This implies the boundedness of the length of the image \( \Gamma^* \) and leads to a contradiction by means of Lemma 2. Thus we have established the following

**Theorem.** Let \( M \) and \( M^* \) be complete product Riemannian manifolds. Then there is no global conformal diffeomorphism of \( M \) onto \( M^* \) such that the associated scalar field \( \rho \) is decomposable, \( \rho = \rho_1 + \rho_2 \), in an open subset \( U \) of \( M \) and the parts \( \rho_1 \) and \( \rho_2 \) are not constants in \( U \).

REFERENCES


DEPARTMENT OF MATHEMATICS
FACULTY OF INTEGRATED SCIENCES
HIROSHIMA UNIVERSITY
HIMEJI TECHNICAL HIGH SCHOOL

(Received June 17, 1980)