Note on Azumaya algebras and H-separable extensions

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H-SEPARABLE EXTENSIONS

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Let \( A/B \) be a ring extension with common identity 1, and \( C \) be the center of \( A \). If \( A \otimes_B A \) is \( A \)-\( A \)-isomorphic to an \( A \)-\( A \)-direct summand of a finite direct sum \( A^n \) then \( A/B \) is called to be \( H \)-separable. As is well known, \( A/C \) is \( H \)-separable if and only if \( A \) is an Azumaya \( C \)-algebra. The purpose of this note is to prove the following theorem, which has an application (Th. 2).

**Theorem 1.** Let \( A \) be an Azumaya \( C \)-algebra, and \( A \supset B \supset C \). If \( A_B \) is projective then \( A/B \) is \( H \)-separable.

**Proof.** Since \( A/C \) is separable, there exists an element \( \sum_i r_i \otimes s_i \) in \( A \otimes_C A \) such that \( \sum_i r_i s_i = 1 \) and \( \sum_i ar_i \otimes s_i = \sum_i r_i \otimes s_i a \) for all \( a \in A \). Further, since \( A_B \) is f. g. projective, there exists a finite number of elements \( t_j \in A \) and \( f_j \in \text{Hom}(A_B, B_B) \) such that \( \sum_j t_j f_j(a) = a \) for all \( a \in A \). Then, the mapping \( \theta: u \otimes v \to \sum_j u t_j \otimes f_j(v) \) of \( A \otimes_C A \) into itself is an endomorphism, and

\[
\sum_{i,j} r_i t_j \otimes f_j(s,ax)y = \theta(\sum_i r_i \otimes s_i ax)y = \theta(\sum_i ar_i \otimes s_i x)y
\]

where \( a, x, y \in A \). This implies that the map \( \phi: A \otimes_B A \to A \otimes_C A \) defined by \( x \otimes y \to \sum_{i,j} r_i t_j \otimes f_j(s,xy) \) is an \( A \)-\( A \)-homomorphism. Obviously, the canonical map \( \psi: A \otimes_C A \to A \otimes_B A \) is an \( A \)-\( A \)-homomorphism and \( \psi \phi \) is the identity map of \( A \otimes_B A \). Hence \( A \otimes_B A \otimes_A \) is \( A \)-\( A \)-homomorphism and \( A/C \) is separable map. Hence \( A/B \) is \( H \)-separable, it follows that \( A/B \) is \( H \)-separable.

Next, we need the following

**Lemma.** Let \( A/B \) be \( H \)-separable, and \( _AM \) a unital \( A \)-module. If \( _BM \) is a generator then so is \( _AM \).

**Proof.** Since \( _BM \) is a generator, \( _B\oplus^n_M^n \) for some integer \( n \geq 0 \). Further, since \( A/B \) is \( H \)-separable, \( _A\otimes_B A \otimes_A \) for some integer \( m \geq 0 \). Then, we obtain \( _A\otimes_B A \otimes_A \otimes_M^n \equiv _A\otimes_B A \otimes_M^n \).

Now, let \( B \) be a commutative ring, \( G \) a finite group of automorphisms of...
of $B$, and $R = B^G$ (the fixed ring of $G$ in $B$). Moreover, $\mathcal{J}(B; G)$ denotes the trivial crossed product $\bigoplus_{\sigma \in G} B u_{\sigma}$ with $u_{\sigma} u_{\tau} = u_{\sigma \tau}$ and $u_{\sigma} b = \sigma(b) u_{\sigma} (\sigma, \tau \in G, b \in B)$. Obviously, the map $j : \mathcal{J}(B; G) \rightarrow \text{Hom}(B_n, B_n)$ defined by $j(b u_{\sigma})(x) = b_{\sigma}(x)$ ($b, x \in B, \sigma \in G$) is a ring homomorphism. If $j$ is an isomorphism and $B_n$ is f. g. projective then $B/R$ is called to be $G$-Galois (cf. [1], [2]). Under this situation, we shall prove the following theorem which contain some characterizations of Galois extensions of commutative rings.

Theorem 2. Let $B$ be a commutative ring, $G$ a finite group of automorphisms of $B$, $R = B^G$, and $\mathcal{J} = \mathcal{J}(B; G)$. Then the following conditions are equivalent.

1. $B/R$ is $G$-Galois.
2. $\mathcal{J}$ is an Azumaya $R$-algebra.
3. $\mathcal{J}/B$ is $H$-separable.

When this is the case, $B$ is a maximal commutative $R$-subalgebra of $\mathcal{J}$ with $\mathcal{J} \otimes_R B = M_m(B)$ and $B \otimes_R \mathcal{J} = M_m(B)$, where $m$ is the order of $G$.

Proof. (1) $\Rightarrow$ (2). It is well known ([2, Prop. 3.1.2 and Prop. 2.4.1]). (2) $\Rightarrow$ (3). Since $u_{\mathcal{J}}$ is free, it follows from Th. 1. (3) $\Rightarrow$ (1). By Lemma, $u_{\mathcal{J}}$ is a generator. Hence $B/R$ is $G$-Galois by [1, Prop. A.1]. Finally, if $B/R$ is $G$-Galois then $B$ coincides with the centralizer of $B$ in $\mathcal{J}$, and hence the last assertion follows immediatly from [3, Lemma 1 (3)].

REFERENCES


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