A note on commutative separable algebras. II

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In this note, we indicate how to employ results concerning descent of projectivity in order to obtain a new proof of the main result in [3, Theorem], namely, that separability for commutative algebras descends by faithful flatness. Following the proof, we comment on the noncommutative case.

Throughout, rings and algebras have identity elements. As usual, if $A$ is a commutative ring and $B$ is an $A$-algebra, then $B$ is said to be a separable $A$-algebra if and only if the multiplication map from $B \otimes_A B^0$ to $B$ induces a projective left $B \otimes_A B^0$-module structure on $B$ [2, p. 40].

Theorem. Let $B$ be a commutative $A$-algebra and $C$ a commutative faithfully flat $A$-algebra. If the $C$-algebra $C \otimes_A B$ is separable, then $B$ is separable over $A$.

Proof. We set $X = B \otimes_A B^0$, $Y = C \otimes_A (B \otimes_A B^0)$, and $Z = (C \otimes_A B) \otimes_C (C \otimes_A B^0)$. Let $p$ and $p'$ be the multiplication maps $X \to B$ and $Z \to C \otimes_A B$ respectively. Moreover, let $g$ and $h$ be the canonical isomorphisms

$$ Z \to Y \text{ and } C \otimes_A B \to Y \otimes_X B $$

respectively. Then, the module $C \otimes_A B$ is a left $Z$-module (under the $p'$-structure), and is also a left $Y$-module (under the $1 \otimes p$-structure). Since $p' = (1 \otimes p)g$ and $h$ is $Y$-linear, the left $Z$-module structure on $C \otimes_A B$ may be identified with the left $Y$-module structure on $Y \otimes_X B$. By hypothesis, $C \otimes_A B$ is a projective left $Z$-module; moreover, it is cyclic and, a fortiori, finitely generated. Hence $Y \otimes_X B$ is a finitely generated projective left $Y$-module. Note that $Y$ is a faithfully flat right $X$-module since faithful flatness is preserved under change of base ring [1, Ch. I, Prop. 5, p. 31]. Therefore, $B$ is a (finitely generated) projective left $X$-module by virtue of the descent result [1, Ch. I, Prop. 12, p. 35].

Remark. The proof of the theorem was phrased in a way that suggests an attack on the more general context in which $B$ is assumed to be noncommutative. One then needs only to show that $B$ is a flat left
$X$-module, given that $Y \otimes_X B$ is a flat (indeed, finitely generated projective) left $Y$-module and $Y$ is faithfully flat over $X$ (on the left and the right). In this generality the problem remains open.

REFERENCES


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(Received September 1, 1980)