Mathematical Journal of Okayama University

Volume 23, Issue 2  1981  Article 14

DECEMBER 1981

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Hisao Tominaga*  Adil Yaqub†

*Okayama University
†University of California

ON GENERALIZED \( n \)-LIKE RINGS AND RELATED RINGS

HISAO TOMINAGA and ADIL YAQUB

Throughout, \( R \) will represent a ring with (Jacobson) radical \( J \), and \( N \) the set of all nilpotent elements in \( R \). A ring \( R \) is called an \( s \)-unital ring if for each \( x \in R \) there holds \( x \in Rx \cap xR \). If \( R \) is an \( s \)-unital ring then for any finite subset \( F \) of \( R \) there exists an element \( e \) in \( R \) such that \( ex = xe = x \) for all \( x \in F \) (see, [4, Lemma 1 (a)]). Such an element \( e \) will be called a pseudo-identity of \( F \). A ring \( R \) is called a generalized \( n \)-like ring if \( R \) satisfies the polynomial identity \((xy)^n - xy^n - x^n y + xy = 0\) for an integer \( n > 1 \). Recently, H. G. Moore [3] showed that if \( n \) is even or 3 then every generalized \( n \)-like ring with identity is commutative.

The present objective is to prove a theorem which generalizes Theorem 4 of [3] and deduces Theorems 2 and 3 of [3]. We begin with the following lemmas.

Lemma 1. Suppose that for each pair of elements \( x, y \) in \( R \) there exists an integer \( n = n(x, y) > 1 \) such that

\[(xy)^n - xy^n - x^n y + xy = 0.\]

Then there holds the following:

1. \((x^{n(x,x)} - x)^2 = x^{2n(x,x)} - 2x^{n(x,x)+1} + x^2 = 0.\)
2. \(x^{k(n(x,x)+1) + 2} = k(x^{n(x,x)+1} - x^2) + x^2\) for all positive integers \( k \).
3. If \( R \) is semi-primitive then \( R \) is commutative.
4. \( N^2 = 0 \) and \( N = J \) contains the commutator ideal of \( R \).

Proof. (1) Setting \( y = x \) in (*), we get (1).
(2) Let \( m = n(x,x) \). Suppose \( x^{k(m-1)+2} = kx^{m+1} - (k-1)x^2 \). Then, by (1),
\[x^{(k+1)(m-1)+2} = x^{m-1} x^{k(m-1)+2} = kx^{2m} - (k-1)x^{m+1}\]
\[= k(2x^{m+1} - x^2) - (k-1)x^{m+1} = (k+1)x^{m+1} - kx^2,\]
which completes the induction.

(3) Note that our hypothesis is inherited by all subrings and homo-
morphic images of \( R \). Note also that no complete matrix ring \((S)_t\) over a division ring \( S \) (\( t > 1 \)) satisfies the hypothesis, as a consideration of \( x = E_{11} + E_{12} \) and \( y = E_{22} \) shows. Because of these facts and the structure
theory of primitive rings, we may assume that \( R \) is a division ring. Then, since \( x^{n(x,x)} - x = 0 \) by (1), a well-known theorem of Jacobson shows that \( R \) is commutative.

(4) Since \( x^2 = x^2(2x^{n(x,x)} - 1 - x^{2(n(x,x)-1)}) \) by (1), we see that \( J \) is a nil ideal and every nilpotent element of \( R \) squares to 0. By (3), \( R/J \) is commutative. Hence \( J \) coincides with \( N \) and contains the commutator ideal of \( R \). Finally, if \( u, v \) are in \( J \) then \( uv = uv^{n(u,v)} + u^{n(u,v)}v - (uv)^{n(u,v)} = 0 \).

**Lemma 2.** Let \( R \) be an \( s \)-unital ring satisfying the hypothesis in Lemma 1. Then there holds the following:

1. For each \( x \in R \) there exists a positive integer \( a \) such that \( x^{a(n(x,x)-1)} \) is an idempotent.
2. Every idempotent of \( R \) is central.

**Proof.** (1) Let \( e \) be a pseudo-identity of \( x \), and set \( a = (2^n e, 2e - 2)^2 \). Then, by Lemma 1 (1), we get \( 0 = ((2^n e, 2e - 2e)^2) = ax \). Thus, Lemma 1 (2) shows that \( x^{a(n(x,x)-1)+2} = x^2 \), whence (1) follows.

(2) Let \( a, b \) be idempotents in \( R \), and \( e \) a pseudo-identity of \( \{a, b\} \). According to (1), we may assume that \( e \) itself is an idempotent. We set \( l = n((e-a)b,a) \) and \( m = n(e-a,b) \). Then, by (*),

\[
((e-a)b)^l a = ((e-a)ba)^l - (e-a)ba^l + (e-a)ba = 0.
\]

But, again by (*),

\[
((e-a)b)^m = (e-a)b^m + (e-a)^m b - (e-a)b = (e-a)b,
\]

and therefore \( ((e-a)b)^m a = (e-a)ba \). Reiterating in the last and using \( ((e-a)b)^l a = 0 \) above, we get \( (e-a)ba = 0 \), and hence \( ba = aba \). Replacing \( a \) by the idempotent \( e-a \) in the above argument, we also have \( b(e-a) = (e-a)b(e-a) \), and hence \( ab = aba \). Combining these, we conclude that \( ab = ba \), and thus all idempotents of \( R \) are central.

**Lemma 3.** (1) \( R \) is a generalized \( n \)-like ring if and only if \( R \) satisfies the polynomial identities \((xy)^n = x^ny^n \) and \((x^n-x)(y^n-y) = 0 \).

(2) If \( R \) is an \( s \)-unital generalized \( n \)-like ring then \((n-1)[u,x] = 0 \) for all \( u \in N \) and \( x \in R \).

**Proof.** (1) If \( R \) is a generalized \( n \)-like ring, then \( R \) satisfies the polynomial identity \( x^ny^n - xy^n - x^ny + xy = (x^n-x)(y^n-y) = 0 \) (Lemma 1 (1) and (4)). Combining this with \((xy)^n - xy^n - x^ny + xy = 0 \), we readily obtain \((xy)^n = x^ny^n \). The converse is trivial.
(2) According to Lemma 1 (4), we have
\[ 0 = [(xu)^n - xu^n - x^n u + xu] - [(ux)^n - ux^n - u^n x + ux] = [u, x^n] = [u, x]. \]

Now, let \( e \) be a pseudo-identity of \( x, u \). Then, by (1) and Lemma 1 (4),
\[ [u, x] = [u, x^n] = (ux + x)^n - (xu + x)^n = (u + e)x^n - x(u + e) = [u + e, x^n] = n[u, x^n] = n[u, x], \]
which implies (2).

We are now in a position to state our main theorem.

**Theorem 1.** Let \( R \) be an \( s \)-unital (directly) indecomposable ring. Suppose that for each pair of elements \( x, y \) in \( R \) there exists an integer \( n = n(x, y) > 1 \) such that \( (xy)^n - xy^n - x^n y + xy = 0 \). Then \( R \) is a local ring whose characteristic is \( p \) or \( p^2 \), \( p \) a prime.

**Proof.** Since \( R \) is indecomposable, Lemma 1 (4) and Lemma 2 show that \( R \) contains 1 and is a local ring. Moreover, noting that \( (2n^2-2)^2 = 0 \) by Lemma 1 (1), we see that the characteristic of \( R \) is a power of a prime \( p \). Since \( p \) is in \( N \), we get \( p^2 = 0 \) (Lemma 1 (4)).

**Corollary 1.** Let \( R \) be an \( s \)-unital ring. Suppose that for each pair of elements \( x, y \) in \( R \) there exists an integer \( n = n(x, y) > 1 \) such that \( (xy)^n - xy^n - x^n y + xy = 0 \). Then \( R \) is a subdirect sum of local rings. If furthermore \( [xy, yx] = 0 \) for all \( x \in N \) and \( y \in N \), then \( R \) is commutative.

**Proof.** In view of Theorem 1, it remains only to prove the latter part. Note that if \( R^* \) is a homomorphic image of \( R \) then \( [x^*, y^*, x^*] = 0 \) for all non-nilpotent elements \( x^*, y^* \) in \( R^* \). Because of this fact, we may assume that \( R \) is subdirectly irreducible, and thus \( R \) is a local ring (Theorem 1). Then, noting that \( N \) is commutative (Lemma 1 (4)), we can easily see that \( [xy, yx] = 0 \) for all \( x, y \in R \). Hence,
\[ [x, [x, y]] = [x(y + 1), [x, y + 1]] - [xy, [x, y]] = 0. \]

Now, by [2, Theorem], we see that \( R \) is commutative.

**Corollary 2.** Let \( R \) be an \( s \)-unital generalized n-like ring. If \( R \) is indecomposable then \( R \) is a local ring whose characteristic is \( p \) or \( p^2 \), \( p \) a prime; if \( p \) does not divide \( n - 1 \) then \( R \) is commutative.

**Proof.** In view of Theorem 1, it remains only to prove that if \( (p, n - 1) = 1 \) then \( R \) is commutative. By Lemma 3 (2), \( (n - 1)[u, x] = 0 \) for all
$u \in N$ and $x \in R$. Combining this with $p^2[u,x] = 0$, we obtain $[u,x] = 0$, and thus $N$ is contained in the center of $R$. Then, using Lemma 1 (1) and [1. Theorem], we see that $R$ is commutative.

The next includes Theorems 2 and 3 of [3].

**Corollary 3.** Let $R$ be an $s$-unital generalized $n$-like ring. If $n$ is even or 3, then $R$ is commutative.

**Proof.** Without loss of generality, we may assume that $R$ is subdirectly irreducible, and therefore $R$ is a local ring by Theorem 1. If $n$ is even, then $4 = ((-1)^n - (-1))^2 = 0$ (Lemma 1 (1)). Hence $R$ is commutative by Corollary 2. Next, we consider the case that $n = 3$. Since $R$ is a local ring, it is enough to show that if $x, y$ are units in $R$ then $xy = yx$. By Lemma 3 (1),

$$x^2y^2 - x^2 - y^2 + 1 = x^{-1}(x^3 - x)(y^3 - y)y^{-1} = 0$$

and $y^2x^2 - y^2 - x^2 + 1 = 0$. Hence $x^2y^2 = y^2x^2$. Using this and Lemma 3 (1), we get

$$(xy)^3 = x^3y^2 = x^2y^2y = xy^2x^2y = (xy)(yx)(xy),$$

whence it follows that $xy = yx$.

**Remark.** H. G. Moore required a theorem of Herstein [1] in the proof of [3. Theorem 3]. However, we can prove the same without making use of Herstein theorem (see the proof of Corollary 3).

**REFERENCES**


**OKAYAMA UNIVERSITY, OKAYAMA, JAPAN**

**UNIVERSITY OF CALIFORNIA, SANTA BARBARA, U.S.A.**

*(Received April 27, 1981)*