On projective killing tensor in a Riemannian manifold

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ON PROJECTIVE KILLING TENSOR IN A
RIEMANNIAN MANIFOLD

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Our purpose is to give some conditions for the non-existence of the
projective Killing tensor. We also show an integrability condition for the
system of the partial differential equations which defines an affine Killing
tensor of degree \( p \).

1. Let \( M^n \) be an \( m \)-dimensional Riemannian manifold with positive
definite metric \( g = (g^{ab}) \) with respect to local coordinate system \( \{ x^a \} \).
We identify a skew symmetric covariant tensor \( \nu = (\nu_{a_1...a_p}) \) with the
differential \( p \)-form
\[
\nu = (1/p!) \nu_{a_1...a_p} \, dx^{a_1} \wedge ... \wedge dx^{a_p}.
\]
d\( \nu \) and \( \delta \nu \) denote the exterior differentiation and the exterior co-differentiation
of \( \nu \) respectively. As is well known, \( d\nu = 0 \) and \( \delta \nu = 0 \) hold,
and \( d \) and \( \delta \) are dual each other with respect to the global scalar product.
Following [2], \( \nu = (\nu_{a_1...a_p}) \) is called a projective Killing tensor if \( \nu \)
is a skew symmetric tensor satisfying
\[
\begin{align*}
\nabla_c \nabla_d \nu_{a_1...a_p} + \frac{1}{2} \sum_{i=1}^{p} R_{cab_i} \nu_{a_1...\hat{a}_i...a_p} \\
- \frac{1}{2} (R_{ca_1} + R_{ca_2}) \nu_{a_2...a_p} \\
- \frac{1}{2} \sum_{i=2}^{p} (R_{bca_i} \nu_{c_1...\hat{a}_i...a_p} + R_{bca_i} \nu_{b_1...\hat{a}_i...a_p}) \\
= \sum_{i=1}^{p} (-1)^{i-1} (g^{ca_i} \nabla_c \theta_{a_1...\hat{a}_i...a_p} + g^{ba_i} \nabla_c \theta_{a_1...\hat{a}_i...a_p})
\end{align*}
\]
where \( \theta = (\theta_{a_1...a_p}) \) is a certain skew symmetric tensor and \( \hat{a}_i \) means that
\( a_i \) is deleted. If the left hand side of (1.1) vanishes identically, \( \nu \) is
called an affine Killing tensor. As was shown in [2], if \( \nu \) is a Killing
tensor then it is an affine one, and if \( \nu \) is an affine Killing tensor then it is a
projective one.

If we transvect (1.1) with \( g^{ba} \) and change the index \( c \) to \( b \), we have

1) The indices \( a, b, c, ... \) run on the range \( \{ 1, 2, ..., m \} \) and the summation convention is
used throughout this paper.
M. KORA

\[ \nabla \theta_{a_1 \ldots a_p} = \frac{1}{(m - p + 2)} \left[ \sum_{i \neq j} (-1)^i g_{a_i \theta a_j} \hat{a}_i \ldots \hat{a}_n \right], \]

that is,

\[ \nabla \theta_{a_1 \ldots \hat{a}_i \ldots a_p} = \frac{1}{(m - p + 2)} \left[ \sum_{i \neq j} (-1)^i g_{a_i \theta a_j} \hat{a}_i \ldots \hat{a}_n \right]. \]

From this, it follows by the straightforward computation that

\[ \sum_{i=1}^n (-1)^{i-1} (g_{e_i \theta} \nabla \theta_{a_1 \ldots \hat{a}_i \ldots a_p} + g_{a_i \theta} \nabla \theta_{a_1 \ldots \hat{a}_i \ldots a_p}) \]

\[ = \frac{1}{(m - p + 2)} \sum_{i=1}^n (-1)^{i-1} (g_{e_i \theta} \nabla \theta_{a_1 \ldots \hat{a}_i \ldots a_p} + g_{a_i \theta} \nabla \theta_{a_1 \ldots \hat{a}_i \ldots a_p}). \]

Thus we have

**Lemma 1.** A skew symmetric tensor \( v \) is projective if and only if it satisfies

\[ \nabla \nabla v_{a_1 \ldots a_p} + \frac{1}{2} \sum_{i=1}^n \mathcal{R}_{e_i a_p} v_{a_1 \ldots \hat{a}_i \ldots a_p} \]

\[ - \frac{1}{2} (\mathcal{R}_{e_i} v_{a_p} + \mathcal{R}_{a_p} v_{e_i}) v_{a_1 \ldots a_p} \]

\[ = \frac{1}{(m - p + 2)} \sum_{i=1}^n (-1)^{i-1} (g_{e_i \theta} \nabla (\partial v)_{a_1 \ldots \hat{a}_i \ldots a_p} + g_{a_i \theta} \nabla (\partial v)_{a_1 \ldots \hat{a}_i \ldots a_p}). \]

If we denote the covariant differentiation by \( \nabla \), we have

**Corollary 2.** A projective Killing tensor \( v \) is affine if and only if \( \nabla \partial v = 0 \) holds.

**Proof.** The if part is obvious. Let \( v \) be affine. The left hand side of (1.2) vanishes. Transvecting this with \( g^{a_1} \), we have \( \nabla_v (\partial v)_{a_2 \ldots a_p} = 0 \). Q. E. D.

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Lemma 3. In $M^n$, a projective Killing tensor $v$ of degree $p$ satisfies
\[
(\partial dv)_{a_1 \ldots a_p} + \frac{m-p}{(m-p+2)} (d\bar{v})_{a_1 \ldots a_p} = (p+2) \left[ R_{a_1}^{\ d} v_{a_2 \ldots a_p} - \sum_{i=2}^{p} R_{a_i a_1}^{\ d} v_{a_2 \ldots a_{i-1} a_{i+1} \ldots a_p} \right]
\]
\[
+ (1/2) \left[ \sum_{i=1}^{p} R_{a_i}^{\ d} v_{a_{i+1} \ldots a_p} - \sum_{i=2, j \neq p}^{p} R_{a_i a_j}^{\ d} v_{a_1 \ldots a_{i-1} a_{i+1} \ldots a_{j-1} a_{j+1} \ldots a_p} \right].
\]

In particular, if $M^n$ is of constant curvature $k$, $v$ satisfies
\[
k(m-p)(p+1)v = \partial dv + \frac{m-p}{(m-p+2)} (d\bar{v}).
\]

(If $k \neq 0$, then (1.4) gives us a decomposition of $v$.)

Proof. It holds that
\[
(\partial dv)_{a_1 \ldots a_p} = -\nabla^b \nabla_b v_{a_1 \ldots a_p} + \sum_{i=1}^{p} \nabla^b \nabla_{a_i} v_{a_1 \ldots a_{i-1} a_{i+1} \ldots a_p}.
\]

From (1.2), we have
\[
-\nabla^b \nabla_b v_{a_1 \ldots a_p} = R_{a_1}^{\ d} v_{a_2 \ldots a_p} - \sum_{j=2}^{p} R_{a_i a_j}^{\ d} v_{a_1 \ldots a_{j-1} a_{j+1} \ldots a_p}
\]
\[
+ \{2/(m-p+2)\} (d\bar{v})_{a_1 \ldots a_p}
\]
and
\[
\sum_{i=1}^{p} \nabla^b \nabla_{a_i} v_{a_1 \ldots a_p} = (p/2) \left[ R_{a_1}^{\ d} v_{a_2 \ldots a_p} - \sum_{i=2}^{p} R_{a_i a_1}^{\ d} v_{a_2 \ldots a_{i-1} a_{i+1} \ldots a_p} \right]
\]
\[
+ (1/2) \left[ \sum_{i=1}^{p} R_{a_i}^{\ d} v_{a_{i+1} \ldots a_p} - \sum_{i=2, j \neq p}^{p} R_{a_i a_j}^{\ d} v_{a_1 \ldots a_{i-1} a_{i+1} \ldots a_{j-1} a_{j+1} \ldots a_p} \right]
\]
\[-(d\bar{v})_{a_1 \ldots a_p}.
\]
The desired result follows from (1.5), (1.6) and (1.7).

If $M^n$ is of constant curvature $k$, (1.3) gives us (1.4). Q. E. D.

2. We deal with non-existence problem of $v$.

Theorem 4. In a Riemannian manifold of constant curvature $k \neq 0$, there exists no closed affine Killing tensor of degree $p$ ($< m$) other than the zero tensor.
Proof. Let \( v \) be closed and affine. From Corollary 2, we can regard \( v \) as a projective Killing tensor satisfying \( \nabla d v = 0 \). Substituting this and \( d v = 0 \) into (1.4), we have \( v = 0 \). Q. E. D.

Next, we treat the problem more generally. First, we see

\[
(2.1) \quad \sum_{i=1}^{p} R_{a_{1}a_{2}...a_{p}}^{a_{1}a_{2}...a_{p}} = \rho R_{a_{1}a_{2}...a_{p}}^{a_{1}a_{2}...a_{p}} v^{a_{1}a_{2}...a_{p}},
\]

\[
(2.2) \quad \sum_{i=1}^{p} R_{a_{1}a_{2}...a_{p}}^{a_{1}a_{2}...a_{p}} = -\frac{(p-1)}{2} R_{a_{1}a_{2}...a_{p}}^{a_{1}a_{2}...a_{p}} v^{a_{1}a_{2}...a_{p}},
\]

and

\[
(2.3) \quad \sum_{i=1}^{p} R_{a_{1}a_{2}...a_{p}}^{a_{1}a_{2}...a_{p}} = \rho \sum_{i=1}^{p} R_{a_{1}a_{2}...a_{p}}^{a_{1}a_{2}...a_{p}} v^{a_{1}a_{2}...a_{p}}.
\]

If we transvect (1.3) with \( v^{a_{1}...a_{p}} \) and use the identities (2.1), (2.2) and (2.3), we get

\[
(\delta d v)_{a_{1}...a_{p}} = [(m-p)/(m-p+2)] (d d v)_{a_{1}...a_{p}} v^{a_{1}...a_{p}}
\]

\[
= (p+1) F(v, v)
\]

where we have put

\[
F(v, v) = R_{a_{1}a_{2}...a_{p}}^{a_{1}a_{2}...a_{p}} v^{a_{1}a_{2}...a_{p}}
\]

\[
+ [(p-1)/2] R_{a_{1}a_{2}...a_{p}}^{a_{1}a_{2}...a_{p}} v^{a_{1}a_{2}...a_{p}}.
\]

If \( M^m \) is compact and orientable, (2.4) gives us

\[
(2.6) \quad \rho ! (d v, d v) + (m-p)/(m-p+2) \int_{M^m} F(v, v)^{*1}
\]

where \( (, ) \) and \( ^{*1} \) denote the global scalar product and the volume element respectively.

Theorem 5. In a compact orientable Riemannian manifold \( M^m \), there exists no projective Killing tensor \( v \) of degree \( p \) \((< m)\) which satisfies \( F(v, v) \leq 0 \), other than the parallel tensor.

Especially, if \( F(v, v) \) is negative definite, then there exists no projective Killing tensor of degree \( p \) \((< m)\) other than the zero tensor.

Proof. The latter part is obvious from (2.6). If \( F(v, v) \leq 0 \), we have from (2.6) that \( d v = 0 \) and \( \delta d v = 0 \). From Corollary 2, we see \( v \) is an affine Killing tensor. As was shown in [2], an affine Killing tensor in a compact orientable Riemannian manifold is a Killing tensor. Thus, \( v \) satisfies
ON PROJECTIVE KILLING TENSOR

\[(dv)_{a_1...a_p+1} = (p+1)\nabla_{a_1}v_{a_2...a_{p+1}}.\]

From \(dv = 0\), we have \(\nabla v = 0\). Q.E.D.

**Corollary 6.** In a compact orientable Riemannian manifold \(M^n\), there exists no Killing tensor of degree \(p (\leq m)\) which satisfies \(F(v, v) \leq 0\), other than the parallel tensor.

Especially, if \(F(v, v)\) is negative definite, then there exists no Killing tensor of degree \(p (\leq m)\) other than the zero tensor.

(In the case \(p \leq m/2\), this was proved by T. Kashiwada [1].)

We can rewrite the condition concerning \(F(v, v)\), variously. For instance, if we substitute Weyl projective curvature tensor

\[W_{abc} = R_{abc} + \{1/(m-1)\} (R_{a0b0}c - R_{c0b0}a)\]

into (2.5), we have

\[2(\mu) F(v, v) = [((p-1)/2) W_{f0\epsilon e} + ((m-p)/(m-1)) R_{f0\epsilon e}v^\epsilon_{a_2...a_p}] v^{de}_{a_2...a_p}.\]

Now, assume that Ricci tensor is negative definite and let \(\lambda_0\) be the least upper bound of the greatest eigen values on \(M^n\) of Ricci tensor. We then have

\[R_{f0\epsilon e}v^\epsilon_{a_2...a_p}v^{de}_{a_2...a_p} \leq \mu \lambda_0 \langle v, v \rangle \quad (\leq 0),\]

where \(\langle, \rangle\) denotes the local scalar product. If we put

\[2W = \sup \{\|W_{f0\epsilon e}\xi^\epsilon\xi^e\|/\langle \xi, \xi \rangle \; ; \text{ for bivector } \xi\} \]

we then have

\[W_{f0\epsilon e}v^\epsilon_{a_2...a_p}v^{de}_{a_2...a_p} \leq \mu W \langle v, v \rangle.\]

Thus, from (2.7) we have

\[F(v, v) \leq [(p-1)/2] W + [(m-p)/(m-1)] \lambda_0 \mu \langle v, v \rangle.\]

Consequently, we have

**Theorem 7.** In a compact orientable Riemannian manifold \(M^n\) with negative definite Ricci tensor, if

\[(p-1)/2] W + [(m-p)/(m-1)] \lambda_0 \leq 0\]

for some \(p (\leq m)\), then there exists no projective Killing tensor of degree
\( p \) other than the parallel tensor.

Especially, if strict inequality holds in (2.8), there exists no projective Killing tensor of degree \( p (\leq m) \) other than the zero tensor.

**Corollary 8.** (i) If \( M^m \) is a compact orientable locally flat Riemannian manifold, there exists no projective Killing tensor of degree \( p (\leq m) \) other than the parallel tensor.

(ii) If \( M^m \) is a compact orientable Riemannian manifold of negative constant curvature, there exists no projective Killing tensor of degree \( p (\leq m) \) other than the zero tensor.

**Proof.** (i) follows from Theorem 5. Since a projectively flat Riemannian manifold \( M^m (m > 1) \) is of constant curvature, (ii) follows from Theorem 7. Q. E. D.

In the case \( p \leq m/2 \), this corollary (ii) was proved by S. Tachibana [2]. This corollary (ii) can be obtained from (1.4) directly.

3. We deal with an integrability condition for the system of the partial differential equations which defines an affine Killing tensor of degree \( p (\leq m) \).

**Theorem 9.** A necessary and sufficient condition for a Riemannian manifold \( M^m (m > 2) \) to be a space of locally flat is that for any point \( Q \) and any constant \( C_{a_1 \ldots a_p} \) (skew symmetric in all indices) and \( C_{a_1 \ldots a_p} \) (skew symmetric in \( a_1, \ldots, a_{p-1} \) and \( a_p \)) there exists locally an affine Killing tensor \( \nu_{a_1 \ldots a_p} \) of degree \( p (\leq m) \) satisfying \( \nu_{a_1 \ldots a_p} (Q) = C_{a_1 \ldots a_p} \) and \( (\nabla_{a_b} \nu_{a_1 \ldots a_p}) (Q) = C_{a_b a_1 \ldots a_p} \).

**Proof.** To consider the equation, the left hand side of (1.1) equal to zero, is equivalent to consider the system of the partial differential equations with unknown functions \( \nu_{a_1 \ldots a_p} \) and \( \nu_{a_1 \ldots a_p} \) as follows:

\[
\nabla_a \nu_{a_1 \ldots a_p} = \nu_{ba_1 \ldots a_p}.
\]

\[
\nabla_a \nu_{ba_1 \ldots a_p} = -(1/2) \left[ \sum_{i=1}^{p} R_{cb_{a_i}} \, \partial_{a_1}^{b_i} \ldots \partial_{a_p}^{b_p} - (R_{ca_{b_i}} + \delta_{ca_{b_i}}) \, \partial_{a_1}^{b_i} \ldots \partial_{a_p}^{b_p} \right.
\]

\[
\left. - \sum_{i=2}^{p} R_{ba_{b_i}} \delta_{a_1}^{b_i} \delta_{a_2}^{b_2} \ldots \delta_{a_p}^{b_p} \right) \nu_{a_1 \ldots a_p}.
\]

(3.1)

(3.2)
and

\[ u_{a_1 \ldots a_j \ldots a_p} = - u_{a_1 \ldots a_j \ldots a_p} \quad (1 \leq i < j \leq p). \]

The necessary part is obvious.

For the system to be completely integrable, it is necessary that

\[ \nabla_c \nabla_{b_1 a_2 a_3 \ldots a_p} + \nabla_c \nabla_{b_2 a_3 a_4 \ldots a_p} = 0 \]

and

\[ \nabla_c \nabla_{b_1 a_2 a_3 \ldots a_p} - \nabla_c \nabla_{b_2 a_3 a_4 \ldots a_p} = - \left[ \sum_{i=1}^{p} R_{a_i c b_1} \delta_{a_2}^{b_1} \ldots \delta_{a_i}^{b_i} \ldots \delta_{a_p}^{b_p} + R_{a_i c b_2} \delta_{a_2}^{b_2} \ldots \delta_{a_i}^{b_i} \ldots \delta_{a_p}^{b_p} \right] v_{b_1 \ldots b_p}. \]

It follows from (3.4) and (3.2) that

\[ \left[ B_{c b_1 a_2 a_3 \ldots a_p}^{b_1 \ldots b_p} + B_{c b_2 a_3 a_4 \ldots a_p}^{b_1 \ldots b_p} \right] v_{b_1 \ldots b_p} = 0 \]

where we have put

\[ B_{c b_1 a_2 a_3 \ldots a_p}^{b_1 \ldots b_p} = \sum_{i=1}^{p} R_{c b_{a_i}} \delta_{a_2}^{b_1} \ldots \delta_{a_i}^{b_i} \ldots \delta_{a_p}^{b_p} - \left( R_{c b_{a_1}} + R_{c b_{a_2}} \right) \delta_{a_2}^{b_1} \ldots \delta_{a_p}^{b_p} - \sum_{i=2}^{p} R_{c b_{a_i}} \delta_{a_2}^{b_2} \ldots \delta_{a_i}^{b_i} \ldots \delta_{a_p}^{b_p} - \sum_{i=2}^{p} R_{c b_{a_i}} \delta_{a_2}^{b_2} \ldots \delta_{a_i}^{b_i} \ldots \delta_{a_p}^{b_p}. \]

Since \( v_{b_1 \ldots b_p} \) are arbitrary and (3.3) holds, it follows from (3.6) that

\[ \sum_{\tau \in \mathfrak{S}_p} \varepsilon_{\tau} B_{c b_{\tau(a_1} a_{\tau(a_2} \ldots a_{\tau(a_p)}}(1 \ldots p) + \sum_{\tau \in \mathfrak{S}_p} \varepsilon_{\tau} B_{c b_{\tau(a_1} a_{\tau(a_2} \ldots a_{\tau(a_p)}}(1 \ldots p) = 0 \]

where \( \varepsilon_{\tau} \) denotes the sign of the permutation \( \tau \in \mathfrak{S}_p \). If we put

\[ \delta_{a_1}^{b_1} \ldots \delta_{a_p}^{b_p} = \sum_{\tau \in \mathfrak{S}_p} \varepsilon_{\tau} \delta_{a_{\tau(a_1)}}^{b_1} \ldots \delta_{a_{\tau(a_p)}}^{b_p} = \det(\delta_{a_i}^{b_i}), \]

we get from the above that

\[ \left( R_{c b_{a_1}} + R_{c b_{a_2}} \right) \delta_{a_2}^{b_1} \ldots \delta_{a_p}^{b_p} + \left( R_{c b_{a_1}} + R_{c b_{a_2}} \right) \delta_{a_2}^{b_1} \ldots \delta_{a_p}^{b_p} + \sum_{i=3}^{p} R_{c b_{a_i}} \delta_{a_2}^{b_1} \ldots \delta_{a_i}^{b_i} \ldots \delta_{a_p}^{b_p} + \sum_{i=3}^{p} R_{c b_{a_i}} \delta_{a_2}^{b_1} \ldots \delta_{a_i}^{b_i} \ldots \delta_{a_p}^{b_p} + \sum_{i=3}^{p} R_{c b_{a_i}} \delta_{a_2}^{b_1} \ldots \delta_{a_i}^{b_i} \ldots \delta_{a_p}^{b_p} + \sum_{i=3}^{p} R_{c b_{a_i}} \delta_{a_2}^{b_1} \ldots \delta_{a_i}^{b_i} \ldots \delta_{a_p}^{b_p} = 0. \]
Contracting this with respect to $a_j$ and $b_j$ ($2 \leq j \leq p$), and taking account of the identity
\[
\delta_{a_1}^{b_1} \cdots \delta_{a_k}^{b_k} = \frac{1}{(m-k)!/(m-p)!} \delta_{a_1}^{b_1} \cdots \delta_{a_p}^{b_p},
\]
we can get $W_{a,b}^c b_i + W_{a,c}^b b_i = 0$. It follows from this that
\[
(3.7) \quad W_{a,c}^b = 0.
\]
On the other hand, from (3.5), (3.2) and (3.1) we have
\[
[B_{c_{b_1} \cdots a_p} h_{b_1} \cdots b_p] - B_{d_{b_1} \cdots a_p} h_{b_1} \cdots b_p
\]
\[-2 \sum_{i=1}^p R_{d_{c_1} \cdots a_p}^{c_{b_1} \cdots b_p} \delta_{a_1}^{c_{b_1}} \cdots \delta_{a_p}^{c_{b_p}} - 2 R_{d_{c_1} \cdots a_p}^{c_{b_1} \cdots b_p} \delta_{a_1}^{c_{b_1}} \cdots \delta_{a_p}^{c_{b_p}}] v_{b_1} \cdots b_p
\]
\[+ [\nabla_c B_{c_{b_1} \cdots a_p} h_{b_1} \cdots b_p] - \nabla_c B_{c_{b_1} \cdots a_p} h_{b_1} \cdots b_p = 0.
\]
From this and the assumption of the theorem, we have
\[
\sum_{\tau \in S_p} \epsilon_{\tau} B_{c_{b_1} \cdots a_p} h_{b_1} \cdots b_p - \sum_{\tau \in S_p} \epsilon_{\tau} B_{d_{a_1} \cdots a_p} h_{b_1} \cdots b_p = 0.
\]
If we contract this with respect to $b_i$ and $d$, $b_i$ and $c$, and $b_i$ and $a_j$ ($2 \leq j \leq p$), we can get $R_{b_1} = 0$. Substituting this into (3.7), we have $R_{a_1} = 0$. Q. E. D.

Last of all, we see the corresponding problem on projective Killing tensor of degree $p$. If in this case we consider the system (3.1), (3.3) and
\[
\nabla_c v_{a_1} \cdots a_p = -(1/2) B_{c_{b_1} \cdots a_p} h_{b_1} \cdots b_p
\]
\[+ (1/(m-p+2)) \sum_{i=1}^p (-1)^{i-1} (g_{c_1} \nabla b_i e_{a_1} \cdots e_{a_p} + g_{b_1} \nabla e_{a_1} \cdots e_{a_p}),
\]
we have the following

Theorem 10. If there exists locally a projective Killing tensor $v_{a_1} \cdots a_p$ of degree $p$ satisfying $v_{a_1} \cdots a_p (Q) = C_{a_1} \cdots a_p$ for any point $Q$ of $M^n (m \geq 3)$ and any skew symmetric constant $C_{a_1} \cdots a_p$, then $M^n$ is a space of projectively flat.
ON PROJECTIVE KILLING TENSOR

Proof. It follows from (3.4) and (3.8) by the same computation as in the proof of Theorem 9 that the equation (3.6) holds. By the assumption, we then have (3.7). Q.E.D.

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