Structure and commutativity of rings with constraints on nilpotent elements. II

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STRUCTURE AND COMMUTATIVITY OF RINGS WITH CONSTRAINTS ON NILPOTENT ELEMENTS. II

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The purpose of this note is to generalize the principal theorem of the previous paper [1] as follows:

**Theorem.** Let $R$ be an associative ring and let $N$ be the set of all nilpotent elements of $R$. Suppose $n$ is a fixed positive integer. Suppose, further, that (i) $N$ is commutative, (ii) for every $x$ in $R$, there exists an element $x'$ in the subring $\langle x \rangle$ generated by $x$ such that $x^n = x^{n+1}x'$ with some positive integer $m = m(x)$, (iii) $x - y \in N$ implies that $x^n - y^n$ is in the center $Z$ of $R$.

(a) If $na = 0$, $a \in N$ imply $a = 0$, then $R$ is a subdirect sum of nil commutative rings and local commutative rings.

(b) If $n$ is a prime, then $R$ is a subdirect sum of nil commutative rings and local commutative rings.

In preparation for the proof, we establish the following lemmas.

**Lemma 1.** Hypothesis (iii) implies that $ab^n = b^n a$ for all $a \in N$ and all $b \in R$, and necessarily all idempotents of $R$ are in $Z$.

**Proof.** Since $(a + b) - b \in N$, by (iii) we have $c = (a+b)^n - b^n \in Z$. Hence $b^n (a + b) = \{(a + b)^n - c\} (a + b) = (a + b) \{(a + b)^n - c\} = (a + b) b^n$, which simplifies to $b^n a = ab^n$. As is well known, every idempotent commuting with all nilpotents is central.

**Lemma 2.** Hypotheses (i), (ii), (iii) imply the following:

(a) $N$ is a commutative nil ideal.

(b) If $e$ is an idempotent and $a$ is in $N$, then $nea \in Z$.

(c) If $\varphi$ is a homomorphism of $R$ onto $R^*$, then $\varphi(N)$ coincides with the set of all nilpotent elements of $R^*$.

**Proof.** (a) and (c) have been proved in Lemma 2 [1]. We shall prove (b). Since $N$ is a commutative nil ideal, it can be easily seen that $a^k \in Z$ for all $k > 1$. By (iii), $(e + a)^n - e^n$ is in $Z$. Hence, $a^n + na^{n-1}e + \cdots + nae \in Z$, since $e$ is central by Lemma 1. This implies that $nae \in Z$. 

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Corollary 1. If $R$ satisfies the hypotheses (i), (ii), (iii), then any subring of $R$ and any homomorphic image of $R$ satisfy (i), (ii), (iii).

Now, we are ready to prove our theorem.

Proof of Theorem. Careful scrutiny of the proof of Theorem 2 [1] shows that it suffices to prove that if $\varphi$ is a homomorphism of $R$ onto a local ring $R^*$ with (nil) radical $N^*$ such that $R^*/N^* = GF(r)$, where $r=p^s$, $p$ prime, $s \geq 1$, then every element $a^*$ in $N^*$ is central.

(a) By (ii) and Lemma 1, we can easily see that there exists a central idempotent $e$ of $R$ such that $\varphi(e) = 1$. Let $b^*$ be an arbitrary element of $R^*$. Then, by Lemma 2, $a^* = \varphi(a)$ with some $a \in N$, and $b^* = \varphi(b)$ with some $b \in R$. Since $ne \in Z$ (Lemma 2 (b)), therefore $ne[a, b] = 0$. By hypothesis, it follows then $e[a, b] = 0$, and therefore $[a^*, b^*] = 0$.

(b) Obviously, $R^*$ is of characteristic $p^s$ for some positive integer $p$. By Lemma 2 (b) and Corollary 1, $na^*$ is central. If $n \neq p$, then it is easy to see that $a^*$ is central. On the other hand, if $n=p$ then Lemma 1 enables us to proceed as in the latter part of the proof of Theorem 2 [1].

The following example was pointed out to us by Prof. H. G. Moore. Let $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in GF(4) \right\}$. It is readily verified that $R$ is not commutative and satisfies all the hypotheses of Theorem (a) except the hypothesis that $na = 0$, $a \in N$ imply $a = 0$ ($n = 6$). Next, we consider the ring $R$ constructed in Remark [1]. Then $R$ is not commutative, and satisfies all the hypotheses of Theorem (b) except the hypothesis that $n$ is prime ($n=6$).

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REFERENCE


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