On s-unital rings

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ON s-UNITAL RINGS

Dedicated to Professor Mikao Moriya on his 70th birthday

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The present paper attempts to generalize several results in [10], [21], [22] and [24] obtained for rings with identity. In fact, we can prove similar ones for left (and right) s-unital rings, where a ring $R (\neq 0)$ is called a left s-unital ring if $Ra \equiv a$ for any $a \in R$. Needless to say, the class of left s-unital rings includes those of rings with identity and of regular rings. In [6], [18] and [23] we treated with left s-unital rings in the connection with regular rings. In the present paper, our attention will be directed towards the classes of left $V$-rings, fully left idempotent rings, and of almost commutative rings, those which are closely related to the class of regular rings. §1 contains a fundamental proposition, a characterization of prime ideals of a left s-unital ring in terms of its right modules as in Beachy [3], and a slight generalization of a result of Hansen [13]. The material of §2 comes from Fisher [10], Michler-Villamayor [21], Ramamurthi [22] and Yue Chi Ming [25], and left $V$-rings will be concerned in regular rings, left $p$-$V$-rings and fully left idempotent rings. In §§3 and 4, almost all results of Wong [24] will be carried over to $s$-unital rings.

For future reference, $R (\neq 0)$ will represent always a ring (with or without identity), and $C$ the center of $R$. The Jacobson radical and the prime radical of $R$ will be denoted by $J(R)$ and $P(R)$, respectively. As for other notations, we follow [18] and [23].

1. s-unital rings. A left $R$-module $M \neq 0$ is defined to be $s$-unital if $Ru \equiv u$ for any $u \in M$. For instance, every irreducible left $R$-module is $s$-unital. Needless to say, if $sM$ is $s$-unital then it is unital, and in case $R$ contains 1 these notions are identical. We can define similarly an $s$-unital right $R$-module.

Theorem 1. If $M (\neq 0)$ is a left $R$-module then the following are equivalent:

1) $sM$ is s-unital.

2) For any $u_1, \cdots, u_n \in M$ there exists an element $e \in R$ such that $eu_i = u_i (i=1, \cdots, n)$. 

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3) For any positive integer \( n \), every \((R)_n\)-submodule of the direct sum \( ^{(n)}M \) of \( n \) copies of \( M \) is of the form \( ^{(n)}N \) with some \( uN \subseteq M \), where \((R)_n \) denotes the \( n \times n \) matrix ring over \( R \).

Proof. 1) \( \iff \) 2). Assume that \( uM \) is \( s \)-unital. Choose an element \( e_n \subseteq R \) such that \( e_n u_n = u_n \), and set \( v_i = u_i - e_n u_i \) (\( i = 1, \ldots, n-1 \)). By induction method, there exists an element \( e' \subseteq R \) such that \( e' v_i = v_i \) (\( i = 1, \ldots, n-1 \)). Then, one will easily see that \( e = e' + e_n - e' e_n \) is an element with the property requested in 2). The converse is trivial.

1) \( \iff \) 3). Given \( a \subseteq R \), \( E_{ij}(a) \) will denote the element of \((R)_n\) with \( a \) in the \((i, j)\)-position and zeros elsewhere. If \( u_1, \ldots, u_n \subseteq M \), then

\[
E_{ii}(a) = \begin{pmatrix}
u_1 & \ldots & au_i \\
\ldots & \ldots & \ldots \\
u_n & \ldots & 0
\end{pmatrix},
\]

whence we can easily see that 1) implies 3). The converse is also easy by the fact that

\[
\begin{pmatrix}
Ru + Zu \\
Ru \\
\ldots \\
Ru
\end{pmatrix}
\]

is an \((R)_n\)-submodule of \( ^{(n)}M \) for any \( u \subseteq M \).

If \( uR \) (resp. \( R u \)) is \( s \)-unital, \( R \) is said to be left (resp. right) \( s \)-unital. To be easily seen, every (non-zero) homomorphic image of a left \( s \)-unital ring is left \( s \)-unital, and any regular ring is left and right \( s \)-unital. (In Ramamurth [22], a left \( s \)-unital ring is cited as a left \( D \)-regular ring.)

Corollary 1. If \( R \) is left \( s \)-unital then so is \((R)_n\) and conversely.

Proof. If \( A = (a_{ij}) \) is an arbitrary element of \((R)_n\), then by Theorem 1 there exists an element \( e \subseteq R \) such that \( e a_{ij} = a_{ij} \) (\( i, j = 1, \ldots, n \)), whence it follows \( \text{diag} \{ e, \ldots, e \} \cdot A = A \). Conversely, if \( Ra \not\supseteq a \) then \((R)_n \cdot \text{diag} \{ a, \ldots, a \} \) does not contain \( \text{diag} \{ a, \ldots, a \} \).

Proposition 1 (cf. [2, Proposition 5]). Let \( \tau \) be a non-zero right ideal of \( R \). Then the following are equivalent:

1) \( \tau \) is a left \( s \)-unital ring.
2) \( \tau \cap I = \tau l \) for any left ideal \( I \) of \( R \).

If \( R \) is right \( s \)-unital then 1) is also equivalent to the following:
3) \( \tau M \cap N = \tau N \) for any left \( R \)-modules \( _{r}N \subseteq _{r}M \).

(In case \( R \) contains 1, it is known that 1) is nothing but to say that \( (R/\tau)_{R} \)
is flat (see for instance [19, Proposition 3, p. 133]).)

**Proof.** 1) \( \iff \) 2) is easy, and in case \( R \) is right \( s \)-unital 2) is obviously a special case of 3).

1) \( \implies \) 3). Let \( u = a_{1}u_{1} + \cdots + a_{n}u_{n} \) \((a_{i} \in \tau, \ u_{i} \in M)\) be an arbitrary element of \( \tau M \cap N \), and choose \( e \in \tau \) with \( ea_{i} = a_{i} \) for all \( i \) (Theorem 1). Then \( u = ea_{1}u_{1} + \cdots + ea_{n}u_{n} = eu \in \tau N \).

The next will play occasionally an important role in our subsequent study.

**Proposition 2.** Let \( R \) be a left (resp. right) \( s \)-unital ring.

1) If \( \alpha \) is a proper ideal of \( R \) then \( \alpha \) is contained in a proper prime ideal.

2) Let \( R' / R \) be a ring extension. If \( \alpha' \) is an ideal of \( R' \) and \( \alpha' \cap R \neq R \) then there exists a maximal left (resp. right) ideal \( m' \) of \( R' \) such that \( m' \supseteq \alpha \) and \( m' \cap R \neq R \). Especially, if \( \alpha \) is a proper ideal of \( R \) then \( \alpha \) is contained in a maximal left (resp. right) ideal of \( R \) (cf. [23, Lemma 1 (a)]).

**Proof.** (1) Let \( r \in R \setminus \alpha \), and choose \( e \in R \) such that \( r = er \). Then \( E = \{e' \mid i = 1, 2, \cdots \} \) is an \( m \)-system excluding \( \alpha \). If \( p \supseteq \alpha \) is an ideal of \( R \) which is maximal with respect to excluding \( E \), then \( p \) is a proper prime ideal.

(2) Let \( r \in R \setminus (\alpha' \cap R) \), and choose \( e \in R \) such that \( r = er \). By Zorn's lemma, there exists a maximal member \( m' \) in the family of left ideals \( b' \) of \( R' \) with \( b' \supseteq \{x' \in R' \mid x'r \in \alpha' (\supseteq \alpha') \} \) and \( b' \neq e \). Obviously \( m' \cap R \neq R \), and one will easily see that \( m' \) is a maximal left ideal of \( R' \).

For a right \( R \)-module \( M_{R} \), we set \( \tau(M_{R}) = \sum_{r} fM \) \((f \in \text{Hom}(M_{R}, R))\) and \( \text{Ann}(M_{R}) = \{x \in R \mid Mx = 0\} \). To be easily seen, \( \tau(M_{R}) \) is an ideal of \( R \) and \( \text{Ann}(M_{R}) \subseteq \text{Ann}(\tau(M_{R})) \).

Now, let \( M_{R} \) and \( M'_{R} \) be non-zero right \( R \)-modules. If for each \( u \neq 0 \in M \) there exists \( f \in \text{Hom}(M_{R}, M'_{R}) \) such that \( fu \neq 0 \), then we write \( M_{R} \rhd M'_{R} \). If \( M_{R} \rhd M'_{R} \) and \( M'_{R} \rhd M_{R} \), then we write \( M_{R} \sim M'_{R} \). It is easy to see that the relations \( \rhd \) and \( \sim \) are transitive. Obviously, \( M_{R} \rhd R_{R} \) is nothing but to say that \( M_{R} \) is torsionless, and then we have \( \text{Ann}(M_{R}) = \text{Ann}(\tau(M_{R})) \). If \( M_{R} \) is faithful then \( R_{R} \rhd M_{R} \), and in case \( R \) is left \( s \)-unital the converse is also true.
In what follows, we shall present a characterization of proper prime ideal of a left s-unital ring in terms of its right modules. If \( R \) is a prime ring and \( M_R \succ R \) then \( \tau(M_R) \) is non-zero and \( \text{Ann}(M_R) = \text{Ann}(\tau(M_R)) \neq 0 \), namely, \( M_R \) is faithful. Conversely, if every torsionless right \( R \)-module is faithful then \( R \) is seen to be prime. Hence, for a left s-unital ring \( R \), we see that \( R \) is prime if and only if \( M_R \succ R \) implies always \( M_R \sim R_R \).

**Theorem 2** (cf. [3, Theorem 2]). If \( \mathfrak{p} \) is a proper ideal of a left s-unital ring \( R \) then the following are equivalent:
1. \( \mathfrak{p} \) is a prime ideal.
2. \( M_R \succ (R/\mathfrak{p})_R \) implies always \( M_R \sim (R/\mathfrak{p})_R \).

**Proof.** If \( M_R \succ (R/\mathfrak{p})_R \) then \( \text{Ann}(M_R) \supseteq \text{Ann}((R/\mathfrak{p})_R) = \mathfrak{p} \), and so \( M_R \) may be regarded as \( M_{R/\mathfrak{p}} \). Hence, \( R/\mathfrak{p} \) is a prime ring if and only if \( M_R \sim (R/\mathfrak{p})_R \) for any \( M_R \succ (R/\mathfrak{p})_R \).

**Corollary 2** (cf. [3, Theorem 3]). Let \( R \) be a left s-unital ring. If \( N_R (\neq 0) \) is a unital module then the following are equivalent:
1. \( M_R \succ N_R \) implies always \( M_R \sim N_R \).
2. \( N_R \sim (R/\mathfrak{p})_R \) for a proper prime ideal \( \mathfrak{p} \).

**Proof.** 1) \( \Rightarrow \) 2). Let \( \mathfrak{p} = \text{Ann}(N_R) (\neq R) \). Since \( N_{R/\mathfrak{p}} \) is faithful, we have \( (R/\mathfrak{p})_{R/\mathfrak{p}} \succ N_{R/\mathfrak{p}} \), and hence \( (R/\mathfrak{p})_R \sim N_R \). If \( M_R \succ (R/\mathfrak{p})_R \) then \( M_R \succ N_R \), and \( M_R \sim N_R \sim (R/\mathfrak{p})_R \), whence it follows that \( \mathfrak{p} \) is a prime ideal (Theorem 2).

2) \( \Rightarrow \) 1). Since \( M_R \succ N_R \sim (R/\mathfrak{p})_R \) and \( \mathfrak{p} \) is prime, Theorem 2 shows that \( M_R \sim (R/\mathfrak{p})_R \sim N_R \).

As was shown in [13], every left Noetherian, left s-unital ring has a left identity. The next is a slight generalization of the result.

**Theorem 3.** If a left Goldie ring \( R \) is left s-unital then \( R \) contains a left identity.

**Proof.** To be easily seen, the left singular ideal \( Z_s(R) \) is contained in \( P(R) \) that is nilpotent by Lanski's theorem (cf. [16, p. 24]). By [9, Theorem 1.3], \( R/Z_s(R) \) satisfies the maximum condition for right annihilators. Then \( R/Z_s(R) \) has a left identity by [14, Proposition 2.1], and hence the semi-prime ring \( R/P(R) \) has the identity. Now, we shall proceed by the induction with respect to the nilpotency index \( n \) of \( P(R) \). The case \( n = 1 \) is obvious by the above. Assume \( n > 1 \). Since \( R/P(R)^{n-1} \) has a left identity by the induction hypothesis and \( R/P(R) \) has the identity,
a result of Herstein (cf. [15, p. 31]) shows that $R$ has a left identity.

**Corollary 3.** If $R$ is left s-unital then the following are equivalent:

1) $R$ is a left Artinian ring.

2) $R$ is a left Noetherian $\pi$-regular ring.

3) $R$ is a fully left Goldie $\pi$-regular ring.

**Proof.** If $R$ is left Artinian then $R$ is left Noetherian by Hopkins' theorem (cf. [17, Theorem 34, p. 134]). Moreover, $R$ being of bounded index, $R$ is $\pi$-regular by [1, Theorem 5]. Since 2) implies 3) obviously, it remains only to prove that 3) implies 1). As was claimed in the proof of Theorem 3, $P(R)$ is nilpotent and $\bar{R}=R/P(R)$ has the identity. Now, let $\bar{a}$ be an arbitrary regular element of $\bar{R}$, and $\bar{a}^n=\bar{a}$. Then, $\bar{a}^n(1-\bar{x}a^n)=0$ implies $\bar{x}a^n=1$, and similarly $\bar{a}^n\bar{x}=1$. Hence, every regular element of $\bar{R}$ is a unit, which means that $\bar{R}$ coincides with its left quotient ring that is Artinian semiprimitive. Recalling here that $R/P(R)^{k+1}$ is a left s-unital, left Goldie ring, one will easily see that $R(P(R)^{k}/P(R)^{k+1})$ is completely reducible and of finite length. It follows therefore that $R$ has a composition series.

**Corollary 4.** Let $R$ be a left s-unital, fully left Goldie ring whose prime factor rings are $\pi$-regular. If $\alpha$ is an ideal of $R$ and $\alpha^\alpha$ is of finite length, then $\alpha^\alpha$ is of finite length.

**Proof.** To our end, it suffices to prove the assertion for a minimal ideal $\alpha$. Obviously, $I(\alpha)$ is a prime ideal of $R$ and $S=R/I(\alpha)$ is Artinian simple by Corollary 3. Since $R$ is left Goldie and $\alpha$ is completely reducible, $\alpha$ is of finite length.

The next is perhaps in the same vein as Corollary 4, and can be proved in the same way as in the proof of [20, Proposition].

**Corollary 5.** Let $R$ be a left s-unital, left Noetherian ring. If $\alpha$ is an ideal of $R$ and $\alpha^\alpha$ is of finite length, then $\alpha^\alpha$ is of finite length, too.

**Remarks.** (1) Every s-unital left $R$-module is a homomorphic image of a direct sum of copies of $R$.

(2) Let $M$ be an s-unital left $R$-module over a left s-unital ring $R$. We consider the map $f: M \to R \otimes_R M$ defined by $u \mapsto e' \otimes u$, where $e'u=u$. If $e'u=u$ ($e'' \subseteq R$) then there exists an element $e \subseteq R$ such that $ee'=e'$ and $ee''=e''$ (Theorem 1) and we have $e' \otimes u = ee' \otimes u = e \otimes e'u$.
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\[ e \otimes e'' u = e'' \otimes u. \] Hence, \( f \) is well-defined and is an \( R \)-homomorphism. 
Now, let \( \sum a_i \otimes u_i \) be an arbitrary element of \( R \otimes_{R} M \). Again by Theorem 1, we can find an element \( a \in R \) such that \( aa_i = a_i \) for all \( i \). 
We have then \( (\sum a_i u_i) f = a \otimes \sum a_i u_i = \sum a a_i \otimes u_i = \sum a_i \otimes u_i \). 
This proves that \( R \otimes_{R} M \) is canonically isomorphic to \( R M \). Similarly, if \( R \) is commutative then we can prove the same for any \( s \)-unital module \( R \).

(3) An \( s \)-unital module \( M \) will be defined to be \( s \)-flat if for each pair of \( s \)-unital left \( R \)-modules \( A \subseteq B \) (with the inclusion map \( i \) \( 1 \otimes i : M \otimes_{R} A \rightarrow M \otimes_{R} B \) is a monomorphism. As a consequence of (2), one will easily see that if \( R \) is left and right \( s \)-unital then \( R \) is \( s \)-flat. Moreover, we can prove the following: Let \( R \) be a left and right \( s \)-unital ring, and \( l \) a left ideal of \( R \). If \( M \) is \( s \)-flat then \( M \otimes_{R} l \) is canonically isomorphic to \( M l \).

2. \textbf{V-rings.} An \( s \)-unital left \( R \)-module \( M \) is defined to be \( s \)-injective if \( M \) has the property that for each pair of \( s \)-unital left \( R \)-modules \( A \subseteq B \) each \( f \in \text{Hom} (A, M) \) can be extended to an element of \( \text{Hom} (B, M) \). 
If \( R \) is \( s \)-injective then \( R M \oplus R M' \) for any \( s \)-unital \( R M' \supseteq M \). Moreover, the proof of [8, Theorem 1.6] enables us to obtain the following:

\textbf{Proposition 3 (Baer Criterion).} Let \( R \) be a left \( s \)-unital ring, and \( M \) an \( s \)-unital left \( R \)-module. Then \( R M \) is \( s \)-injective if and only if for each left ideal \( l \) of \( R \) each \( f \in \text{Hom} (A, M) \) can be extended to an element of \( \text{Hom} (R, M) \).

An \( s \)-unital left (resp. right) \( R \)-module \( M \) is called a \textbf{V-module} if every \( R \)-submodule of \( M \) is an intersection of maximal \( R \)-submodules. If \( R \) (resp. \( R l \)) is a \( V \)-module, \( R \) is called a \textbf{left} (resp. \textbf{right}) \textbf{V-ring} (cf. [5]). As was mentioned in [18, Remark], we obtain the following which corresponds to [21, Theorem 2.1]:

\textbf{Theorem 4.} The following are equivalent:

1) \( R \) is a \textbf{left} \textbf{V-ring}.
2) \( R \) is left \textbf{s-unital} and every irreducible \textbf{left} \textbf{module} is \textbf{s-injective}.
3) \( R \) is left \textbf{s-unital} and every \textbf{s-unital} \textbf{left} \textbf{module} is a \textbf{V-module}.
4) \( R \) is left \textbf{s-unital}, and for any \textbf{s-unital} \textbf{left} \textbf{module} \( M \) the intersection of all maximal \textbf{R-submodules} is \( 0 \); \( \text{rad } R M = 0 \).
5) For any positive integer \( n \), \( (R) \) is a \textbf{left} \textbf{V-ring}.

\textbf{Proof.} First, we shall prove the equivalence of 1)\( \rightarrow 4) \). Obviously, 
4) \( \iff 3) \iff 1) \).
2) \( \rightarrow 4) \). Let \( M \) be an arbitrary \textbf{s-unital} \textbf{left} \textbf{R-module}. If 0 \( \neq u \in \text{rad } M \),

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M, then there exists an R-submodule Y of M which is maximal with respect to Y \not\supseteq u. Let S be the set of R-submodules of M properly containing Y, and D = \bigcap_{x \in S} X (\ni u). Since D/Y is an irreducible R-module, by 2) there exists an R-submodule K of M containing Y such that M/Y = D/Y \oplus K/Y. Then u \not\in K, and hence Y = K, namely, M = D. This means that Y is a maximal R-submodule of M and rad uM = 0.

1) \implies 2). Let M be an irreducible left R-module, and I a left ideal of R. If f is a non-zero element of \text{Hom}(u, uM), then I' = \text{Ker} f \subseteq I. By 1), there exists a maximal left ideal m such that m \supseteq I' and m \supseteq I. Since \text{rad}_R M \supseteq (I'/I') is irreducible and I \supseteq m \cap I \supseteq I', we have m \cap I = I'. Now, taking this into mind, we can well-define an extension g \in \text{Hom}(u, uM) of f by I + m \longrightarrow I f (I \subseteq I, m \subseteq m). Hence \text{rad}_R M is s-injective by Proposition 3.

Next, we shall prove 1) \implies 5) \implies 4).

1) \implies 5). The direct sum R^{(n)} of n copies of R is an s-unital left R-module (Theorem 1), and we have seen that \text{rad}_R R^{(n)} is a V-module. Again by Theorem 1, every (R)_s-submodule of (R)_n = (R^{(n)}) is of the form (N) with some \text{rad}_R N \subseteq R^{(n)}. Since \text{rad}_R R^{(n)} is a V-module, N = \bigcap \text{rad}_R M, with maximal submodules \text{rad}_R M \subseteq \text{rad}_R R^{(n)}. Hence (a)N = \bigcap \text{rad}_R M, and (R)_n is a left V-ring.

5) \implies 4). Again by Theorem 1, given an s-unital uM, the left (R)_s-module (a)M is s-unital and rad (a)M = 0, whence it follows rad uM = 0.

A left R-module M is said to be p-injective if for any principal left ideal (a) of R and f \in \text{Hom}(u, uM) there exists an element u \in M such that xf = xu for all x \in (a). As was noted in [6], R is regular if and only if every left R-module is p-injective (cf. also [25]). In connection with Theorem 3, a left s-unital ring R is defined to be a left p-V-ring if every irreducible left R-module is p-injective. We can define a right p-V-ring in an obvious way. In case R contains 1, a left V-ring is a left p-V-ring. More generally we have

**Proposition 4.** If R is a right s-unital, left V-ring then it is a left p-V-ring.

**Proof.** Let uM be irreducible, and (a) = Ra an arbitrary principal left ideal of R. Choose an element e \in R with ae = a. If f \in \text{Hom}(u, uM) and g \in \text{Hom}(u, uM) is an extension of f, then for any x \in R there holds (xf) = (xa)g = (xag) = xa \cdot eg.

If every left (resp. right) ideal of R is idempotent, R is said to be fully left (resp. right) idempotent. (In [22], a fully left idempotent ring is cited as a left weakly regular ring.) On the other hand, R is said to be fully idempotent if every ideal of R is idempotent.

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Proposition 5. (1) The following are equivalent:
1) $\mathcal{R}$ is fully left idempotent.
2) $(Ra)^2 \equiv a$ for any $a \in \mathcal{R}$.
3) For each pair of left ideals $I \subseteq I'$ of $\mathcal{R}$, there holds $I'I = I$.
4) For any positive integer $n$, $(\mathcal{R})_n$ is fully left idempotent.
If $\mathcal{R}$ is right $s$-unital then 1) is also equivalent to each of the following:
5) For each ideal $\alpha$ and each left ideal $I$ of $\mathcal{R}$ there holds $\alpha \cap I = \alpha I$.
6) For each ideal $\alpha$ of $\mathcal{R}$ and each pair of left $\mathcal{R}$-modules $\_R N \subseteq _R M$ there holds $\alpha M \cap N = \alpha N$.

(2) The following are equivalent:
1) $\mathcal{R}$ is fully idempotent.
2) $(Ra\mathcal{R})^2 \equiv a$ for any $a \in \mathcal{R}$.
3) Every ideal of $\mathcal{R}$ is semiprime.
4) For each pair of ideals $\alpha$, $\alpha'$ of $\mathcal{R}$ there holds $\alpha \cap \alpha' = \alpha \alpha'$.
5) For any positive integer $n$, $(\mathcal{R})_n$ is fully idempotent.

Proof. The assertion (2) is given in [7]. Concerning (1), the equivalence of 1)–3) is given in [22, Proposition 1] and 4) $\Rightarrow$ 1) is trivial. Moreover, the latter part will be obvious by Proposition 1.

1) $\Rightarrow$ 4). We shall modify slightly the proof of [12, Theorem 4]. At first, we consider the case $n = 2$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an arbitrary element of $\mathcal{R} = (\mathcal{R})_2$. If $a = \sum_i w_i a w_i; a (w_i, w_i \in \mathcal{R})$, then $A = X = \begin{pmatrix} 0 & b' \\ c & 0 \end{pmatrix}$ for $X = \sum_i \begin{pmatrix} w_i & 0 \\ 0 & 0 \end{pmatrix} A \begin{pmatrix} w_i & 0 \\ 0 & 0 \end{pmatrix}. \quad$ Next, if $d = \sum_j x_j d x_j; d (x_j, x_j \in \mathcal{R})$, then $A X = Y = \begin{pmatrix} 0 & b' \\ c' & 0 \end{pmatrix}$ for $Y = \sum_j \begin{pmatrix} 0 & 0 \\ x_j & 0 \end{pmatrix} (A - X) \begin{pmatrix} 0 & 0 \\ x_j & 0 \end{pmatrix} (A - X)$. Finally, if $b' = \sum_k y_k b' y_k b'$ and $c' = \sum_k z_k c' z_k c'$ then $A X Y = \sum_k \begin{pmatrix} y_k & 0 \\ z_k & 0 \end{pmatrix} (A - X - Y) \begin{pmatrix} 0 & z_k' \\ y_k & 0 \end{pmatrix} (A - X - Y)$. We obtain therefore $A = (A - X - Y) + Y + X \in (3(A - X - Y))^2 + (3(A - X))^2 + (3A)^2 = (3A)^2$, namely, $\mathcal{R}$ is fully left idempotent.

Since $(\mathcal{R})^* \equiv (\mathcal{R})_{2^*}$, one will easily see that $(\mathcal{R})^*$ is fully left idempotent. Given arbitrary $n$, we choose $k$ so that $2^* \geq n$. If $A \in (\mathcal{R})_n$, we choose $A' \in (\mathcal{R})^*$ with $A$ in the upper left-hand corner and zeros elsewhere. Now, $A' \in ((\mathcal{R})^* A)^2$ and a brief computation gives $A \in ((\mathcal{R}) A)^2$.

Proposition 6. Every left $p$-$V$-ring is fully left idempotent, and so every right $s$-unital, left $V$-ring is fully left idempotent.
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Proof. If not, there exists a non-zero element $a \in R$ such that $(a)^2 \neq (a) (-Ra)$. Let $m$ be a maximal member in the family of left ideals $I$ of $R$ such that $(a)^2 \subseteq I \subseteq (a)$. Since the irreducible left $R$-module $(a)/m$ is $p$-injective, there exists an element $b \in (a)$ such that $z + m = xb + m$ for all $x \in (a)$. But this implies a contradiction $(a) = m$. The latter part is evident by Proposition 4.

As a direct consequence of Proposition 6 and [11, Theorem 1.1], we obtain the following:

Corollary 6 (cf. [10, Theorem 13]). If $R$ is a left $V$-ring then the following are equivalent:
1) $R$ is a regular ring.
2) $R$ is right $s$-unital and every prime factor ring of $R$ is a regular ring.

Proposition 7 (cf. [21], [22]). Let $R$ be fully left idempotent.
1) $R$ is right non-singular; $Z(R) = 0$.
2) $R$ is semiprimitive; $f(R) = 0$.
3) If $a \in R$ is left regular then $R = RaR$.
4) $C$ is a regular ring.$^{11}$

Proof. (1) Let $z \in Z_c(R)$, and choose an element $y \in Z_c(R)$ such that $z = yz$ (Proposition 5 (1)). If $nz + zx$ ($n$ an integer and $x \in R$) is an arbitrary element of $|z| \cap r(y)$, then $0 = y(nz + zx) = nyz + yzx = nz + zx$. Hence, $|z| \cap r(y) = 0$, which means $z = 0$.

(2) Let $z \in f(R)$, and choose $y \in f(R)$ such that $z = yz$. Since $\{xy - x | x \in R\} = R$, it follows $Rz = 0$, namely, $z = 0$.

(3) This is evident by $Ra = RaR$.

(4) If $c$ is an arbitrary element of $C$ then $c \in (Rc) = Rc$ by Proposition 5 (1). Hence, $C$ is regular by [1, Lemma 1].

Theorem 5 (cf. [10, Theorem 14]). If $R$ is right $s$-unital then the following are equivalent:
1) $R$ is a left $V$-ring.

$^{11}$ If $R$ is fully idempotent then it is almost evident that $C$ is still regular and the centroid $G$ of $R$ is commutative. Moreover, as was shown by R. Courter [Proc. Amer. Math. Soc. 43 (1974), 293–295], $G$ is a regular ring. In fact, given an arbitrary element $\gamma$ of $G$, one will easily see that $R^2 = (R)^2 \cap (R) = R^2 \cap (R)$ and $R \cap Ker \gamma = (R \cap Ker \gamma)^2 = 0$. Hence, $R = R \cap Ker \gamma$ and $\gamma$ induces an automorphism of $\mathfrak{g} R \gamma$. We can then find an element $\gamma'$ of $G$ such that $\gamma = \gamma' \gamma$. 

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2) \( R \) is fully left idempotent and every left primitive factor ring of \( R \) is a left \( V \)-ring.

Proof. 1) \( \implies \) 2). This is a consequence of Proposition 6.
2) \( \implies \) 1). Let \( M \) be an irreducible left \( R \)-module, and \( I \) a left ideal of \( R \). Let \( f \) be a non-zero element of \( \text{Hom}(aI, R/M) \). Obviously \( a = \text{Ann}(aM) \) is a left primitive ideal of \( R \). Noting that \( a \cap I = aI \) (Proposition 5 (1)), one will easily see the map defined by \( l + a \mapsto lf (l \subseteq I, a \subseteq a) \) is an extension of \( f \) in \( \text{Hom}(a(I + a), R/M) \). Now, the rest of the proof proceeds in the same way as for 1) \( \implies \) 2) of Theorem 4.

Corollary 7 (cf. [10, Corollary 15]). If \( R \) is a regular ring then the following are equivalent:
1) \( R \) is a left \( V \)-ring.
2) Every left primitive factor ring of \( R \) is a left \( V \)-ring.

A left (resp. right) \( s \)-unital ring is said to be left (resp. right) semiartinian if every \( s \)-unital left (resp. right) \( R \)-module contains an irreducible \( R \)-submodule.

Theorem 6 (cf. [10, Theorem 17]). If \( R \) is left semiartinian then the following are equivalent:
1) \( R \) is a regular ring.
2) \( R \) is fully idempotent.
3) \( R \) is fully left idempotent.
3') \( R \) is fully right idempotent.
4) \( R \) is a left \( p \)-\( V \)-ring.
4') \( R \) is a right \( p \)-\( V \)-ring.

When this is the case, \( R \) is right semiartinian.

Proof. 1) \( \implies \) 4) (resp. 4') \( \implies \) 3) (resp. 3') \( \implies \) 2). These are obvious by the remark mentioned before Proposition 4 and Proposition 6.
2) \( \implies \) 1). Let \( S \neq 0 \) be the left socle of \( R \). If \( I \) is a left ideal of \( S \) then it is easy to see that \( RI = RI \cdot R I \subseteq SI \subseteq I \), namely, \( I \) is a left ideal of \( R \). (Note that \( R \) is semiprime.) Hence \( sS \) is completely reducible. Since \( S \) is also semiprime, each homogeneous component of \( sS \) is (non-trivial) simple and regular. Hence \( S \) is regular. Now, let \( m \) (\( \supseteq S \)) be the maximal regular ideal of \( R \) (cf. [4]). Suppose \( m \neq R \). Then \( R/m \) is fully idempotent and has non-zero left socle. By the above argument, we see that the maximal regular ideal of \( R/m \) is non-zero, which contradicts the maximality of \( m \). We have seen thus \( R = m \). Finally, noting that \( S \) coincides with the right socle of \( R \), one will easily see that \( R \) has a right
socle sequence, namely, $R$ is right semiartinian.

3. **$AC$-rings.** $R$ is called an **$AC$-ring** (almost commutative ring) if for any proper prime ideal $\mathfrak{p}$ of $R$ and $a \notin \mathfrak{p}$ there exists $x$ such that $ax \in C \setminus \mathfrak{p}$. Any $P_1$-ring is obviously an $AC$-ring (cf. [6]), and the next will be easily seen (cf. [24, Theorem 1]).

**Proposition 8.** Let $R$ be an $AC$-ring.

1. Every homomorphic image of $R$ is an $AC$-ring.
2. Every prime ideal of $R$ is completely prime. In particular, $R$ is a prime ring if and only if it is a domain.
3. Every semiprime ideal of $R$ is completely semiprime. In particular, $R$ is a semiprime ring if and only if it is a reduced ring.
4. For any proper prime ideal $\mathfrak{p}$ of $R$ and $a \notin \mathfrak{p}$ there exists $y$ such that $ya \in C \setminus \mathfrak{p}$. (The notion of $AC$ is right-left symmetric.)

By Proposition 8 (3), the prime radical of an $AC$-ring coincides with the set of all nilpotent elements. If $R$ is an $AC$-ring and $R \neq P(R)$ then $(R)_n$ cannot be an $AC$-ring for $n > 1$.

**Proposition 9.** The following are equivalent:

1. $R$ is a division ring.
2. $aR = R$ for any $a \neq 0$ in $R$.
3. $Ra = R$ for any $a \neq 0$ in $R$.
4. $R$ is a (non-trivial) simple $AC$-ring.
5. $R$ is a prime $AC$-ring with minimum condition on ideals.
6. $R$ is a fully idempotent, prime $AC$-ring.

**Proof.** Obviously, 1) implies each of 2)—6) and 6) implies 5). Next, assume 2). Since $R$ is strongly regular, there exists $x$ such that $axa = a$ and $ax = xa$. By $axR = aR = R$, we see that the central idempotent $ax$ is the identity of $R$, and 1) is obvious. Similarly, 1) $\Rightarrow$ 2') $\Rightarrow$ 1). Finally, assume one of 3)—5). For any $a \neq 0$ there exists $x$ such that $ax$ is a non-zero central element. Since $R$ is a domain (Proposition 8), one will easily see $R = axR = aR$.

**Corollary 9** (cf. [24, Theorem 3]). The following are equivalent:

1. $R$ is a finite direct sum of division rings.
2. $R$ is a semiprime $AC$-ring with minimum condition on ideals.
3. $R$ is a semiprimitive $AC$-ring with minimum condition on ideals.
Proof. It suffices to prove 2) \(\Rightarrow\) 1). For any proper prime ideal \(\mathfrak{p}\), \(R/\mathfrak{p}\) is a division ring (Proposition 9). Hence, \(R\) is a subdirect sum of division rings. As is well known, by the minimum condition on ideals, \(R\) is then a finite direct sum of division rings.

**Theorem 7** (cf. [24, Theorem 2]). If \(R\) is a left (resp. right) \(s\)-unital AC-ring and \(\mathfrak{n}\) is a submodule of \(R\), then the following are equivalent:

1) \(\mathfrak{n}\) is a maximal right (resp. left) ideal.
2) \(\mathfrak{n}\) is a maximal ideal.
3) \(\mathfrak{n}\) is a right (resp. left) primitive ideal.

Proof. Since \(R\) is left \(s\)-unital, \(R^2 = R\) and any maximal ideal of \(R\) is a prime ideal. Moreover, if \(\mathfrak{n}\) is a right ideal of \(R\) then \((\mathfrak{n}: R) = \{x \in R | Rx \subseteq \mathfrak{n}\}\) coincides with the largest ideal contained in \(\mathfrak{n}\).

1) \(\Rightarrow\) 2). If \(a = (\mathfrak{n}: R) (\subseteq \mathfrak{n} 
\neq R)\) is not maximal, then \(a\) is properly contained in a proper prime ideal \(\mathfrak{p}\) (Proposition 2 (1)). Evidently, there exists an element \(a \in \mathfrak{n}\backslash \mathfrak{p}\), and \(ax \in (C \cap \mathfrak{n}) \backslash \mathfrak{p}\) for some \(x\). But this is impossible by \(ax \subseteq a \subseteq \mathfrak{p}\). This proves that \(a\) is a maximal ideal. Hence, \(R/\mathfrak{a}\) is a division ring (Proposition 9), and \(\mathfrak{n} = \mathfrak{a}\).

2) \(\Rightarrow\) 3). Since \(R/\mathfrak{n}\) is a division ring (Proposition 9), \(\mathfrak{n}\) is primitive.

3) \(\Rightarrow\) 1). There exists a maximal right ideal \(\mathfrak{m}\) of \(R\) such that \(\mathfrak{m} \supseteq \mathfrak{n}\) and \((\mathfrak{m}/\mathfrak{n}: R/\mathfrak{n}) = 0\). Since \(R/\mathfrak{n}\) is a left \(s\)-unital AC-ring, as was shown in 1) \(\Rightarrow\) 2), we obtain \(\mathfrak{m}/\mathfrak{n} = (\mathfrak{m}/\mathfrak{n}: R/\mathfrak{n}) = 0\), i.e., \(\mathfrak{m} = \mathfrak{n}\).

By Theorem 7, if \(R\) is a left (resp. right) \(s\)-unital AC-ring then \(J(R)\) is the intersection of maximal ideals, and so a left (resp. right) \(s\)-unital semiprimitive AC-ring is a subdirect sum of division rings.

**Theorem 8.** The following are equivalent:

1) \(R\) is a strongly regular ring.
2) \(R\) is a regular AC-ring.
3) \(R\) is an AC-ring and a left (or right) \(p\cdot V\)-ring.
4) \(R\) is a fully idempotent AC-ring.
5) \(R\) is an AC-ring whose ideals are semiprime.
6) \(R\) is a reduced ring such that \(R/\mathfrak{p}\) is regular (in fact a division ring) for any proper prime ideal \(\mathfrak{p}\).
7) \(R\) is a reduced ring whose proper completely prime ideals are maximal left ideal.

Proof. 1) \(\iff\) 6) \(\iff\) 7) are given in [5] (and also in [11]), 1) \(\Rightarrow\) 2) \(\Rightarrow\) 3) are trivial, 3) \(\Rightarrow\) 4) by Proposition 6, and 4) \(\Rightarrow\) 6) is a consequence of Propositions 8 and 9. Finally, 4) \(\iff\) 5) is contained in Propo-
Following [24], $R$ is primary if every zero-divisor is nilpotent, and is local if it has exactly one maximal ideal.

**Theorem 9** (cf. [24, Theorem 5]). (1) If $R$ is an AC-ring then the following are equivalent:

1) $R$ is primary.
2) Every right zero-divisor is nilpotent.
3) Every left zero-divisor is nilpotent.
4) There exists a minimal prime ideal $\mathfrak{p}$ of $R$ which contains all zero-divisors.

(2) If $R$ is a left s-unital AC-ring then the following are equivalent:

1) $R$ has a unique prime ideal $\mathfrak{p} \neq R$.
2) $R$ is local and $P(R) = J(R)$.
3) $R/P(R)$ is a division ring.

**Proof.** (1) (2) $\implies$ (3). Let $xy = 0$, $y \neq 0$. If $x \notin P(R)$ then $x \notin \mathfrak{p}_0$ for some prime ideal $\mathfrak{p}_0$. Choose $u \in R$ such that $ux \in C \setminus \mathfrak{p}_0$ (cf. Proposition 8). But, by 2), $0 = uxy = yux$ yields a contradiction $ux \in P(R)$. Similarly, we have (3) $\implies$ (2). Obviously, $P(R)$ is a prime ideal.

1) $\implies$ 2) $\implies$ 4). Trivial.

4) $\implies$ 1). It suffices to show that if $x$ is non-nilpotent then $x \notin \mathfrak{p}$. To be easily seen, $T = \{x^s | k \geq 0, s \in R \setminus \mathfrak{p}\} \cup \{x^s | k > 0\}$ is an $m$-system such that $x \in T$ and $0 \notin T$. Then there exists a prime ideal $\mathfrak{p}_0$ such that $\mathfrak{p}_0 \cap T = \emptyset$. Since $\mathfrak{p}$ is a minimal prime ideal and $\mathfrak{p}_0 \subseteq R \setminus T \subseteq \mathfrak{p}$, we have $\mathfrak{p} = \mathfrak{p}_0 \neq x$.

(2) 1) $\implies$ 2). Every maximal ideal of $R$ is a prime ideal. If $\mathfrak{p}$ is not maximal then it is properly contained in a proper prime ideal (Proposition 2 (1)), a contradiction.

2) $\implies$ 3). Since $P(R) = J(R)$ is a unique maximal ideal, $R/P(R)$ is a division ring (Proposition 9).

3) $\implies$ 1). Trivial.

4. **Integral extensions of s-AC-rings.** In [24], $R$ with 1 is called an SAC-ring if for any proper ideal $\mathfrak{a}$ and $x \notin \mathfrak{a}$ there exists $y$ such that $xy \in C \setminus \mathfrak{a}$. However, in our present study, we shall employ a somewhat weaker (but right-left symmetric) definition: An AC-ring is called an s-AC-ring if for any non-prime ideal $\mathfrak{a}$ and $x \notin \mathfrak{a}$ there holds $R \cdot x \cap C \subseteq \mathfrak{a}$. To be easily seen, every s-AC-ring has the following property:

(*) For any proper ideal $\mathfrak{a}$ and $x \notin \mathfrak{a}$ there holds $R \cdot x \cap C \subseteq \mathfrak{a}$.
Any strongly regular ring is $s$-$AC$, and conversely any $P_1$-ring with the property (*) is strongly regular.

For a while, we assume that $R$ is a ring with the property (*). By a routine manner, we can show that an ideal $a$ is prime (resp. semiprime) if and only if $a \cap C$ is prime (resp. semiprime) in $C$. Accordingly, a ring is strongly regular if and only if it is an $s$-$AC$-ring whose center is regular (Theorem 8). We assume further that $R' \vert R$ is a ring extension such that $C$ is contained in the center $C'$ of $R'$. Then we can easily see that if $a'$ is a prime (resp. semiprime) ideal of $R'$ then $a' \cap R$ is a prime (resp. semiprime) ideal of $R$.

In what follows, $R' \vert R$ will mean a ring extension, and $C'$ the center of $R'$. $R' \vert R$ is called a left integral extension if $C \subseteq C'$ and for each $x \in R'$ there exist $a_0, \ldots, a_{n-1}$ in $R$ such that $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$.

Concerning "going up" we have the following:

Theorem 10 (cf. [24, Theorem 7 and Corollary 2]). Let $R$ be an $s$-$AC$-ring, $R'$ a left (or right) $s$-unital ring, and let $R' \vert R$ be a left integral extension. If $a'$ is an ideal of $R'$ and $\mathfrak{p}$ is a proper prime ideal of $R$ containing $a' \cap R$, then there exists a proper prime ideal $\mathfrak{p}'$ of $R'$ such that $\mathfrak{p}' \supseteq a'$ and $\mathfrak{p}' \cap R = \mathfrak{p}$.

Proof. Let $M$ be the non-empty $m$-system $R\setminus\mathfrak{p}$, and $\mathfrak{p}' \supseteq \mathfrak{a}'$ an ideal of $R'$ which is maximal with respect to excluding $M$. Then $\mathfrak{p}'$ is a proper prime ideal and $\mathfrak{p}' \cap R \subseteq \mathfrak{p}$. If $\mathfrak{p}' \cap R \subseteq \mathfrak{p}$ then there exists $c \in (C \cap \mathfrak{p}) \setminus (\mathfrak{p}' \cap R)$. Since $(cR' + \mathfrak{p}') \cap M \neq \emptyset$, $cx + \mathfrak{p}' = m$ with some $x \in R'$, $\mathfrak{p}' \subseteq \mathfrak{p}'$, $m \in M$. Suppose $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ ($a_i \in R$). Then $0 = x^n + a_{n-1}x^{n-1}c + \cdots + a_0c^n = (m - \mathfrak{p})^n + a_{n-1}c + \cdots + a_0c^n$. There exist therefore $r \in R$ and $q' \in \mathfrak{p}'$ such that $m^n + rc + q' = 0$. This shows $q' \in \mathfrak{p}' \cap R \subseteq \mathfrak{p}$, and hence $m^n \in \mathfrak{p}$, whence it follows a contradiction $m \in \mathfrak{p}$ (Proposition 8).

Corollary 9. Let $R$ be a left $s$-unital $s$-$AC$-ring, $R'$ a left $s$-unital ring, and let $R' \vert R$ be a left integral extension. If $\mathfrak{a}$ is a proper ideal of $R$ then $\mathfrak{a}R'$ is a proper ideal of $R'$.

Proof. By Proposition 2 (1), $\mathfrak{a}$ is contained in a proper prime ideal $\mathfrak{p}$ of $R$, and then there exists a proper prime ideal $\mathfrak{p}'$ of $R'$ such that $\mathfrak{p}' \cap R = \mathfrak{p}$ (Theorem 10). Hence $\mathfrak{a}R' \subseteq \mathfrak{p}' \neq R'$. Next, to be easily seen, $R(\mathfrak{a} \cap C) = \mathfrak{a}$. It follows therefore $R'(\mathfrak{a}R') = R'R(\mathfrak{a} \cap C)R' \subseteq \mathfrak{a}R'$.

Lemma 1. Let $R' \vert R$ be a left integral extension. If a completely prime ideal $\mathfrak{p}'$ of $R'$ is contained in a left ideal $\mathfrak{w}'$ and $\mathfrak{w}' \cap R = \mathfrak{p}' \cap R$, then

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\[ n' = p'. \]

**Proof.** Suppose there exists \( x \in n' \setminus p' \). Let \( n \) be the smallest integer such that \( x^n + a_{n-1}x^{n-1} + \cdots + a_0 = p' \subseteq p' \) \((a_i \in R)\). This implies \( a_0 \in n' \cap R = \psi' \cap R \) and \( n > 1 \). Since \( \psi' \) is completely prime, \((x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1)x = p' - a_0 \in \psi' \) yields a contradiction \( x^{n-1} + \cdots + a_1 \in \psi' \).

**Theorem 11** (cf. [24, Corollary 4]). Let \( R \) be a right and left s-unital s-AC-ring, \( R' \) a left s-unital ring, and let \( R'/R \) be a left integral extension. Let \( \psi' \) be a completely prime ideal of \( R' \). Then \( \psi' \) is a maximal left ideal if and only if \( \psi' \cap R \) is a maximal ideal of \( R \).

**Proof.** Suppose \( \psi' \cap R \) is a maximal ideal. By Proposition 2 (2), there exists a maximal left ideal \( m' \) of \( R' \) such that \( m' \supseteq \psi' \) and \( m' \cap R \neq R \). Since \( \psi' \cap R \) is a maximal left ideal by Theorem 7, we have \( m' \cap R = \psi' \cap R \), whence it follows \( m' = \psi' \) (Lemma 1). Conversely, suppose \( \psi' \) is a maximal left ideal. We claim here \( \psi' \cap R \neq R \). In fact, if \( x \in R \setminus \psi' \) and \( x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0 \) \((a_i \in R)\) then \( \psi' \supseteq R \) gives a contradiction \( x^n \in \psi' \). Now, suppose \( \psi' \cap R \) is not maximal. Then \( \psi' \cap R \) is properly contained in a proper prime ideal \( \psi_0 \) of \( R \) (Proposition 2 (1)), and we can find a proper prime ideal \( \psi_0' \supseteq \psi' \) of \( R' \) such that \( \psi_0' \cap R = \psi_0 \) (Theorem 10). Since \( \psi_0' \) has to be equal to \( \psi' \), we have a contradiction \( \psi' \cap R = \psi_0' \cap R = \psi_0 \).

**Theorem 12** (cf. [24, Theorem 9]). Let \( R \) be a left and right s-unital s-AC-ring, \( R' \) a left and right s-unital reduced ring, and let \( R'/R \) be a left integral extension. Then, \( R \) is regular if and only if so is \( R' \).

**Proof.** If \( R \) is (strongly) regular then every proper prime ideal of \( R \) is a maximal left ideal (Theorem 8). By the proof of Theorem 11, for any proper completely prime ideal \( \psi' \) of \( R' \), \( \psi' \cap R \) is a proper prime ideal of \( R \), and so it is a maximal left ideal. Hence \( \psi' \) is a maximal left ideal by Theorem 11, and again by Theorem 8 \( R' \) is a regular ring. Conversely, if \( R' \) is a regular ring, then for any proper prime ideal \( \psi \) of \( R \) there exists a proper prime ideal \( \psi' \) of \( R' \) such that \( \psi' \cap R = \psi \) (Theorem 10) and \( \psi' \) is a maximal left ideal (Theorem 8). Hence, \( \psi \) is a maximal ideal by Theorem 11, and so it is a maximal left ideal (Theorem 7). Theorem 8 proves therefore that \( R \) is regular.

**Theorem 13** (cf. [24, Theorem 10]). Let \( R'/R \) be a left integral extension. If \( R \) is strongly regular then for each \( x \in R' \) there exist \( y \in R' \) which can be expressed as a (left) polynomial in \( x \) over \( R' = R + Z \) and a natural number \( n \) such that \( yx^{n+1} = x^n \).
Proof. Let \( A(x) = \{ p(x) \mid p(x) \text{ is a monic polynomial of positive degree in } x \text{ over } R \text{ such that } p(x)x^m = 0 \text{ for some } m \} \). In \( A(x) \) we choose \( p(x) = x^n + a_1 x^{n-1} + \cdots + a_n \) of the least degree; \( p(x)x^{n-1} = 0 \) \((n > 1)\). By [1, Lemma 1], there exists (uniquely) an element \( a \in R \) such that \( a_0 = a_0a, \ a_1a = a_0 \) and \( a^n = a_0 \). Obviously, \( e = a_0a \) is a central idempotent with \( ea_0 = a_0 \) and \( ea = a \). If \( k = 1 \) then \( x^2 + a_0 x^{n-1} = 0 \) implies \( 0 = x^3 + a_0 x^{n-1} - e(x^3 + a_0 x^{n-1}) = x^3 - e x^3 \), i.e., \( ex^3 = x^3 \). Hence, \( 0 = a(x^3 + a_0 x^{n-1}) x = ax^{n+1} + ex^3 = ax^{n+1} + x^n \), whence it follows \( -ax^{n+1} = x^n \). Next, we shall consider the case \( k > 1 \), and set \( x_0 = x - ex \). Since \( 0 = p(x)x^{n-1} - ep(x)x^{n-1} = (x^n + a_1 x^{n-1} + \cdots + a_n x_0 x^{n-1} - e(x^n + a_1 x^{n-1} + \cdots + a_1 x_0) x^{n-1} = (x^n + a_1 x^{n-2} + \cdots + a_1 x_0 x^{n-2} + \cdots + a_1 x_0^2) x^{n-1}, \ A(x_0) \) contains a polynomial of degree \( k - 1 \). By induction method, there exists a polynomial \( f(x_0) \) over \( R \) such that \( f(x_0) x^{n+1} = x_0^m \) for some \( m \). Since \( x^{n+1} = (-a)p(x)x^{n-1} + x^{n-1} = (-a)(x^n + a_1 x^{n-2} + \cdots + a_1 x_0) x^n + x_0^{n-1} \), we obtain \( x^{n+m} = (-a)(x^{n-1} + a_1 x^{n-2} + \cdots + a_1 x_0) x^{n+1} + f(x_0) - ef(x_0) \) \( x^{n+m} = x^n \). Finally, we shall prove the following, which will enable us to readily obtain [24, Theorem 11].

Corollary 10. Let \( R'/R \) be a left and right integral extension. If \( R \) is strongly regular then for any \( x \in R \) there exists a quasi-regular element \( u \) in \( R[x] \) such that \( x^m - ux^{n-1} = x^n \) and \( ux^n = ux \) for some \( n \).

Proof. By Theorem 13, there exist \( s \) and \( t \) in \( R[x] \) such that \( sx^{n+1} = x^n t \) for some \( n \). To be easily seen, \( s^nx^n = x^n t \). If we set \( a = x \) and \( b = s^nx^n \), then it is known that \( ab = ba, a^2b = a \) and \( ab^2 = b \) (cf. the proof of [1, Lemma 1]). Obviously, \( e = ab \) is an idempotent and \( u = e - b \) is a quasi-regular element in \( R[x] \) with quasi-inverse \( e - a \). Now, it is easy to see that \( a^2 - ua^2 = a \).

References

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Added in proof. Recently, Theorem 3 has been proved also by F. Hansen [Proc. Amer. Math. Soc. 55 (1976), 281—286].