A Characterization of Anti-Integral Extensions

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In this paper, we mean by a ring a commutative ring with identity and by an integral domain (or a domain) a ring which has no non-trivial zero-divisors. Our unexplained technical terms are standard and are seen in [1].

Let $R$ be a Noetherian domain and $R[X]$ a polynomial ring. Let $\alpha$ be a non-zero element of an algebraic field extension $L$ of the quotient field $K$ of $R$ and let $\pi: R[X] \to R[\alpha]$ be the $R$-algebra homomorphism sending $X$ to $\alpha$. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of $\alpha$ over $K$ with $\deg \varphi_\alpha(X) = d$ and write

$$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d.$$ 

Then $\eta_i$ ($1 \leq i \leq d$) are uniquely determined by $\alpha$. Let $I_{n_i} := R : R \eta_i$ and $I_{[\alpha]} := \bigcap_{i=1}^d I_{n_i}$, the latter of which is called a denominator ideal of $\alpha$. We say that $\alpha$ is an anti-integral element if and only if $\text{Ker} \pi = I_{[\alpha]} \varphi_\alpha(X) R[X]$. The concept of anti-integrality is given in [2] in the birational case, and the higher degree case appears in [3]. For $f(X) \in R[X]$, let $C(f(X))$ denote the ideal of $R$ generated by the coefficients of $f(X)$. For an ideal $J$ of $R[X]$, let $C(J)$ denote the ideal generated by the coefficients of the elements in $J$. If $\alpha$ is an anti-integral element, then $C(\text{Ker} \pi) = C(I_{[\alpha]} \varphi_\alpha(X) R[X]) = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. Put $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. If $J_{[\alpha]} \not\subseteq p$ for all $p \in DP_1(R) := \{ p \in \text{Spec}(R) \mid \text{depth} R_p = 1 \}$, then $\alpha$ is called a super-primitive element. It is known that a super-primitive element is an anti-integral element (cf. [3,(1.12)]). By definition, the super-primitive is characterized by the set of $DP_1(R)$. In this paper, we shall show that the anti-integrality is also characterized by the set $DP_1(R)$. In fact, we prove the following:

Let $R$ be a Noetherian domain with quotient field $K$ and let $\alpha$ be an element of an algebraic field extension $L$ of $K$. Then the following statements are equivalent:

1. $\alpha$ is an anti-integral element over $R$,
2. $\alpha$ is an anti-integral element over $R_p$ for all $p \in DP_1(R)$.
In what follows, we use the notation as above.
We start with the following theorem, which characterizes anti-integrality.

**Theorem 1.** The following statements are equivalent:
1. $\alpha$ is an anti-integral element of degree $d$ over $R$,
2. the ideal $I_{[\alpha]}\eta_d$ of $R$ is generated by the set $\{g(0) \mid g(X) \in \text{Ker} \pi \}$.

**Proof.** (1) $\Rightarrow$ (2): Let $J$ be the ideal of $R$ generated by the set $\{g(0) \mid g(X) \in \text{Ker} \pi \}$. Since $I_{[\alpha]}\varphi_{\alpha}(X) \subseteq \text{Ker} \pi$ and the constant term of $I_{[\alpha]}\varphi_{\alpha}(X)$ is $I_{[\alpha]}\eta_d$, it follows that $I_{[\alpha]}\eta_d \subseteq J$. Conversely take $a \in J$, and let
\[ a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a = 0 \]
be a relation, where $a_i \in R$ and $a_n \neq 0$. Put $f(X) = a_nX^n + \cdots + a_1X + a$. Then $f(X) \in \text{Ker} \pi$. Since $\alpha$ is an anti-integral element of degree $d$ over $R$, we have $\text{Ker} \pi = I_{[\alpha]}\varphi_{\alpha}(X)R[X]$. Hence $f(X) = \sum h_i(X)g_i(X)$ for some $h_i(X) \in I_{[\alpha]}\varphi_{\alpha}(X)$ and $g_i(X) \in R[X]$. Thus $a = f(0) = \sum h_i(0)g_i(0) \in I_{[\alpha]}\eta_d$, as desired.

(2) $\Rightarrow$ (1): Let $0 \neq f(X) \in \text{Ker} \pi$ and write $f(X) = a_nX^n + \cdots + a_1X + a$. Since $[K(\alpha) : K] = d$, we have $n \geq d$. By the assumption that $a \in J = I_{[\alpha]}\eta_d$, it follows that $a = b\eta_d$ for some $b \in I_{[\alpha]}$. Put $g(X) = bX^d + (b\eta_1)X^{d-1} + \cdots + (b\eta_d)$. Note that $g(X) \in \text{Ker} \pi$. As $f(0) = g(0)$, we get $f(X) - g(X) = X(h(X)) \in \text{Ker} \pi$ for some $h(X) \in R[X]$. Since $R[\alpha]$ is an integral domain, $\text{Ker} \pi$ is a prime ideal of $R[X]$, and hence $h(X) \in \text{Ker} \pi$. Since $\deg h(X) \leq n - 1$, we can prove $f(X) \in I_{[\alpha]}\varphi_{\alpha}(X)R[X]$ by induction. Therefore $\alpha$ is an anti-integral element of degree $d$ over $R$.

Under this preparation, we obtain the following result mentioned before.

**Theorem 2.** The following statements are equivalent to each other.
1. $\alpha$ is an anti-integral element of degree $d$ over $R$,
2. $\alpha$ is an anti-integral element of degree $d$ over $R_p$ for all $p \in D\pi_1(R)$.

**Proof.** (1) $\Rightarrow$ (2): By assumption, we have the following exact sequence:
\[ 0 \rightarrow I_{[\alpha]}\varphi_{\alpha}(X)R[X] \rightarrow R[X] \rightarrow R[\alpha] \rightarrow 0. \]
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Take $p \in Dp_1(R)$. Tensoring $\otimes_R R_p$, we have an exact sequence:

$$0 \rightarrow I_{[\alpha]} \varphi_\alpha(X) R_p[X] \rightarrow R_p[X] \rightarrow R_p[\alpha] \rightarrow 0.$$  

This exact sequence implies that $\alpha$ is an anti-integral element of degree $d$ over $R_p$.

(2) $\implies$ (1): Consider the following canonical exact sequence:

$$0 \rightarrow \text{Ker } \pi \rightarrow R[X] \rightarrow R[\alpha] \rightarrow 0.$$  

Let $J$ denote the ideal generated by the set $\{g(0) | g(X) \in \text{Ker } \pi\}$. We need to show that $I_{[\alpha]} \eta_d = J$ by Theorem 1. Since $I_{[\alpha]} \varphi_\alpha(X) \subseteq \text{Ker } \pi$, we have $I_{[\alpha]} \eta_d \subseteq J$. We shall show the converse inclusion. Since $\alpha$ is an anti-integral element of degree $d$ over $R_p$ by assumption, we conclude that $(I_{[\alpha]} \eta_d)_p = J_p$ for all $p \in Dp_1(R)$ by Theorem 1. Thus we have $J \subseteq J_p = (I_{[\alpha]} \eta_d)_p$. Let $q \in \text{Spec}(R)$ be a prime divisor of $I_{[\alpha]} \eta_d$. Since $I_{[\alpha]} \eta_d$ is a divisorial ideal, we see that $q \in Dp_1(R)$ ([4, Proposition 1.10]). Hence $J \subseteq \bigcap_{p \in Dp_1(R)} (I_{[\alpha]} \eta_d)_p = I_{[\alpha]} \eta_d$ ([4, Proposition 5.6]). This completes the proof.

REFERENCES

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