A Non-immersion Result for Lens Spaces $L_n(2m)$

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1. Introduction. The lens space $L^n(2^m)$ is the quotient of the sphere $S^{2n+1}$ by the free action of the cyclic group $\mathbb{Z}/2^m$ given by:

$$\zeta^k z = (\zeta^k z_0, \zeta^k z_1, \ldots, \zeta^k z_n),$$

where $\zeta = \exp(i\pi/2^{m-1})$ is the generator of $\mathbb{Z}/2^m$, and $z = (z_0, z_1, \ldots, z_n) \in C^{n+1}$ is such that $\sum_{i=0}^{n} |z_i|^2 = 1$. A classical question is to determine the smallest integer $k$ such that $L^n(2^m)$ immerses into $R^{2n+1+k}$. In [3], we have seen that for $m$ sufficiently large, $k$ is greater or equal than $2n-2\alpha(n)$, where $\alpha(n)$ denotes the number of 1 in the dyadic expansion of $n$. More precisely, we have proved the following theorem

**Theorem 1.1.** For $m \geq [\log_2 n] + [n/2]$, $L^n(2^m)$ does not immerse into $R^{4n-2\alpha(n)}$.

Here $[x]$ denotes the integer part of $x$. Some other results have been published in the same direction, (see [1], [5], [6] and [7]). In this note, we are completing theorem 1.1 for the case $m \leq [\log_2 n] + [n/2] - 1$. Let $l(n)$ be the integer

$$l(n) = \max \{1 \leq i \leq n - 1 \text{ such that } \binom{n + i + 1}{n} \neq 0 \pmod{4}\}.$$

We prove:

**Theorem 1.2.** Let $m \geq 2$.

a) If $n \neq 2^s + 1$ and $n \geq 2$, $L^n(2^m)$ does not immerse in $R^{2n+1+2l(n)}$.

b) If $n = 2^s + 1$, with $s \geq 1$, $L^n(2^m)$ does not immerse into $R^{2n+2l(n)} = R^{4n-4}$.

We apply theorem 1.2 to some particular values of $n$, and we obtain

**Corollary 1.1.** Let $m \geq 2$.

a) If $n = 2^s$ with $s \geq 1$, $L^n(2^m)$ does not immerse in $R^{4n-1}$.

b) If $n = 2^s + 2^t$, with $s > t \geq 1$, $L^n(2^m)$ does not immerse in $R^{4n-3}$. 

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This improves for these two cases the results obtained by theorem 1.1. Recalling that for \( n = 2^q \), the space \( L^n(2^m) \) immerses in \( \mathbb{R}^{dn} \), we note that our result is the best possible for this case.

2. Preliminaries. In this section we establish some cohomology properties of the spaces \( B(n, k) \) defined in [4] (see also [2]). This properties will be used to prove theorem 1.2. We begin with a result about spherical fibrations and recall that for any sphere bundle \( S^k \rightarrow E \rightarrow B \) there is long exact sequence of \( H^*(B; \mathbb{Z}) \)-modules called the Gysin sequence (see [8] p.143 or [9] p.356)

\[
\cdots \rightarrow H^q(B; \mathbb{Z}) \xrightarrow{p^*} H^q(E; \mathbb{Z}) \xrightarrow{\phi} H^{q-k}(B; \mathbb{Z}) \xrightarrow{\cup e} H^{q+1}(B; \mathbb{Z}) \rightarrow \cdots
\]

where \( e \) is the Euler-class of the fibration. In particular we have:

**Lemma 2.1.** If in the above spherical fibration, \( B \) is connected and the Euler-class \( e \) is zero, then

\[ H^*(E; \mathbb{Z}) \cong H^*(B; \mathbb{Z}) \oplus a \cup H^*(B; \mathbb{Z}) \]

as an \( H^*(B; \mathbb{Z}) \)-module, where \( a \) is an element of \( H^k(E; \mathbb{Z}) \) such that \( \phi(a) \) is a generator of \( H^0(B; \mathbb{Z}) \cong \mathbb{Z} \).

The proof of this lemma is straightforward.

We now turn to the space \( B(n, k) \) which by definition is the pull-back space of the diagram

\[
\begin{array}{ccc}
BSO(k) & \xrightarrow{} & BSO(2n) \\
\downarrow & & \\
BU(n) & \longrightarrow & BSO(2n)
\end{array}
\]

Inductively we can identify the space \( B(n, k) \) with the pull-back of the diagram

\[
\begin{array}{ccc}
BSO(k) & \xrightarrow{} & BSO(k + 1) \\
\downarrow & & \\
B(n, k + 1) & \longrightarrow & BSO(k + 1)
\end{array}
\]

Let be \( V_{2n,2n-2j} \) the Stiefel manifold \( SO(2n)/SO(2j) \), and let

\[
(2.1) \quad V_{2n,2n-2j} \xrightarrow{i_j} B(n,2j) \xrightarrow{p} BU(n)
\]
be the fibration induced from
\[ V_{2n,2n-2j} \xrightarrow{i_2} BSO(2j) \rightarrow BSO(2n) \]
by the canonical map \( BU(n) \xrightarrow{r_n} BSO(2n) \).

Let \( u_j \) be the generator of \( H^{2j}(V_{2n,2n-2j}; \mathbb{Z}) \cong \mathbb{Z} \) such that
\[ i_2^*(e_j) = -2u_j, \]
where \( e_j \in H^{2j}(BSO(2j); \mathbb{Z}) \) is the universal Euler-Poincaré class. By the pull-back property, there is a map \( BU(j) \xrightarrow{h} B(n,2j) \) and a commutative diagram

\[
\begin{array}{ccc}
BU(j) & \xrightarrow{h} & B(n,2j) \\
\downarrow{g_j} & \downarrow{f_{2j}} & \downarrow{r_n} \\
B(n,2j) & \xrightarrow{f_{2j}} & BSO(2j) \\
\downarrow{p} & & \downarrow{r_n} \\
BU(n) & \xrightarrow{r_n} & BSO(2n)
\end{array}
\]

where all the others maps are canonical maps.

**Lemma 2.2.** For every \( n \geq 1 \) and \( 1 \leq j \leq n - 1 \), there is an element \( a_j \) in the abelian group \( H^{2j}(B(n,2j); \mathbb{Z}) \) such that
\[ f_{2j}^*(e_j) = p^*(e_j) - 2a_j, \quad i_1^*(a_j) = u_j, \quad h^*(a_j) = 0. \]

**Proof.** There is an exact sequence coming from the Serre spectral sequence of the fibration (2.1)
\[
0 \rightarrow H^{2j}(BU(n); \mathbb{Z}) \xrightarrow{p^*} H^{2j}(B(n,2j); \mathbb{Z}) \xrightarrow{i_1^*} H^{2j}(V_{2n,2n-2j}; \mathbb{Z}) \rightarrow 0
\]
since \( V_{2n,2n-2j} \) is \((2j - 1)\)-connected and \( BU(n) \) is \(1\)-connected without cohomology in odd degree. Let \( x \in H^{2j}(B(n,2j); \mathbb{Z}) \) be such that \( i_1^*(x) = u_j \). Since the map \( g_j^* \) is an isomorphism in degree \( \leq 2j \) we can replace \( x \) by \( a_j = x - p^*((g_j^*)^{-1}(h^*(x))) \) so that \( h^*(a_j) = 0 \). The above exact sequence splits and we have an isomorphism
\[ H^{2j}(B(n,2j); \mathbb{Z}) \cong \text{im}(p^*) \oplus Za_j \cong H^{2j}(BU(n); \mathbb{Z}) \oplus Za_j. \]

On the other hand \( h^*(f_{2j}^*(e_j)) = r_j^*(e_j) = c_j \) so \( f_{2j}^*(e_j) = p^*(e_j) + ma_j \). As \( i_1^*((f_{2j}^*(e_j))) = i_2^*(e_j) = -2u_j \) we see that \( m = -2 \) and \( f_{2j}^*(e_j) = p^*(e_j) - 2a_j \).
Let now
\[ S^r - 1 \longrightarrow B(n, r - 1) \xrightarrow{pr^{-1}} B(n, r) \]
be the spherical fibration induced from
\[ S^r - 1 \longrightarrow BSO(r - 1) \longrightarrow BSO(r) \]
by the map \( B(n, r) \xrightarrow{fr} BSO(r) \). We consider the Gysin sequence of (2.2) which becomes, using lemma 2.2,
\[ \cdots \longrightarrow H^0(B(n, 2j); Z) \xrightarrow{\cup (p^*(c_j) - 2a_j)} H^{2j}(B(n, 2j); Z) \xrightarrow{p_{2j-1}^*} H^{2j}(B(n, 2j - 1); Z) \longrightarrow \cdots. \]

By exactness, \( p_{2j-1}^*(p^*(c_j)) = 2p_{2j-1}^*(a_j) \).

In the following, we note \( p_{2j-1}^*(a_j) = b_j \), more generally, \( (p_k^* \circ p_{k+1}^* \circ \cdots \circ p_{2j-1}^*)(a_j) = b_j \) and for simplicity \( (p_k^* \circ p_{k+1}^* \circ \cdots \circ p_{2j-1}^*)(p^*(c_j)) = c_j \). So we have for every space \( B(n, k) \) a family of elements \( b_i, [k/2] + 1 \leq i \leq n - 1 \), such that \( 2b_i = c_i \). We can now give the additive structure of \( H^*(B(n, k); Z) \). This result already appears in [2] and [4].

**Theorem 2.1.** \( H^*(B(n, k); Z) \) is a free \( Z \)-module determined by the isomorphism
\[ H^*(B(n, k); Z) \cong \begin{cases} Z[c_1, \ldots, c_t] \otimes \Delta(a_t, b_{t+1}, \ldots, b_{n-1}) & \text{if } k = 2t \\ Z[c_1, \ldots, c_t] \otimes \Delta(b_{t+1}, \ldots, b_{n-1}) & \text{if } k = 2t + 1 \end{cases} \]
where \( \Delta(x_1, \ldots, x_m) \) is the free abelian group generated by the elements
\[ x_{i_1}x_{i_2} \cdots x_{i_s}, \quad 1 \leq i_1 < i_2 < \cdots < i_s \leq m. \]

**Proof of theorem 2.1.** We proceed by induction descending over \( k \), beginning with \( k = 2n - 1 \). In this case the result is valid since \( B(n, 2n - 1) = BU(n - 1) \). Next we examine the case \( k = 2n - 2 \). Here we consider the spherical fibration (2.2) with \( r = 2n - 1 \). As \( H^{2n-1}(BU(n - 1); Z) = 0 \), the Euler class of this fibration is 0 and the Gysin sequence splits into short exact sequences
\[ 0 \longrightarrow H^{2q}(BU(n - 1); Z) \xrightarrow{p_{2n-2}^*} H^{2q}(B(n, 2n - 2); Z) \xrightarrow{\phi} H^{2q-2n+2}(BU(n - 1); Z) \longrightarrow 0. \]
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In lemma 2.2 we have seen that

$$H^{2n-2}(B(n,2n-2); \mathbb{Z}) \cong \text{im}(p^*) \oplus \mathbb{Z}a_{n-1}.$$ 

But $\text{im}(p^*) \subset \text{im}(p^*_{2n-2}) = \ker(\phi)$, so the element $\phi(a_{n-1})$ is a generator of $H^0(BU(n-1); \mathbb{Z}) = \mathbb{Z}$. Under the map $p^*_{2n-2}$ we have a $H^*(BU(n-1); \mathbb{Z})$-module structure over $H^*(B(n,2n-2); \mathbb{Z})$. With the help of lemma 2.1, we can see that this structure is given by the isomorphism

$$H^*(B(n,2n-2); \mathbb{Z}) \cong H^*(BU(n-1); \mathbb{Z}) \oplus a_{n-1} \cup H^*(BU(n-1); \mathbb{Z})$$

$$\cong \mathbb{Z}[c_1, \ldots, c_{n-1}] \otimes \Delta(a_{n-1})$$

this achieves the proof in this case. Moreover the multiplicative structure is well-known in this case, since

$$(c_{n-1} - 2a_{n-1})^2 = f^*_{2n-2}(e_{n-1}^2) = f^*_{2n-2}(P_{n-1})$$

$$= p^*(r_n^*(P_{n-1})) = p^*(c_{n-1}^2 - 2c_{n-2}c_n)$$

$$= c_{n-1}^2$$

where $P_{n-1}$ is the $(n-1)^{th}$-Pontrjagin class in $H^{4n-4}(BSO(2n-2); \mathbb{Z})$ or in $H^{4n-4}(BSO(2n); \mathbb{Z})$. The relations used here are proved for example in [8] (see also proof of lemma (2.3)). So we have $a_{n-1}^2 = c_{n-1}a_{n-1} - 1.$

Now we suppose that the result is valid for $r \leq k \leq 2n - 1$. We consider the Gysin sequence of the sphere bundle (2.2):

$$\cdots \rightarrow H^q(B(n,r); \mathbb{Z}) \stackrel{p^*_r}{\rightarrow} H^q(B(n,r-1); \mathbb{Z}) \stackrel{\phi}{\rightarrow} H^{q-r+1}(B(n,r); \mathbb{Z}) \stackrel{\text{cup}}{\rightarrow} H^{q+1}(B(n,r); \mathbb{Z}) \rightarrow \cdots.$$

1) If $r$ is odd, say $r = 2j + 1$, we prove exactly as above that

$$H^*(B(n,2j); \mathbb{Z}) \cong H^*(B(n,2j+1); \mathbb{Z}) \oplus a_j \cup H^*(B(n,2j+1); \mathbb{Z})$$

$$\cong H^*(B(n,2j+1); \mathbb{Z}) \otimes \Delta(a_j)$$

$$\cong \mathbb{Z}[c_1, \ldots, c_j] \otimes \Delta(a_j, b_{j+1}, \ldots, b_{n-1}).$$

Moreover the group homomorphism

$$\psi_n : \mathbb{Z}[c_1, \ldots, c_{j-1}, c_j - 2a_j] \otimes \Delta(a_j, b_{j+1}, \ldots, b_{n-1})$$

$$\rightarrow H^*(B(n,2j); \mathbb{Z})$$

defined by $\psi_n(x \otimes y) = x \cup y$, is an isomorphism.
We proceed by induction on $n$, beginning with $n = j + 1$. In this case, the morphism $\psi_n$ becomes

$$\psi_{j+1} : Z[c_1, \ldots, c_{j-1}, c_j - 2a_j] \otimes \Delta(a_j) \longrightarrow H^*(B(j + 1, 2j); Z).$$

We have seen above that $a_j^2 = a_jc_j$ in $H^*(B(j + 1, 2j); Z)$, so

$$c_j = \psi_{j+1}((c_j - 2a_j) \otimes 1 + 2(1 \otimes a_j))$$
$$c_j^2 = \psi_{j+1}((c_j - 2a_j)^2 \otimes 1)$$

and $\psi_{j+1}$ is surjective. As we have a bijection between the $Z$-module basis, $\psi_{j+1}$ is an isomorphism.

Suppose now that the result is true for $n - 1$, and let $h : B(n - 1, 2j) \rightarrow B(n, 2j)$ the map induced by the pull-back property of $B(n, 2j)$. Let

$$A = Z[c_1, \ldots, c_{j-1}, c_j] \otimes \Delta(a_j, b_{j+1}, \ldots, b_{n-2})$$

and

$$B = Z[c_1, \ldots, c_{j-1}, c_j - 2a_j] \otimes \Delta(a_j, b_{j+1}, \ldots, b_{n-2}).$$

By definition of $h^*$ we have a short exact sequence

$$\ker(h^*) \longrightarrow H^*(B(n, 2j); Z) \xrightarrow{h^*} H^*(B(n - 1, 2j); Z)$$

where $\ker(h^*) = A \cup b_{n-1}$. We also have a $Z$-modules isomorphism

$$H^*(B(n, 2j); Z) \cong A \oplus A \cup b_{n-1}.$$

If $(x_q)_{q \geq 1}$ is the canonical $Z$-module basis of $A$, $(x_q \cup b_{n-1})_{q \geq 1}$ is a basis of $A \cup b_{n-1}$. Since $h^*$ is a ring homomorphism, we have the commutative diagramm

$$\begin{array}{ccc}
B & \xrightarrow{\psi_n|B} & H^*(B(n, 2j); Z) \\
\downarrow & & \downarrow h^* \\
H^*(B(n - 1, 2j); Z) & \xrightarrow{\psi_{n-1}|B} & H^*(B(n - 1, 2j); Z)
\end{array}$$

By the induction hypothesis, $\psi_{n-1}$ is an isomorphism and so $\psi_n|B$ is a monomorphism and there is a basis $(y_q)_{q \geq 1}$ of $B$, such that

$$\psi_n(y_q) = x_q + z_q \cup b_{n-1} \quad \text{for} \quad q \geq 1, \quad \text{with} \quad z_q \in A.$$
As \( b^2_{n-1} = 0 \) in \( H^*(B(n,2j); \mathbb{Z}) \), \( \psi_n|B \cup b_{n-1} \) is injective and \( \psi_n(B \cup b_{n-1}) = A \cup b_{n-1} \).

2) If \( r \) is even, say \( r = 2j \), we know by lemma 2.2 that the Euler-class of the spherical fibration (2.2) is the element \( c_j - 2a_j \) and since \( \psi_n \) is injective, we can say that the multiplication by the Euler-class is injective, so the map \( \phi = 0 \) in the Gysin sequence of (2.2) and we have the group isomorphisms

\[
H^*(B(n,2j-1); \mathbb{Z}) \cong H^*(B(n,2j); \mathbb{Z})/(c_j - 2a_j) \\
\cong \mathbb{Z}[c_1, \ldots, c_{j-1}] \otimes \Lambda(a_j, b_{j+1}, \ldots, b_{n-1}).
\]

We can now describe the multiplicative structure of \( H^*(B(n,2j); \mathbb{Z}) \) as follows.

**Lemma 2.3.** For every \( n \geq 1 \) and \( 1 \leq j \leq n-1 \), the element \( a_j \) in the abelian group \( H^*(B(n,2j); \mathbb{Z}) \) satisfies the relation

\[
a_j^2 = a_j c_j + (-1)^j \sum_{r=j+1}^{\min(2j,n-1)} (-1)^r b_r c_{2j-r}.
\]

**Proof.** Recall that the universal Euler-Poincaré class \( e_j \in H^{2j}(BSO(2j); \mathbb{Z}) \), satisfies the relation

\[
e_j^2 = P_j
\]

where \( P_j \) is the \( j \)th universal Pontrjagin class in \( H^{4j}(BSO(2j); \mathbb{Z}) \), and that

\[
r_n^*(P_j) = e_j^2 + (-1)^j \sum_{r=j+1}^{\min(2j,n)} (-1)^r 2c_r c_{2j-r}
\]

in \( H^{4j}(BU(n); \mathbb{Z}) \), here \( P_j \) is the \( j \)th universal Pontrjagin class in \( H^{4j}(BSO(2n); \mathbb{Z}) \), (see [8]). From the definition of \( a_j \) and the above relations, we see that

\[
(f_j^*(e_j^2)) = c_j^2 - 4a_j c_j + 4a_j^2
\]

and

\[
p^*(r_n^*(P_j)) = c_j^2 + (-1)^j \sum_{r=j+1}^{\min(2j,n-1)} (-1)^r 2c_r c_{2j-r}
\]

\[
= c_j^2 + (-1)^j \sum_{r=j+1}^{\min(2j,n-1)} (-1)^r 4b_r c_{2j-r}.
\]
Since $H^*(B(n, 2j); \mathbb{Z})$ has no torsion, the relation (2.3) is valid.

Using the relation (2.3) we can now give the Steenrod squares of the mod 2 reduction of the elements $a_j$ in $H^{2j}(B(n, 2j); \mathbb{Z}/2)$.

**Theorem 2.2.** For every $n \geq 1$, every $1 \leq j \leq n - 1$ and every $0 \leq k \leq j$ the following relation is valid in $H^*(B(n, 2j); \mathbb{Z}/2)$.

\[
Sq^{2k}(a_j) = \sum_{r = \max(0, k + j + 1 - n)}^{k-1} {j - r \choose k - r} b_{k+j-r}c_r + a_jc_k.
\]

**Proof.** We proceed by an induction argument over $n$. We begin with the case $n = 1$ where all relations are empty. For $n = 2$, $j = 1$ and $k = 0$ or 1, so the only non trivial relation in $H^*(B(2, 2); \mathbb{Z}/2)$ is $Sq^2(a_1) = a_1^2 = a_1c_1$ which is compatible with (2.4).

Now we suppose the result is valid for $n \geq 2$. First we observe that (2.4) is still true for $k + j \leq n - 1$ in $H^*(B(n + 1, 2j); \mathbb{Z}/2)$ since

\[
H^q(B(n + 1, 2j); \mathbb{Z}/2) \cong H^q(B(n, 2j); \mathbb{Z}/2) \quad q < 2n.
\]

If $k + j \geq n$, we consider the following diagram, where all the arrows are canonical.

\[
\begin{array}{ccc}
B(n, 2j - 2) \times CP^\infty & \rightarrow & BSO(2j - 2) \times CP^\infty \\
\downarrow B(n + 1, 2j) & & \downarrow BSO(2j) \\
BU(n) \times CP^\infty & \rightarrow & BU(n + 1) \rightarrow BSO(2n + 2)
\end{array}
\]

In particular the next square is homotopy commutative

\[
(2.5) \quad B(n, 2j - 2) \times CP^\infty \rightarrow BSO(2j)
\]

\[
\downarrow BU(n + 1) \rightarrow BSO(2n + 2)
\]

and replacing if necessary $B(n, 2j - 2) \times CP^\infty \rightarrow BSO(2j)$ by a map homotopy equivalent, we can suppose that the diagramm (2.5) is commutative since the map $BSO(2j) \rightarrow BSO(2n + 2)$ is a fibration.
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So there is a map $f: B(n, 2j - 2) \times CP^\infty \to B(n + 1, 2j)$ such that the squares

$$
\begin{array}{ccc}
B(n, 2j - 2) \times CP^\infty & \longrightarrow & BSO(2j - 2) \times CP^\infty \\
\downarrow f & & \downarrow \\
B(n + 1, 2j) & \longrightarrow & BSO(2j)
\end{array}
$$

and

$$
\begin{array}{ccc}
B(n, 2j - 2) \times CP^\infty & \xrightarrow{f} & B(n + 1, 2j) \\
\downarrow & & \downarrow \\
BU(n) \times CP^\infty & \longrightarrow & BU(n + 1)
\end{array}
$$

are still commutative. We can easily see that

$$
\begin{align*}
f^*(c_i) &= c_i + c_{i-1}z \quad (1 \leq i \leq j), \\
f^*(a_j) &= b_j + a_{j-1}z, \\
f^*(b_i) &= b_i + b_{i-1}z \quad (j + 1 \leq i \leq n - 1), \\
f^*(b_n) &= b_{n-1}z
\end{align*}
$$

in $H^*(B(n, 2j - 2) \times CP^\infty; \mathbb{Z}) \cong H^*(B(n, 2j - 2); \mathbb{Z}) \otimes H^*(CP^\infty; \mathbb{Z})$, where $z$ is the canonical generator of $H^2(CP^\infty; \mathbb{Z})$.

Let be $G = \mathbb{Z}/2[c_1, \ldots, c_{j-1}] \otimes (\mathbb{Z}/2\langle a_j \rangle \oplus \mathbb{Z}/2\langle b_{j+1} \rangle \oplus \cdots \oplus \mathbb{Z}/2\langle b_n \rangle)$, where $\mathbb{Z}/2(x)$ is the group of order two with generator $x$. It is clear that $G$ is a subgroup of $H^*(B(n + 1, 2j); \mathbb{Z}/2)$ and we can easily see that the restriction of $f^*$ to $G$ is injective. Let $h: B(n, 2j) \to B(n + 1, 2j)$ be the canonical map as in the proof of theorem 2.1. For $j > 1$, $k < j$ and $k + j \geq n$,

$$
h^*(Sq^{2k}(a_j)) = Sq^{2k}(h^*(a_j)) = Sq^{2k}(a_j)
$$

by the induction hypothesis, and since $\ker(h^*) = b_n \cup H^*(B(n, 2j); \mathbb{Z}/2)$, we have

$$
Sq^{2k}(a_j) = \sum_{r=k+j+1-n}^{k-1} \binom{j-r}{k-r} b_{k+j-r}c_r + a_jc_k
$$

where $p(c_1, \ldots, c_{j+k-n}) \in \mathbb{Z}/2[c_1, \ldots, c_{j-1}]$. Then, the element $Sq^{2k}(a_j)$ is in $G$ and we can give its image under $f^*$

$$
f^*(Sq^{2k}(a_j)) = Sq^{2k}(f^*(a_j)) = Sq^{2k}(b_j + a_{j-1}z)
$$

$$
= Sq^{2k}(b_j) + Sq^{2k}(a_{j-1})z + Sq^{2k-2}(a_{j-1})z^2.
$$
Applying once more the induction hypothesis,
\[
f^*(S^{q^k}(a_j)) = \sum_{r=k+j+1-n}^{k} \binom{j-r}{k-r} b_{k+j-r} c_r \]
\[
+ \sum_{r=k+j-n}^{k-1} \binom{j-1-r}{k-r} b_{k+j-1-r} c_r z \]
\[
+ \sum_{r=\max(0,k+j-1-n)}^{k-2} \binom{j-1-r}{k-1-r} b_{k+j-2-r} c_r z^2 \]
\[
+ a_{j-1} c_k z + a_{j-1} c_{k-1} z^2 \]

and since \( \binom{j-1-r}{k-r} \equiv \binom{j-1-r}{k-1-r} + \binom{j-r}{k-r} \) (mod 2),
\[
f^*(S^{q^k}(a_j)) = \sum_{r=k+j+1-n}^{k} \binom{j-r}{k-r} b_{k+j-r} (c_r + c_{r-1} z) \]
\[
+ \sum_{r=k+j-n}^{k-1} \binom{j-r}{k-r} b_{k+j-1-r} c_r z \]
\[
+ \sum_{r=\max(1,k+j-n)}^{k-1} \binom{j-r}{k-r} b_{k+j-1-r} c_{r-1} z^2 \]
\[
+ a_{j-1} z(c_k + c_{k-1} z). \]

If \( k + j > n \) we have
\[
f^*(S^{q^k}(a_j)) = \sum_{r=k+j+1-n}^{k} \binom{j-r}{k-r} b_{k+j-r} (c_r + c_{r-1} z) \]
\[
+ \sum_{r=k+j-n}^{k-1} \binom{j-r}{k-r} b_{k+j-1-r} z(c_r + c_{r-1} z) \]
\[
+ a_{j-1} z(c_k + c_{k-1} z) \]
\[
= \sum_{r=k+j+1-n}^{k} \binom{j-r}{k-r} (b_{k+j-r} + b_{k+j-1-r} z)(c_r + c_{r-1} z) \]
\[
+ (b_j + a_{j-1} z)(c_k + c_{k-1} z) \]
\[
+ \binom{n-k}{n-j} b_{n-1} z(c_{j+k-n} + c_{j+k-1-n} z) \]
\[
= f^* \left( \sum_{r=\max(0,k+j-n)}^{k-1} \binom{j-r}{k-r} b_{k+j-r} c_r + ajc_k \right) \]

as expected since \( f^*|G \) is injective. If \( k + j = n \), we proceed exactly as above. It remains two cases, the first is for \( j = 1 \), but the only non trivial
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Steenrod operations are $Sq^0(a_1) = a_1$ and $Sq^2(a_1) = a_1^2 = a_1c_1 + b_2$ by lemma 2.3. The second is for $k = j$ but in this case

$$Sq^{2j}(a_j) = a_j^2 = a_jc_j + \sum_{r=2j-n}^{j-1} b_{2j-r}c_r$$

$$= \sum_{r=2j-n}^{j-1} \binom{j-r}{j-r} b_{2j-r}c_r + a_jc_j$$

always by lemma 2.3.

3. Proof of Theorem 1.2. The integral cohomology and the mod 2 cohomology of $L^n(2^m)$ are well known, they are given by the isomorphisms of abelian groups:

$$H^q(L^n(2^m); \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & \text{if } q = 0, \ 2n + 1 \\
\mathbb{Z}/2^m & \text{if } q = 2i, \ 0 < i \leq n \\
0 & \text{otherwise,}
\end{cases}$$

$$H^q(L^n(2^m); \mathbb{Z}/2) \cong \begin{cases} 
\mathbb{Z}/2 & \text{if } 0 \leq q \leq 2n + 1 \\
0 & \text{otherwise.}
\end{cases}$$

Let be $\pi: L^n(2^m) \to CP^n$ the natural projection, $\mu$ the canonical complex line bundle over $CP^n$, and let denote $z = c_1(\pi^*(\mu)) = \pi^*(c_1(\mu)) \in H^2(L^n(2^m); \mathbb{Z})$. We observe that $z^i$ is an additive generator of $H^{2i}(L^n(2^m); \mathbb{Z})$ for every $1 \leq i \leq n$.

Let us still write $z^i$ for the mod 2 reduction of the additive generator above, we see readily that

$$Sq^2(z^i) = iz^{i+1}$$

$$Sq^4(z^i) = \left(\frac{i}{2}\right)z^{i+2}.$$  

(3.1)
(3.2)

Finally, let $l(n)$ denote the integer

$$l(n) = \max \left\{ 0 \leq i \leq n - 1 \text{ such that } \binom{n+i+1}{n} \neq 0 \pmod{4} \right\}.$$  

Recall the 2-divisibility of $\binom{n+i+1}{n}$:

$$\nu_2(\binom{n+i+1}{n}) = \alpha(n) + \alpha(i+1) - \alpha(n+i+1).$$

We observe that for $\alpha(n) = 1$ and $i = n - 1$, we get $\nu_2(\binom{n+i+1}{n}) = \alpha(n) = 1$ and so $l(n) = n - 1$. For $\alpha(n) = 2$ we obtain, likewise, $l(n) = n - 2$. 

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For \( \alpha(n) \geq 3 \), we have the next result where we relate \( l(n) \) with the dyadic expansion of \( n \).

**Lemma 3.1.** If \( n = 2^{s_1} + 2^{s_2} + \cdots + 2^{s_k} \) with \( s_1 > s_2 > \cdots > s_k \geq 0 \) and \( k \geq 3 \), \( l(n) = 2^{s_1} + 2^{s_2} - 2 - 2^{s_3} - \cdots - 2^{s_k} \).

**Proof.** The 2-divisibility of \( \binom{n+i+1}{n} \) is 0 or 1 if \( n \) and \( i+1 \) have at most one common term in their dyadic expansion. So \( i+1 \) is greatest possible, if there is one common term of highest 2-valuation, here \( 2^{s_1} \). The rest of the expansion of \( i+1 \) contains all powers \( 2^r \) with \( r < s_2 \), except \( r = s_3, \ldots, s_k \).

This description of \( l(n) \) gives for \( \alpha(n) \geq 3 \):

\[
(3.3) \quad l(n) = \begin{cases} 
2 \pmod{4} & \text{if } n \equiv 0 \pmod{4} \\
1 \pmod{4} & \text{if } n \equiv 1 \pmod{4} \\
0 \pmod{4} & \text{if } n \equiv 2 \pmod{4} \\
3 \pmod{4} & \text{if } n \equiv 3 \pmod{4}.
\end{cases}
\]

We come back to the immersion problem for \( L^n(2^m) \). We know that the stable class of the tangent bundle of \( L^n(2^m) \) is \( r(n+1)\sigma \) (see [10]), where \( r \) denotes the realification. So, if \( L^n(2^m) \) immerses in \( R^{2n+1+k} \), the stable class of the normal bundle of this immersion is \(-r(n+1)\sigma\) and its classifying map

\[-r(n+1)\sigma: L^n(2^m) \rightarrow BSO(2n+2)\]

lifts to \( BU(n+1) \) and to \( BSO(k) \). Therefore, this map also lifts to \( B(n+1,k) \), and we obtain the commutative diagram

\[
\begin{array}{ccc}
B(n+1,k) & \xrightarrow{p} & BU(n+1) \\
\downarrow j_k & & \\
L^n(2^m) & \xrightarrow{g} & BU(n+1)
\end{array}
\]

where \( g: L^n(2^m) \rightarrow BU(n+1) \) denotes a lifting of \(-r(n+1)\sigma\) to \( BU(n+1) \) and \( j_k \) a lifting of \( g \) in \( B(n+1,k) \). We also note that

\[
g^*(c_i) = c_i(-r(n+1)\sigma) = \binom{-n-1}{i}z^i = (-1)^i \binom{n+i}{n}z^i
\]
since for the total Chern class of $-(n+1)\sigma$ we find
\[
c(-(n+1)\sigma) = c(\sigma)^{-(n-1)} = (1 + c_1(\sigma))^{-(n-1)} = (1 + z)^{-(n-1)} = \sum_{i\geq 0} \binom{-n-1}{i} z^i.
\]
For $i \geq [k/2] + 1$, we have $p^*(c_i) = 2b_i$ in $H^*(B(n+1,k); \mathbb{Z})$, hence
\[
2\tilde{f}_k^*(b_i) = \tilde{f}_k^*(2b_i) = \tilde{f}_k^*(p^*(c_i)) = g^*(c_i)
\]
and therefore, if $\binom{n+i}{i} \not\equiv 0 \pmod{2^m}$,
\[
\tilde{f}_k^*(b_i) = \frac{1}{2} \binom{-n-1}{i} z^i = \pm \frac{1}{2} \binom{n+i}{n} z^i.
\]

Now, if $k = 2i$, and $a_i \in H^{2i}(B(n+1,2i); \mathbb{Z})$ as in the previous section, $\tilde{f}_2^*(a_i)$ is an element $\lambda_i z^i$ of $H^{2i}(L^n(2^m); \mathbb{Z}) \cong \mathbb{Z}/2^m$ where $\lambda_i \in \mathbb{Z}/2^m$.

The Steenrod squares are natural and so with the help of relations (2.4), (3.1) and (3.2), we deduce for $i \leq n - 2$

(3.4) \[ i\lambda_i = (n+1)\lambda_i + i \frac{1}{2} \binom{n+i+1}{n} \pmod{2}, \]

(3.5) \[ \binom{i}{2} \lambda_i = \binom{n+2}{2} \lambda_i + (i-1)(n+1) \frac{1}{2} \binom{n+i+1}{n} \]
\[ + \binom{i}{2} \frac{1}{2} \binom{n+i+2}{n} \pmod{2}. \]

We shall note that (3.4) is still valid for $i = n - 1$. In the following we shall take $i = l(n)$ and $m \geq 2$.

First we suppose $n = 2^s$ with $s \geq 1$. In this case, $i = l(n) = n - 1$ and (3.4) becomes
\[ \frac{1}{2} \binom{n+i+1}{n} \equiv 0 \pmod{2} \]
which is impossible.

When $n$ is even with $\alpha(n) \geq 2$, $i = l(n)$ is even and $\lambda_i \equiv 0 \pmod{2}$. Using (3.5) we deduce
\[ 0 \equiv \frac{1}{2} \binom{n+i+1}{n} + \binom{i}{2} \frac{1}{2} \binom{n+i+2}{n} \equiv \frac{1}{2} \binom{n+i+1}{n} \pmod{2}, \]
since \( i + 1 > l(n) \), which is in contradiction with the definition of \( l(n) \).

When \( n \) is odd with \( \alpha(n) \geq 3 \), \( i = l(n) < n - 3 \) is odd, the relation (3.4) becomes

\[
\lambda_i \equiv \frac{1}{2} \left( \frac{n + i + 1}{n} \right) \pmod{2}.
\]

Now, using (3.5) and (3.3) we obtain

\[
\frac{1}{2} \left( \frac{n + i + 1}{n} \right) \equiv \begin{cases} 
0 & \text{if } n \equiv 1 \pmod{4} \\
\frac{1}{2} \left( \frac{n + i + 2}{n} \right) & \text{if } n \equiv 3 \pmod{4}.
\end{cases}
\]

As before we have a contradiction since \( i + 1 > l(n) \) and so we have proved part a) of theorem 1.2.

Finally, if \( n = 2^s + 1 \) with \( s \geq 1 \), and if \( L^n(2^m) \) immerses in \( R^{2n+1+2(n-2)-1} \), the classifying map \( g \) of \( -(n+1) \sigma \) lifts to \( B(n+1,2(n-2)-1) \), and also to \( B(n+1,2(n-2)) \). With the same notations, relation (3.4) becomes in this case

\[
\lambda_{n-2} \equiv \frac{1}{2} \left( \frac{2n-1}{n} \right) \pmod{2}
\]

and so

\[
\lambda_{n-2} \equiv 1 \pmod{2}.
\]

However, if the map \( g \) lifts to \( B(n+1,2(n-2)-1) \), we have

\[
\lambda_{n-2}z^{n-2} = \tilde{f}^*(n-2)(a_{n-2})
\]

\[
= \tilde{f}^*(n-2-1)(p_{2n-5}^*(a_{n-2}))
\]

\[
= \tilde{f}^*(n-2-1)(b_{n-2})
\]

\[
= \frac{1}{2} \left( \frac{2n-2}{n} \right) z^{n-2}
\]

\[
\equiv 0 \pmod{2}
\]

where \( p_{2n-5} \) denotes the canonical map \( B(n+1,2(n-2)-1) \rightarrow B(n+1,2(n-2)) \). So, we have proved part b) of theorem 1.2.

References

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