Dual-bimodules and Finitely Cogenerated Modules

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DUAL-BIMODULES AND FINITELY COGENERATED MODULES

In memory of Professor Hisao Tominaga

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Let $R$ and $S$ be rings with identity and $RQ_S$ an $(R, S)$-bimodule. We shall call $Q$ a left dual-bimodule provided that $\ell_R r_Q(A) = A$ for every left ideal $A$ of $R$ and $r_Q \ell_R(Q') = Q'$ for every $S$-submodule $Q'$ of $Q$ (see [5]).

In this note, first we shall show that a left dual-bimodule $RQ_S$ defines a duality between the finitely generated left ideals of $R$ and the finitely cogenerated factor modules of $Q_S$. Then, as an application of this duality, we shall give necessary and sufficient conditions for $R$ to be left semihereditary or left coherent.

For notations and definitions we shall follow [1] and [5].

1. Let $R$ and $S$ be rings with identity and $RQ_S$ an $(R, S)$-bimodule. Suppose that $Q_S$ is quasi-injective and the natural homomorphism $\lambda: R \rightarrow \text{End}(Q_S)$ is an isomorphism. Then by [7, Theorem 2.1], for each $S$-module $N_S$, there is a bijection between the finitely generated submodules of the $Q$-dual $R N^*$ of $N$ and the finitely closed submodules of $N_S$ given by

$$L \rightarrow r_N(L)$$

with the inverse $K \rightarrow \ell_{N^*}(K)$. Here, a submodule $K$ of $N_S$ is said to be finitely closed (with respect to $Q_S$) if there exists an integer $m > 0$ such that

$$0 \rightarrow N/K \rightarrow Q^m$$

is exact, or equivalently, there exist $f_1, f_2, \ldots, f_m$ in $N^*$ such that

$$\bigcap_{j=1}^m \text{Ker} f_j = K.$$

In case $Q_S$ is finitely cogenerated, $K$ finitely closed means that $N/K$ is finitely cogenerated $Q$-torsionless.

Using this theorem, Miller and Turnidge pointed out that, under the same assumption as above, $R$ is left Noetherian (right perfect) if and only if $Q_S$ has DCC (ACC) on finitely closed submodules.
If, in particular, $RQ_S$ is a left dual-bimodule with $Q_S$ quasi-injective and $\lambda$ surjective, then the bijection yields one between the finitely generated left ideals of $R$ and the finitely closed submodules of $Q_S$ given by

$$A \rightarrow r_Q(A)$$

with the inverse $Q' \rightarrow \ell_R(Q')$. Hence, in this case, $R$ is left Noetherian (right perfect) if and only if $Q_S$ has DCC (ACC) on the submodules $r_Q(A)$ of $Q_S$ with $A$ a finitely generated left ideal of $R$ and $R$ is regular if and only if every submodule of $Q_S$ of the above form is a direct summand of $Q_S$ (cf. [4, Proposition 4.2 and Theorem 4.3]). On the other hand, since the mapping $Ra \rightarrow r_Q(a)$ is a bijection between the principal left ideals of $R$ and the submodules $r_Q(a)$ of $Q_S$ with $a$ in $R$, it follows that $R$ is right perfect if and only if $Q_S$ has ACC on the submodules $r_Q(a)$ of $Q_S$ with $a$ in $R$ and that $R$ is regular if and only if every submodule of $Q_S$ of the last form is a direct summand of $Q_S$ (cf. [4, Theorem 3.1 and Proposition 4.1]).

2. For an $(R, S)$-bimodule $RQ_S$, as was shown in [5, Theorem 3.3], if $Q_S$ is quasi-injective and $\lambda$ is surjective, then the pair $(H', H'')$ of functors

$$H' = \text{Hom}_R(-, Q) : \text{RM} \rightarrow \text{NS},$$

$$H'' = \text{Hom}_S(-, Q) : \text{NS} \rightarrow \text{RM}$$

defines a duality between the full subcategory $\text{RM}$ of $R$-mod of finitely generated $Q$-torsionless $R$-modules and the full subcategory $\text{NS}$ of $\text{mod-S}$ whose objects are all the $S$-modules $N_S$ such that

$$0 \rightarrow N \rightarrow Q^n \rightarrow Q^I$$

is exact for some integer $n > 0$ and a set $I$. Assume further that $Q_S$ is finitely cogenerated, then by [5, Proposition 3.4]

$$\text{RM} = \{RM \mid M \text{ is finitely generated and } Q\text{-reflexive}\}$$

and

$$\text{NS} = \{N_S \mid N \text{ is finitely cogenerated and } Q\text{-reflexive}\}.$$

If, in addition, $Q_S$ is a self-cogenerator, then by [6, Proposition 4]

$$\text{NS} = \{N_S \mid 0 \rightarrow N \rightarrow Q^n \text{ is exact for some } n > 0\}.$$
Using the bijection in Section 1, we shall now show that $(H', H'')$ defines a duality between more restricted subcategories of $\mathcal{R}_M$ and $\mathcal{N}_S$.

**Theorem 1.** Let $\mathcal{R}_Q S$ be a left dual-bimodule with $Q S$ quasi-injective and $\lambda$ surjective. Then $(H', H'')$ defines a duality between the finitely generated left ideals of $R$ and the finitely cogenerated factor modules of $Q S$.

**Proof.** Let $A$ be a finitely generated left ideal of $R$. Then $A$ belongs to $\mathcal{R}_M$ and $A^*$ is isomorphic to a finitely cogenerated factor module $Q / r_Q(A)$ of $Q S$ by [5, Lemma 1.13]. On the other hand, for each finitely cogenerated factor module $Q / Q'$ of $Q S$, $Q'$ is finitely closed and hence $\ell_R(Q')$ is finitely generated and is $Q$-reflexive. Again by [5, Lemma 1.13], $Q / Q' \cong \ell_R(Q')^*$.

Thus, $Q / Q'$ is in $\mathcal{N}_S$ and $(Q / Q')^*$ is isomorphic to $\ell_R(Q')$.

**Corollary 2.** Let $\mathcal{R}_Q S$ be a left dual-bimodule with $Q S$ quasi-injective and $\lambda$ surjective. If $R$ is left Noetherian, then $(H', H'')$ defines a duality between the left ideals of $R$ and the factor modules of $Q S$.

In contrast with Corollary 2, $(H', H'')$ always defines a duality between the factor modules of $R$ and the submodules of $Q S$ under the same assumption of Corollary 2. Indeed, for each left ideal $A$ of $R$, $R / A$ is $Q$-reflexive by [5, Theorem 3.2] and $(R / A)^* \cong r_Q(A)$. On the other hand, for each submodule $Q'$ of $Q S$, $Q' = r_Q(\ell_R(Q')) \cong (R / \ell_R(Q'))^*$. Hence, $Q'$ is $Q$-reflexive by [1, Proposition 20.14] and $Q'^* \cong R / \ell_R(Q')$.

If $\mathcal{R}_Q S$ is a dual-bimodule with both $\mathcal{R}_Q$ and $Q S$ injective, then using [1, Exercise 23.7] $(H', H'')$ defines a duality between the left $R$-modules of finite length and the right $S$-modules of finite length by [5, Theorem 2.1]. However, we have

**Theorem 3.** Let $\mathcal{R}_Q S$ be a left dual-bimodule with $Q S$ quasi-injective and $\lambda$ surjective. Then $(H', H'')$ defines a duality between the $Q$-torsionless left $R$-modules of finite length and the $Q$-torsionless right $S$-modules of finite length.

**Proof.** Let $\mathcal{R}_M$ be a $Q$-torsionless $R$-module of finite length and let $M = M_0 > M_1 > \cdots > M_n = 0$ be a composition series of $M$. Then

$$0 = r_M(M_0) \leq r_M(M_1) \leq \cdots \leq r_M(M_n) = M^*$$


is a series of $S$-submodules of $M^*$, where $r_{M^*}(M_i) = \{ f: M \to Q | M_i \leq \text{Ker} f \}$ (see [1, p.281]). For each $i$, each element of $r_{M^*}(M_{i+1})$ induces an $R$-homomorphism from $M_i/M_{i+1}$ to $Q$ and hence $r_{M^*}(M_{i+1})/r_{M^*}(M_i)$ can be seen as an $S$-submodule of $(M_i/M_{i+1})^*$. Since $M_i/M_{i+1}$ is simple, $(M_i/M_{i+1})^*$ is isomorphic to a simple submodule of $Q_S$, as is seen from the proof of [5, Theorem 2.1]. Hence, $r_{M^*}(M_{i+1})/r_{M^*}(M_i)$ is zero or simple. Thus, $M_S^*$ is a module of finite length and $c(M^*) \leq c(M)$, where $c(-)$ denotes the composition length. Furthermore, by [1, Proposition 20.14], $M_S^*$ is $Q$-torsionless.

Using [1, Exercise 16.18], for a $Q$-torsionless $S$-module $N_S$ of finite length, $R N^*$ is a $Q$-torsionless $R$-module of finite length and $c(N^*) \leq c(N)$ holds.

Clearly each $Q$-torsionless $R$-module $R M$ of finite length is $Q$-reflexive and we have $c(M) = c(M^*)$. On the other hand, each $Q$-torsionless $S$-module $N_S$ of finite length is finitely cogenerated. Hence it is $Q$-reflexive. Thus we have $c(N) = c(N^*)$.

**Corollary 4.** Let $R Q_S$ be a left dual-bimodule with $Q_S$ quasi-injective and $\lambda$ surjective. Then $(H', H'')$ defines a duality between the simple left $R$-modules and the $Q$-torsionless simple right $S$-modules.

In case $R Q_S$ is a dual-bimodule, however, $(H', H'')$ defines a duality between the simple left $R$-modules and the simple right $S$-modules, as is seen from [5, Theorem 2.1].

**3.** It is shown by [5, Proposition 1.12] that for a left dual-bimodule $R Q_S$, $R$ is semisimple if and only if $Q_S$ is semisimple. On the other hand, we have

**Theorem 5.** Let $R Q_S$ be a left dual-bimodule with $\lambda$ surjective. Then $R$ is simple Artinian if and only if $Q_S \cong Q_n^1$ for some integer $n > 0$ and some simple right $S$-module $Q_1$.

**Proof.** Suppose that $R$ is simple Artinian. Then $Q_S$ is semisimple and is finitely generated by [5, Proposition 1.8]. Let $Q_1$ be a simple submodule of $Q_S$. Then $\ell_R(R Q_1)$ is a proper ideal of $R$ and hence it must be zero by assumption. Therefore, $R Q_1 = r_Q \ell_R(R Q_1) = Q$. However, $R Q_1 = \sum_{a \in R} a Q_1$ and each $a Q_1$ is either zero or isomorphic to $Q_1$. Thus we have $Q \cong Q_n^1$ for some integer $n > 0$. 

http://escholarship.lib.okayama-u.ac.jp/mjou/vol37/iss1/9
Conversely, suppose that \( Q_S \cong Q_1^n \) for some integer \( n > 0 \) and some simple right \( S \)-module \( Q_1 \). Then since \( R \cong \lambda \text{End}(Q_S) \), \( R \) is isomorphic to the ring of all \( n \times n \) matrices over the division ring \( \text{End}(Q_1) \). Thus, \( R \) is simple Artinian.

Now, using Theorem 1, we shall give a necessary and sufficient condition for \( R \) to be left semihereditary (cf. [2, Corollary 2.4] and [8, Proposition 2.1]).

**Theorem 6.** Let \( RQ_S \) be a left dual-bimodule with \( Q_S \) quasi-injective and \( \lambda \) surjective. Then the following conditions are equivalent:

1. \( R \) is left semihereditary.
2. Every finitely cogenerated factor module of \( Q_S \) is \( Q \)-injective.
3. For every finitely generated left ideal \( A \) of \( R \), \( A^* \) is \( Q \)-injective.

**Proof.** Let \( A \) be a finitely generated left ideal of \( R \) and let \( R^n \to A \to 0 \) be exact for some integer \( n > 0 \). Then the sequence

\[
0 \to A^* \to Q^n
\]

is also exact. Since \( A \) is \( Q \)-reflexive and \( R \cong \lambda \text{End}(Q_S) \), \( A \) is projective if and only if \((*)\) is split exact and this is so if and only if \( A^* \) is \( Q \)-injective. Thus, (1) and (3) are equivalent. By Theorem 1, (2) and (3) are also equivalent.

**Theorem 7.** For a dual ring \( R \) the following conditions are equivalent:

1. \( R \) is left semihereditary.
2. \( R \) is semisimple.

**Proof.** (1) \(\Rightarrow\) (2). Suppose that \( R \) is left semihereditary. Since \( R/\text{rad}(R) \) is semisimple by [5, Theorem 1.10], \( 0 \to R/\text{rad}(R) \to \text{soc}(R)^n \) is split exact for some integer \( n > 0 \). By [5, Proposition 1.8] \( \text{soc}(R) \) is projective. Hence, \( R/\text{rad}(R) \) is also projective. Thus, \( \text{rad}(R) \) must be a direct summand of \( R \), from which it follows that \( \text{rad}(R) = 0 \) and \( R \) is semisimple. (2) \(\Rightarrow\) (1) is trivial.

As is easily seen, a ring \( R \) is left coherent if and only if for every integer \( n > 0 \) and every \( R \)-homomorphism \( f: R^n \to R \) there exist an integer \( m > 0 \) and an \( R \)-homomorphism \( g: R^m \to R^n \) such that

\[
R^m \xrightarrow{g} R^n \xrightarrow{f} R
\]
is exact. For a left dual-bimodule $RQ_S$, using $Q_S$ instead of $R$, a similar characterization for $R$ to be left coherent can be obtained (cf. [2, Theorem 2.6 and Corollary 2.7]).

**Theorem 8.** For a left dual-bimodule $RQ_S$ with $Q_S$ quasi-injective and $\lambda$ surjective, the following conditions are equivalent:

1. $R$ is left coherent.
2. For every finitely cogenerated factor module $Q/Q'$ of $Q_S$, there exist integers $n, m > 0$ such that
   
   \[ 0 \rightarrow Q/Q' \rightarrow Q^n \rightarrow Q^m \]

   is exact.
3. For every finitely generated left ideal $A$ of $R$, there exist integers $n, m > 0$ such that
   
   \[ 0 \rightarrow A^* \rightarrow Q^n \rightarrow Q^m \]

   is exact.
4. For every integer $n > 0$ and every $S$-homomorphism $f: Q \rightarrow Q^n$ there exist an integer $m > 0$ and an $S$-homomorphism $g: Q^n \rightarrow Q^m$ such that
   
   \[ Q \xrightarrow{f} Q^n \xrightarrow{g} Q^m \]

   is exact.

**Proof.** It is easy to see that (1), (2) and (3) are equivalent.

(1) $\Rightarrow$ (4). Assume (1) and let $f: Q \rightarrow Q^n$ be an $S$-homomorphism. Then $0 \rightarrow Q/K \xrightarrow{f} Q^n$ is exact, where $K = \text{Ker} f$ and $\bar{f}$ is the homomorphism induced by $f$. Hence $Q/K$ is finitely cogenerated $Q$-reflexive. By Theorem 1, $(Q/K)^*$ is a finitely generated left ideal of $R$ and $R^n \rightarrow (Q/K)^* \rightarrow 0$ is exact. Since $R$ is left coherent, there exists an integer $m > 0$ such that $R^n \rightarrow R^m \rightarrow (Q/K)^* \rightarrow 0$ is exact. Thus, $0 \rightarrow Q/K \xrightarrow{f} Q^n \xrightarrow{g} Q^m$ is exact for some $S$-homomorphism $g$, which shows that

\[ Q \xrightarrow{f} Q^n \xrightarrow{g} Q^m \]

is exact.

(4) $\Rightarrow$ (2). Assume (4) and let $Q/Q'$ be any finitely cogenerated factor module of $Q_S$. Then $Q'$ is finitely closed and hence there exists an integer $n > 0$ such that $0 \rightarrow Q/Q' \xrightarrow{\bar{f}} Q^n$ is exact for some $S$-homomorphism $f$. Let $\pi: Q \rightarrow Q/Q'$ be the canonical epimorphism. Then by (4) there exist
an integer \( m > 0 \) and an \( S \)-homomorphism \( g \) such that \( Q^{\oplus m} \xrightarrow{f} Q^n \xrightarrow{g} Q^m \) is exact and thus so is \( 0 \to Q/Q' \xrightarrow{f} Q^n \xrightarrow{g} Q^m \).

It is to be noted that if \( R \) is a dual ring with \( RR \) injective, then the bimodule \( RR \) defines a Morita duality by [1, Exercise 24.10] and [6, Corollary 6]. However, this is not the case for a left dual-bimodule in general. For example, let \( R = Q = \mathbb{Z}/(p) \), \( p \) a prime number, and \( S = \mathbb{Z} \). Then the bimodule \( QR \) is a left dual-bimodule with \( QS \) quasi-injective and \( \lambda \) surjective, but does not define any Morita duality.

For this left dual-bimodule, \( R \) is left Noetherian, right perfect and is also regular. Furthermore, it is left semihereditary and left coherent, too.

References


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