Ko-Group of $\text{PSp}(2^{4n})$

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Dedicated to Professor Teiichi Kobayashi on his 60th birthday

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Let $Sp(n)$ be the symplectic group of degree $n$ and $PSp(n)$ be the projective group associated with $Sp(n)$, that is, $PSp(n) = Sp(n)/C$ where $C$ denotes the center of $Sp(n)$ which is generated by the scalar matrix with all diagonal entries $-1$.

Our purpose here is to compute the real $K$-group $KO^*(PSp(2^{4n}))$. As for the complex $K$-group, $K^*(PSp(\ell))$ has been determined in [7,9] for any $\ell \geq 1$. But we begin with the calculation of $K^*(PSp(2^{4n}))$ by our method for convenience of calculation. The way getting these groups is quite parallel to that of [12]. As it turns out that there is a $\mathbb{Z}/2$-map from $S^{8n+3}$ to $Sp(2^{4n})$ where the generator of $\mathbb{Z}/2$ acts on $S^{8n+3}$ as antipodal involution and on $Sp(2^{4n})$ as the generator of $C$ respectively, the multiplicative structures of the $K$-groups of $PSp(2^{4n})$ can be reduced to those of the $K$-groups of $P^{8n+3}$ and $Sp(2^{4n})$ just as in the case of $SO(8\ell)$ [12] by making use of this $\mathbb{Z}/2$-map and applying a device to the equivariant $K$-theories associated with $\mathbb{Z}/2$.

This paper is arranged as follows. Section 1 consists of preparations for the subsequent sections. Sections 2 and 3 deal with the computation of $K^*(PSp(2^{4n}))$ and $KO^*(PSp(2^{4n}))$ respectively.

1. Let $\Gamma$ denote the multiplicative group generated by $-1$ and $H$ denote the canonical non-trivial 1-dimensional real representation of $\Gamma$.

We write $nH$ for the direct sum of $n$ copies of $H$. And by $B(pH \oplus R^q)$ and $S(pH \oplus R^q)$ we denote the unit ball and the unit sphere in $pH \oplus R^q$ centered at the origin $o$, and let $\Sigma^{n,q} = B(pH \oplus R^q)/S(pH \oplus R^q)$ with the collapsed $S(pH \oplus R^q)$ as base point. Here $R$ denotes the field of real numbers. Also, for later use we fix the notations $C$ and $H$ for the fields of complex numbers and quaternions as usual.

Let $\Delta^+: Spin(8n + 4) \to U(2^{4n+1})$ be one of the half-spin representations of $Spin(8n + 4)$. It is known [10], §13 that $\Delta^+$ is the restriction of a quaternionic representation of $Spin(8n + 4)$, denoted by

$$\bar{\Delta}^+: Spin(8n + 4) \to Sp(2^{4n})$$
below. Assume that the generator of $\Gamma$ acts on $\text{Spin}(8n + 4)$ and $\text{Sp}(2^{4n})$ as the elements $-1$ and $-I$ of these groups respectively where $I$ is the unit matrix, and thus consider these two groups as $\Gamma$-spaces. Then $\Delta^+$ becomes a $\Gamma$-map obviously. Moreover we know [6] that $\text{Spin}(8n + 4)$ contains $S^{8n+4,0}$ as an invariant subspace. This follows from the fact that $\text{Spin}(8n + 4)$ is a subgroup of the Clifford algebra $C_{8n+3}$ multiplicatively generated by the elements of the unit sphere $S^{8n+3}$ ([10], §11). Therefore we have the following result similar to [6],(1.14).

(1.1) There exists a $\Gamma$-map $\iota: S^{8n+4,0} \to \text{Sp}(2^{4n})$, so that we have a homeomorphism

$$(S^{8n+4,0} \times \text{Sp}(2^{4n}))/\Gamma \approx P^{8n+3} \times \text{Sp}(2^{4n}).$$

In fact, this homeomorphism is induced by the assignment $(x,g) \mapsto (\pi(x),\iota(x)^{-1}g)$ for $x \in S^{8n+4,0}$ and $g \in \text{Sp}(2^{4n})$, where $P^{8n+3} = S^{8n+4,0}/\Gamma$, the real projective space of dimension $8n + 3$, and $\pi$ is the canonical projection from $S^{8n+4,0}$ to $P^{8n+3}$.

A Real $(\Gamma^\cdot)$-vector bundle is a complex $(\Gamma^\cdot)$-vector bundle together with a conjugate (equivariant) involutive automorphism and a quaternionic $(\Gamma^\cdot)$-vector bundle is a complex $(\Gamma^\cdot)$-vector bundle together with a conjugate (equivariant) anti-involutive automorphism. It is clear by definition that the external tensor product $E \hat{\otimes}_C F$ of two quaternionic $(\Gamma^\cdot)$-vector bundles $E$ and $F$ admits an obvious Real structure.

Let $KR$ and $KSp$ denote the Real and quaternionic $K$-theories and let $KR_{\Gamma}$ and $KSp_{\Gamma}$ denote the equivariant ones associated with $\Gamma$. But $KR(X) \cong KO(X)$ and $KR_{\Gamma}(X) \cong KO_{\Gamma}(X)$ canonically if $X$ has a trivial Real structure. Since all spaces of this note are such ones, we identify these isomorphisms throughout this paper. Then the above external tensor product $x \hat{\otimes}_C y$ defines uniquely an element $x \wedge_C y$ of either $KO(X \wedge Y)$ or $KO_{\Gamma}(X \wedge Y)$ according as $x \in \overline{KSp}(X)$, $y \in \overline{KSp}(Y)$ or $x \in \overline{KSp}_{\Gamma}(X)$, $y \in \overline{KSp}_{\Gamma}(Y)$.

Considering $S^{0,3}$ to be the unit quaternions $Sp(1)$ yields a generator of $\overline{KSp}(\Sigma^{0,4})$ in a canonical way. We write $\alpha$ for this element. Then

$$\overline{KSp}(\Sigma^{0,4}) = Z \cdot \alpha$$

and also $\alpha$ satisfies

(1.2) $\quad \alpha \otimes_C H = \eta_4, \; \alpha \wedge_C \alpha = \eta_8 \; \text{and} \; s(\alpha) = \mu^2$
where \( \eta_4, \eta_8 \) and \( \mu \) denote the canonical generators of \( \overline{KO}(\Sigma^{0,4}) \), \( \overline{KO}(\Sigma^{0,8}) \) and \( \overline{K}(\Sigma^{0,2}) \), (the last two generators are called the Bott class), and \( s \) denotes the natural complexification \( KS \to K \).

From [3,11,14] we now recall the equivariant Thom isomorphism theorems. Consider the isomorphism \( S^{8n+4,0} \times H^{2^{4n}} \cong S^{8n+4,0} \times H^{2^{4n}} \otimes_R H \) of \( \Gamma \)-quaternionic vector bundles over \( S^{8n+4,0} \) given by the assignment \( (x,v) \mapsto (x,\iota(x)v) \) for \( x \in S^{8n+4,0} \), \( v \in H^{2^{4n}} \) where \( \iota \) is as in (1.1). Then, in a canonical manner, this isomorphism yields a generator \( \tau_H \) of \( \overline{KSp}_{\Gamma}(\Sigma^{8n+4,0}) \) such that its restriction to \( o \in B((8n+4)H) \) is \( 2^{4n}(H - H \otimes_R H) \in KSp_{\Gamma}(o) (= RS_{\Gamma}(\Gamma), \) the quaternionic representation ring of \( \Gamma \)).

Set

\[
\sigma = s(\tau_H) \in \overline{K}\Gamma_{s}(\Sigma^{8n+4,0}) \quad \text{and} \quad \omega = \sigma \wedge \alpha \in \overline{KO}_{s}(\Sigma^{8n+4,4}).
\]

Then their restrictions to \( o \) and \( \Sigma^{0,4} \) are \( 2^{4n+1}(1 - L) \in K_{s}(o) = R(\Gamma) \) and \( 2^{4n}(1 - H) \eta_4 \in \overline{KO}_{s}(\Sigma^{0,4}) = RO(\Gamma) \cdot \eta_4 \) respectively where \( L = C \otimes_R H \), and multiplications by \( \sigma \) and \( \omega \) give isomorphisms \( \overline{K}_{s}(X) \cong \overline{K}_{s}(\Sigma^{8n+4,0} \wedge X) \) and \( \overline{KO}_{s}(X) \cong \overline{KO}_{s}(\Sigma^{8n+4,4} \wedge X) \) for any \( \Gamma \)-space \( X \) with base-point respectively. Here \( R(\Gamma) \) and \( RO(\Gamma) \) are the complex and real representation rings of \( \Gamma \) and \( R \cdot g \) denotes an \( R \)-module generated by a single element \( g \) for a ring \( R \).

By \( h \) we denote the \( K \)- or \( KO \)-functor. For \( X = + \) (a point), \( Sp(2^{4n}) \) we consider the exact sequence of the pair \( B((8n+4)H) \times X, S((8n+4)H) \times X) \) in \( h_{s} \)-theory. In general if \( \Gamma \) acts on \( X \) freely then there is a natural isomorphism \( h_{s}^{*(X)}(X/\Gamma) \cong h^{*(X/\Gamma)} \). Combining this with (1.1) and (1.3) gives rise to the following exact sequences.

\[
\begin{align*}
(1.4a) \quad & \cdots \xrightarrow{\delta} h_{s}^{*(+)}(X) \xrightarrow{J} h_{s}^{*(+)}(X) \xrightarrow{I} h^{*(P^{8n+3})} \xrightarrow{\delta} \cdots, \\
(1.4b) \quad & \cdots \xrightarrow{\delta} h^{*(PG)(X)} \xrightarrow{J} h^{*(PG)(X)} \xrightarrow{I} h^{*(P^{8n+3} \times G)} \xrightarrow{\delta} \cdots
\end{align*}
\]

where \( G = Sp(2^{4n}) \) and there holds the equality \( \delta(xI(y)) = \delta(x)y \) in either case.

We write \( G \) for \( Sp(2^{4n}) \) for simplicity in the subsequent sections.

2. By the same symbol \( \sigma \) we denote the reduced bundles of the canonical line bundles \( (S^{8n+4,0} \times H)/\Gamma \to P^{8n+3} \) and \((G \times H)/\Gamma \to PG\). And we write \( \sigma = c(\sigma) \) where \( c \) denotes the complexification \( KO \to K \). Since
$H^2 = 1$ in $RO(\Gamma)$ there hold obviously

$$\sigma^2 + 2\sigma = 0 \quad \text{and} \quad \sigma^2 + 2\sigma = 0.$$ 

Let $\nu = p^*(\eta_8^{n+1}) \in \tilde{KO}^{-5}(P^{8n+3})$ and $\nu = p^*(\mu^{4n+2}) \in \tilde{K}^{-1}(S^{8n+3})$ where $p$ is the map $P^{8n+3} \to S^{8n+3}$ obtained by collapsing the outside of a top dimensional cell in $P^{8n+3}$ to a point. Then the equalities

$$c(\nu) = \mu^2 \nu \quad \text{and} \quad r(\nu) = \eta_4 \nu$$

follow from the relations $c(\eta_4) = 2 \mu^2$ and $\eta_4^2 = 4$.

We consider the complex and real $K$-theories the $Z/2$-and $Z/8$-graded cohomology theories with the coefficient rings $K^*(+) = Z[\mu]/(\mu^2 - 1)$ and $KO^*(+) = Z[\eta_1, \eta_4, \eta_8]/(2\eta_1, \eta_3^2, \eta_1 \eta_4, \eta_4^2 - 4, \eta_8 - 1)$ respectively where $\eta_1 \in KO^{-1}(+)$ and the others are as in Section 1. But the complex $K$-theory is viewed as $Z/8$-graded, so that $K^*(+) = Z[\mu]/(\mu^4 - 1)$, when we discuss the relation between these two kinds of $K$-theories.

Here we calculate $K^*(P^{8n+3})$ and $KO^*(P^{8n+3})$ whose additive structures are given in [2,5]. Consider the exact sequence of (1.4a). First note that $h^*_f(+) \cong h^*(+) [t]/(t^2 - 1)$ because of $\Gamma \cong Z/2$ where $t = L$ or $H$ according as $h = K$ or $KO$. From inspecting the definitions of the maps it follows that

$$\delta(\nu) = 1 + L, \quad J(1) = 2^{4n+1}(1 - L) \quad \text{and} \quad I(L) = \sigma + 1 \quad \text{for} \ h = K,$$

$$\delta(\nu) = 1 + H, \quad J(1) = 2^{4n} \eta_4(1 - H) \quad \text{and} \quad I(H) = \tilde{\sigma} + 1 \quad \text{for} \ h = KO.$$

Moreover we have a unique element $\zeta$ of $KO^{-6}(P^{8n+3})$ satisfying $\delta(\zeta) = \eta_1$.

Using this and the equality $\delta(xI(y)) = \delta(x)y$ we obtain by the exactness of (1.4a) the following.

With the notation as above

$$\tilde{K}(P^{8n+3}) = Z/2^{4n+1} \cdot \sigma, \quad \tilde{K}^{-1}(P^{8n+3}) = Z \cdot \nu$$

where the ring structure is given by

$$\sigma^2 + 2\sigma = 0, \quad \nu^2 = 0,$$
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$$\tilde{K}\tilde{O}(p^{8n+3}) = \mathbb{Z}/2^{4n+2} \cdot \tilde{\sigma},$$
$$\tilde{K}\tilde{O}^{-1}(p^{8n+3}) = \mathbb{Z}/2 \cdot \eta_1 \tilde{\sigma} \oplus \mathbb{Z} \cdot \eta_4 \tilde{\nu},$$
$$\tilde{K}\tilde{O}^{-2}(p^{8n+3}) = \mathbb{Z}/2 \cdot \eta_1^2 \tilde{\sigma},$$
$$\tilde{K}\tilde{O}^{-3}(p^{8n+3}) = 0,$$
$$\tilde{K}\tilde{O}^{-4}(p^{8n+3}) = \mathbb{Z}/2^{4n} \cdot \eta_4 \tilde{\sigma},$$
$$\tilde{K}\tilde{O}^{-5}(p^{8n+3}) = \mathbb{Z} \cdot \tilde{\nu},$$
$$\tilde{K}\tilde{O}^{-6}(p^{8n+3}) = \mathbb{Z}/2 \cdot \eta_1 \tilde{\nu} \oplus \mathbb{Z}/2 \cdot \zeta,$$
$$\tilde{K}\tilde{O}^{-7}(p^{8n+3}) = \mathbb{Z}/2 \cdot \eta_1^2 \tilde{\nu} \oplus \mathbb{Z}/2 \cdot \eta_4 \zeta.$$

(2.2b)

where the ring structure is given by

$$\tilde{\sigma}^2 + 2\tilde{\sigma} = 0, \quad \tilde{\nu}^2 = 0, \quad \zeta^2 = 0, \quad \eta_4 \zeta = 0,$$
$$\eta_1 \zeta = \eta_1 \tilde{\nu}, \quad \eta_1^2 \zeta = 2^{4n+1} \tilde{\sigma}.$$

Now we are ready for computing the $K$-groups of $PG$.

Let $\rho$ be the canonical, non-trivial, $2^{4n}$-dimensional complex representation of $G$ and $\lambda^i \rho$ be the $i$-th exterior power of $\rho$. Since the restriction of $\lambda^i \rho$ to the center of $G$ is trivial clearly, it factors through the canonical projection $\pi: \rightarrow PG$. So we view $\lambda^i \rho$ also as a representation of $PG$ below. Moreover, as is well known, an element of $K^{-1}(PG)$ is represented as the homotopy class of a map from $PG$ to the infinite dimensional unitary group $U$. Hence we see that $\lambda^i \rho$ yields naturally an element $\beta(\lambda^i \rho)$ of $K^{-1}(PG)$, which is called the $\beta$-construction of $\lambda^i \rho$ [8]. Because $\dim \mathbb{C} \lambda^i \rho = \binom{2^{4n+1}}{2i+1}$ and $2^{4n+1} \parallel \binom{2^{4n+1}}{2i+1}$, $d_{2i+1} = \frac{\binom{2^{4n+1}}{2i+1}}{2^{4n+1}}$ is odd. Let $\ell \rho$ denote the direct sum of $\ell$ copies of $\rho$. The map $PG \rightarrow U \left( \binom{2^{4n+1}}{2i+1} \right)$ given by the assignment $\pi(g) \mapsto (d_{2i+1} \rho(g))\lambda^{2i+1} \rho(g)$ defines a similar element $\beta(d_{2i+1} \rho + \lambda^{2i+1} \rho)$ of $K^{-1}(PG)$.

We describe explicitly the image of $\beta(\rho) \in K^{-1}(G)$ by the transfer map $\pi_*: K^{-1}(G) \rightarrow K^{-1}_\Gamma(G) = K^{-1}(PG)$. Let us view $E = G \times (\mathbb{C}^{2^{4n+1}} \oplus C^{2^{4n+1}})$ as a product $\Gamma$-vector bundle over $G$ provided with the $\Gamma$-action given by $(g, u, v) \mapsto (-g, v, u)$ for $g \in G, u, v \in \mathbb{C}^{2^{4n+1}}$. Then the assignment $(g, u, v) \mapsto (g, \rho(g)u, -\rho(g)v)$ gives an equivariant bundle automorphism of $E$. In a canonical way this gives rise to an element of $K^{-1}_\Gamma(G)$ which is just $\pi_*(\beta(\rho))$ and is written $\beta(\rho, \Gamma)$ below.

Then we have

**Theorem 2.3 ([7,9]).** With the notation as above
\[ K^*(PSp(2^{4n})) = Z[\sigma]/(2^{4n+1}\sigma, \sigma^2 + 2\sigma) \]
\[ \otimes \Lambda(\beta(d_{2i-1}\rho + \lambda^{2i-1}\rho), \beta(\lambda^{2j}\rho), \beta(\rho, \Gamma)) \]
\[ (2 \leq i \leq 2^{4n-1}, 1 \leq j \leq 2^{4n-1}))/I \]

as a ring where \( I \) is the ideal generated by
\[ \sigma\beta(\rho, \Gamma). \]

**Proof.** We observe the exact sequence of (1.4b). According to [8]
\[ K^*(G) = \Lambda(\beta(\rho), \beta(\lambda^2\rho), \cdots, \beta(\lambda^{2n}\rho)). \]

Since \( K^*(G) \) is torsion-free we have the Künneth isomorphism
\[ K^*(P^{8n+3} \times G) \cong K^*(P^{8n+3}) \otimes K^*(G). \]

Then we get similarly to (2.1) the following.

(2.4) \[ \delta(\nu \times 1) = \sigma + 2, \quad J(1) = -2^{4n+1}\sigma \quad \text{and} \quad I(\sigma) = \sigma + 1. \]

Now \( 2^{4n+1}\sigma = 0 \) follows because of \( \rho(-1) = -I \). Hence (1.4b) becomes a short exact sequence
\[ 0 \rightarrow K^*(PG) \overset{I}{\rightarrow} K^*(P^{8n+3} \times G) \rightarrow \delta K^*(PG) \rightarrow 0 \]
provided with \( \delta(xI(y)) = \delta(x)y \). Further by inspecting definition we have
\[ I(\beta(\lambda^{2i}\rho)) = 1 \times \beta(\lambda^{2i}\rho), \]
\[ I(\beta(d_{2i-1}\rho + \lambda^{2i-1}\rho)) = (\sigma + 1) \times d_{2i-1}\beta(\rho) + 1 \times \beta(\lambda^{2i-1}\rho) + d_{2i-1}\nu \times 1, \]
\[ I(\beta(\rho, \Gamma)) = (\sigma + 2) \times \beta(\rho) + \nu \times 1, \]
\[ \delta(1 \times \beta(\rho)) = -1. \]

Let \( R \) denote the ring on the right-hand side of the equality of the theorem. Using the last formula of (2.4) and the first three formulas of (2.5), the injectivity of \( I \) shows that \( R \) is a subring of \( K^*(PG) \).

To prove the theorem it therefore suffices to verify that \( \text{Im} \delta = R \) since \( \delta \) is surjective. The images of generators of \( K^*(P^{8n+3} \times G) \) as a
module by $\delta$ can be calculated by using (2.5) together with the equality $\delta(xI(y)) = \delta(x)y$. For example, we have
\[
\begin{align*}
\delta(1 \times \beta(\lambda^{2i-1}\rho)) &= -d_{2i-1}(\sigma + 1), \\
\delta(\nu \times 1) &= -(\sigma + 2), \\
\delta(\nu \times \beta(\rho)) &= \beta(\rho, \Gamma), \\
\delta(1 \times \beta(\rho)\beta(\lambda^{2i-1}\rho)) &= -\beta(d_{2i-1}\rho + \lambda^{2i-1}\rho) - d_{2i-1}\beta(\rho, \Gamma).
\end{align*}
\]
Thus by repeating such a computation inductively we get $\text{Im} \delta = R$, which completes our proof.

3. In this section we compute $KO^*(PG)$. First we consider the exact sequence (1.4b) for $KO$-theory. The complex representation $\rho$ of $G$ is, of course, the complexification of the $2^{4n}$-dimensional quaternionic representation, for which we write $\bar{\rho}$. Clearly $\bar{\rho}$ yields an isomorphism $G \times H^{2^{4n}} \otimes_R H \cong G \times H^{2^{4n}}$ of $\Gamma$-quaternionic vector bundles over $G$. Now we have $J(1) = 2^{4n-1}\eta_4\bar{\rho}$ similarly to the 2nd formula of (2.1) and also $\alpha \otimes_C H = \eta_4$ by (1.2). Hence we see that $J(1) = 0$, so that (1.4b) becomes a short exact sequence
\[(3.1) \quad 0 \rightarrow KO^*(PG) \overset{1}{\rightarrow} KO^*(P^{8n+3} \times G) \overset{\delta}{\rightarrow} KO^*(PG) \rightarrow 0\]
provided with $\delta(xI(y)) = \delta(x)y$.

Using this exact sequence we proceed as the same way as for $K^*(PG)$.

Let $\lambda_C^{2i}\bar{\rho}$ be the exterior power $\bar{\rho} \wedge_C \cdots \wedge_C \bar{\rho}$ of $\bar{\rho}$ over $C$. Then in general $\lambda_C^{2i}\bar{\rho}$ is quaternionic. But if $k$ is even then it has a natural Real structure. So we consider $\lambda_C^{2i}\bar{\rho}$ to be real. By the $\beta$-construction we have
\[
\beta(\lambda_C^{2i-1}\bar{\rho}) \in \overline{KO}^{-1}(G) \quad \text{and} \quad \beta(\lambda_C^{2i}\bar{\rho}) \in \overline{KO}^{-1}(G)
\]
and we set
\[
\bar{\beta}(\lambda_C^{2i-1}\bar{\rho}) = \alpha \wedge_C \beta(\lambda_C^{2i-1}\bar{\rho}) \in \overline{KO}^{-1}(\Sigma^{0,4} \wedge G) = \overline{KO}^{-5}(G).
\]
Then, according to [15], Theorem 5.6,
\[(3.2) \quad KO^*(G) = \Lambda KO^*(+)\langle \bar{\beta}(\lambda_C^{2i-1}\bar{\rho}), \beta(\lambda_C^{2i}\bar{\rho}) \rangle \quad (1 \leq i \leq 2^{4n-1})
\]
as a $KO^*(+)$-module. Further by [4], §6 and [13], Corollary 2.3 we see that its generators satisfy the relations
\[(3.3) \quad \bar{\beta}(\lambda_C^{2i-1}\bar{\rho})^2 = \eta_1\beta(\lambda_C^{4i-2}\bar{\rho}), \quad \beta(\lambda_C^{2i}\bar{\rho})^2 = \eta_1\beta(\lambda_C^{4i}\bar{\rho}).
\]
Here we note that $\lambda_C^{k} \tilde{\rho} = \lambda_C^{2^{k+1}-k} \tilde{\rho}$ for $1 \leq k \leq 2^{4n}$. Of course this equality holds for $\lambda_C^{2^{k}} \tilde{\rho}$ viewed as a representation of $PG$.

Because $KO^*(G)$ is torsion-free, there holds the Künneth isomorphism $KO^*(P^{8n+3} \times G) \cong KO^*(P^{8n+3}) \otimes_{KO^*(+)} KO^*(G)$. Therefore by using (2.2b), (3.2) and (3.3), the multiplicative structure of $KO^*(P^{8n+3} \times G)$ centered in the sequence (3.1) can be described explicitly.

In order to state our theorem we provide generators of $KO^*(PG)$. Similarly to the complex case we have

$$\beta(d_{2i-1} \tilde{\rho} + \lambda_C^{2i-1} \tilde{\rho}), \quad \beta(\tilde{\rho}, \Gamma) \in \overline{KSp}^{-1}(PG) \quad \text{and}$$

$$\beta(\lambda_C^{2i} \tilde{\rho}) \in \overline{KO}^{-1}(PG)$$

and so we set

$$\tilde{\beta}(d_{2i-1} \tilde{\rho} + \lambda_C^{2i-1} \tilde{\rho}) = \alpha \wedge_C \beta(d_{2i-1} \tilde{\rho} + \lambda_C^{2i-1} \tilde{\rho}),$$

$$\tilde{\beta}(\tilde{\rho}, \Gamma) = \alpha \wedge_C \beta(\tilde{\rho}, \Gamma) \in \overline{KO}^{-5}(PG).$$

Moreover we see that

(3.4) There exists an element $\tilde{\zeta} \in KO^{-6}(PG)$ such that

$$I(\tilde{\zeta}) = \eta_1 \times \tilde{\beta}(\tilde{\rho}) + \zeta \times 1.$$

This is shown below.

Then we obtain the following.

**Theorem 3.5.** With the notation as above

$$KO^*(PSp(2^{4n})) = \mathbb{Z}[(\tilde{\sigma})/(\tilde{\sigma}^2 + 2\tilde{\sigma}) \otimes E \otimes \Lambda_{Z/2}(\tilde{\zeta})]/I$$

as a ring where $E$ is a $KO^*(+)$-module

$$\Lambda_{KO^*(+)}(\tilde{\beta}(d_{2i-1} \tilde{\rho} + \lambda_C^{2i-1}), \beta(\lambda_C^{2i} \tilde{\rho}), \tilde{\beta}(\tilde{\rho}, \Gamma))$$

$$\quad (2 \leq i \leq 2^{4n-1}, 1 \leq j \leq 2^{4n-1})$$

with the relations

$$\tilde{\beta}(d_{2i-1} \tilde{\rho} + \lambda_C^{2i-1} \tilde{\rho})^2 = \eta_1(\beta(\lambda_C^{2i} \tilde{\rho}) + \beta(\lambda_C^{4i-2} \tilde{\rho})),$$

$$\beta(\lambda_C^{2i} \tilde{\rho})^2 = \eta_1 \beta(\lambda_C^{4i} \tilde{\rho}),$$

$$\tilde{\beta}(\tilde{\rho}, \Gamma)^2 = 0$$
and \(I\) is the ideal generated by
\[
2^{4n}\bar{\sigma}\eta_4, \quad \bar{\sigma}\bar{\beta}(\bar{\rho}, \Gamma), \quad \eta_4\bar{n}, \quad \bar{\sigma}\bar{\zeta} - \eta_1\beta(\bar{\rho}, \Gamma), \quad \eta_1^2\bar{\zeta} - 2^{4n+1}\bar{\sigma}
\]
(the \(\otimes\)'s are omitted).

Proof. Observe (3.1). By looking at the definitions of the maps and elements we have

(i) \(I(\bar{\sigma}) = \bar{\sigma} \times 1,\)
(ii) \(I(\beta(\lambda^2_2\bar{\rho})) = 1 \times \beta(\lambda^2_2\bar{\rho}),\)
(iii) \(I(\bar{\beta}(d_{2i-1}\bar{\rho} + \lambda^{2i-1}_C\bar{\rho})) = (\bar{\sigma} + 1) \times d_{2i-1}\bar{\beta}(\bar{\rho}) + 1 \times \bar{\beta}(\lambda^{2i-1}_C\bar{\rho}) + d_{2i-1}\bar{\nu} \times 1,\)
(iv) \(I(\bar{\beta}(\bar{\rho}, \Gamma)) = (\bar{\sigma} + 2) \times \bar{\beta}(\bar{\rho}) + \bar{\nu} \times 1,\)
(v) \(I(1 \times \bar{\beta}(\bar{\rho})) = -1,\)
(vi) \(\delta(\bar{\nu} \times 1) = (\bar{\sigma} + 2) \times 1,\)
(vii) \(\delta(\zeta \times 1) = \eta_1.\)

(3.4) is immediate from (v) and (vii). Let \(\bar{R}\) denote the ring on the right-hand side of the equality of Theorem 3.5. Then using (i)–(iv) and (3.4) we see that \(\bar{R} \subset KO^*(PG)\) because of the injectivity of \(I\), and by using (v)–(vii) and the equality \(\delta(xI(y)) = \delta(x)y\) in addition we can verify that \(\bar{R}\) fills \(KO^*(PG)\) because of the surjectivity of \(\delta\). This completes the proof of the theorem.

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