On the Ko-theory of Lie Groups and Symmetric Spaces

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ON THE KO-THEORY OF LIE GROUPS
AND SYMMETRIC SPACES

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0. Introduction. Let $G$ be a compact, 1-connected simple Lie group of rank 2. Then $G$ is one of the following: $SU(3)$, $Sp(2)$ and $G_2$. In this note we shall describe explicitly the $KO$-theory of $G$, together with the action of the Adams operations $\psi^k$ on it, and also describe the $KO$-theory of symmetric spaces $SU(2n + 1)/SO(2n + 1)$ and $SU(2n)/Sp(n)$ for $n \geq 1$. In particular, for the first topic, the following fact should be noted. For a compact, connected, semisimple and simply-connected Lie group $G$, Seymour [16, Theorem 5.6] described theoretically the $(\mathbb{Z}/(8)$-graded) ring $KO^*(G)$; its additive structure was determined completely and its multiplicative structure was almost done. However, it seems that papers containing an explicit description of $KO^*(G)$ are [13], [14] and [15].

This paper is arranged as follows. In section 1 we compute the Adams operations in $K^*(G)$. Section 2 consists of preparations for subsequent sections and involves a review of Seymour’s work. The $(\mathbb{Z}$-graded) ring $KO^*(G)$ will be described in section 3, and the rings $KO^*(SU(2n + 1)/SO(2n + 1))$ and $KO^*(SU(2n)/Sp(n))$ in section 4.

We shall deal with the $\mathbb{Z}$-graded objects, simultaneously with the associated $\mathbb{Z}/(2)$- or $\mathbb{Z}/(8)$-objects.

1. The Adams operations in $K^*(G)$. Since the Chern character of $G$ was described explicitly in [17], the Adams operations in $K^*(G)$ should be computed. This is what we put into practice in this section.

We begin by recalling some facts on complex $K$-theory needed in the sequel. For details, see [2], [3] and [9]. We will use the following notation: $R$ is the field of real numbers; $C$ is the field of complex numbers; $H$ is the algebra of quaternions; $K$ is the algebra of Cayley numbers. Let $X$ be a space with nondegenerate base point. The Adams operations $\psi^k: K(X) \to K(X)$, $k \in \mathbb{Z}$, are homomorphisms of rings. They are closely related with the Chern character $ch: K(X) \to H^*(X; \mathbb{Q})$. That is, if

$$ch(x) = \sum_{q \geq 0} ch_q(x), \quad ch_q(x) \in H^{2q}(X; \mathbb{Q}),$$

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for $x \in K^*(X)$, then

$$\text{ch}(\psi^k(x)) = \sum_{q \geq 0} k^q \text{ch}_q(x).$$

This $\text{ch}$ extends to a multiplicative natural transformation of $\mathbb{Z}/(2)$-graded cohomology theories $\text{ch}: K^*(\ ) \to H^*(\ ; \mathbb{Q})$. The coefficient ring of reduced $\mathbb{Z}$-graded $K$-theory is

$$\tilde{K}^*(S^0) = \mathbb{Z}[g, g^{-1}]/(gg^{-1} - 1),$$

where $g \in \tilde{K}^{-2}(S^0)$ is the Bott generator. The action of $\psi^k$ on $\tilde{K}^*(S^0)$ is given by

$$\psi^k(g) = k g, \quad \text{in particular} \quad \psi^{-1}(g) = -g.$$  

The $K$-ring $K^*(X)$ and the complex representation ring $R(G)$ are $\lambda$-rings (see [9, 12(1.1)]); roughly speaking, they possess the exterior power operations $\lambda^k$ for $k \geq 0$. Let $\beta: R(G) \to \tilde{K}^{-1}(G)$ be the homomorphism of abelian groups, introduced in [8], called the beta-construction. Notice that $\beta$ does not commute with $\lambda^k$.

We now consider the case $G = SU(3)$. The space $C^3$ becomes a $SU(3)$-$C$-module in the usual way. We write $\lambda_1$ for the class of $C^3$ in $R(SU(3))$, and put $\lambda_k = \lambda^k(\lambda_1) \in R(SU(3))$. Then $R(SU(3))$ equals the polynomial algebra $\mathbb{Z}[\lambda_1, \lambda_2]$ (see [1, Theorem 7.4] or [9, 13(3.1)]). Therefore, by the theorem of Hodgkin [8, Theorem A], $K^*(SU(3))$ equals the exterior algebra $\Lambda_{\mathbb{Z}}(\beta(\lambda_1), \beta(\lambda_2))$ as a $\mathbb{Z}/(2)$-graded Hopf algebra over $\mathbb{Z}$. On the other hand, $H^*(SU(3); \mathbb{Z}) = \Lambda_{\mathbb{Z}}(x_3, x_5)$, where $x_i \in H^i(SU(3); \mathbb{Z})$. With this notation, we may set

$$\text{ch}(\beta(\lambda_1)) = ax_3 + bx_5,$$
$$\text{ch}(\beta(\lambda_2)) = cx_3 + dx_5$$

for some $a, b, c, d \in \mathbb{Q}$ (as seen below, these numbers are known). Using the relations $x_3^2 = 0$, $x_5 x_3 = -x_3 x_5$ and $x_5^2 = 0$, we have

$$\text{ch}(\beta(\lambda_1) \beta(\lambda_2)) = ad x_3 x_5 + bc x_5 x_3$$
$$= (ad - bc)x_3 x_5$$

in the $\mathbb{Z}/(2)$-graded ring $H^{**}(SU(3); \mathbb{Q})$. Since $\{\beta(\lambda_1), \beta(\lambda_2)\}$ is a basis for $\tilde{K}^{-1}(SU(3)) = \tilde{K}(\Sigma SU(3)) \cong \mathbb{Z} \oplus \mathbb{Z}$, we may set

$$\psi^k(\beta(\lambda_1)) = e \beta(\lambda_1) + f \beta(\lambda_2)$$
for some $e, f \in \mathbb{Z}$. Let $s : \tilde{H}^*(G; \mathbb{Q}) \to \tilde{H}^{*+1}(\Sigma G; \mathbb{Q})$ denote the suspension isomorphism. Then, by (1.3), $ch: \tilde{K}(\Sigma SU(3)) \to \tilde{H}^*(\Sigma SU(3); \mathbb{Q})$ satisfies

\[
ch(\beta(\lambda_1)) = as(x_3) + bs(x_5),
\]

\[
ch(\beta(\lambda_2)) = cs(x_3) + ds(x_5).
\]

It follows from this and (1.1) that

\[
ch(\psi^k(\beta(\lambda_1))) = ak^2s(x_3) + bk^3s(x_5),
\]

while

\[
ch(\psi^k(\beta(\lambda_2))) = ch(e\beta(\lambda_1) + f\beta(\lambda_2)) = (ae + cf)s(x_3) + (be + df)s(x_5).
\]

Since $a = -1, b = 1/2, c = -1$ and $d = -1/2$ by [17, Theorem 2], we obtain $-e - f = -k^2$ and $e - f = k^3$. These reduce to $e = k^2(k+1)/2$ and $f = -k^2(k-1)/2$. We call the reader's attention to the fact that these numbers belong to $\mathbb{Z}$ (see also part (i) of Theorems 1 to 3).

Similarly, if we set

\[
\psi^k(\beta(\lambda_2)) = e\beta(\lambda_1) + f\beta(\lambda_2)
\]

for some $e, f \in \mathbb{Z}$, we obtain $-e - f = -k^2$ and $e - f = -k^3$, which reduce to $e = -k^2(k-1)/2$ and $f = k^2(k+1)/2$.

Using these results and the relations $\beta(\lambda_1)^2 = 0$, $\beta(\lambda_2)\beta(\lambda_1) = -\beta(\lambda_1)\beta(\lambda_2)$ and $\beta(\lambda_2)^2 = 0$, we have

\[
\psi^k(\beta(\lambda_1)\beta(\lambda_2)) = \psi^k(\beta(\lambda_1))\psi^k(\beta(\lambda_2)) = \frac{k^4(k+1)^2}{4}\beta(\lambda_1)\beta(\lambda_2) + \frac{k^4(k-1)^2}{4}\beta(\lambda_2)\beta(\lambda_1)
\]

\[= k^5\beta(\lambda_1)\beta(\lambda_2).
\]

**Theorem 1.** The action of $\psi^k$ on $\tilde{K}^*(SU(3))$ is given by:

(i) ([18, (2.5)]) In $\tilde{K}^{-1}(SU(3)) = \mathbb{Z}\{\beta(\lambda_1), \beta(\lambda_2)\}$,

\[
\psi^k(\beta(\lambda_1)) = \frac{k^2(k+1)}{2}\beta(\lambda_1) - \frac{k^2(k-1)}{2}\beta(\lambda_2),
\]

\[
\psi^k(\beta(\lambda_2)) = -\frac{k^2(k-1)}{2}\beta(\lambda_1) + \frac{k^2(k+1)}{2}\beta(\lambda_2).
\]
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(ii) In \( \overline{K}^{-2}(SU(3)) = \mathbb{Z}\{\beta(\lambda_1)\beta(\lambda_2)\} \),

\[ \psi^k(\beta(\lambda_1)\beta(\lambda_2)) = k^5\beta(\lambda_1)\beta(\lambda_2). \]

We move on to consider the case \( G = Sp(2) \). Since the space \( H^2 \) is a left \( Sp(2) \)-\( H \)-module, its complex restriction \( (H^2)_C \) becomes a \( Sp(2) \)-\( C \)-module. We write \( \mu'_1 = [(H^2)_C] \in R(Sp(2)) \), and put \( \mu'_k = \lambda^k(\mu'_1) \). Then \( R(Sp(2)) \) equals the polynomial algebra \( \mathbb{Z}[\mu'_1, \mu'_2] \) (see [1, Theorem 7.6] or [9, 13(6.1)]). Therefore, by the theorem of Hodgkin [8], \( K^*(Sp(2)) = \Lambda_{\mathbb{Z}}(\beta(\mu'_1), \beta(\mu'_2)) \). On the other hand, \( H^*(Sp(2); \mathbb{Z}) = \Lambda_{\mathbb{Z}}(x_3, x_7) \), where \( x_i \in H^i(Sp(2); \mathbb{Z}) \). With this notation, [17, Theorem 3] tells us that

\[ ch(\beta(\mu'_1)) = x_3 - \frac{1}{6}x_7, \]
\[ ch(\beta(\mu'_2)) = 2x_3 + \frac{2}{3}x_7. \]

By a calculation similar to the case of \( SU(3) \) we have

**Theorem 2.** The action of \( \psi^k \) on \( \overline{K}^*(Sp(2)) \) is given by:

(i) In \( \overline{K}^{-1}(Sp(2)) = \mathbb{Z}\{\beta(\mu'_1), \beta(\mu'_2)\} \),

\[ \psi^k(\beta(\mu'_1)) = \frac{k^2(k^2+2)}{3}\beta(\mu'_1) - \frac{k^2(k^2-1)}{6}\beta(\mu'_2), \]
\[ \psi^k(\beta(\mu'_2)) = -\frac{4k^2(k^2-1)}{3}\beta(\mu'_1) + \frac{k^2(2k^2+1)}{3}\beta(\mu'_2). \]

(ii) In \( \overline{K}^{-2}(Sp(2)) = \mathbb{Z}\{\beta(\mu'_1)\beta(\mu'_2)\} \),

\[ \psi^k(\beta(\mu'_1)\beta(\mu'_2)) = k^6\beta(\mu'_1)\beta(\mu'_2). \]

Finally we consider the case \( G = G_2 \). The automorphism group of \( K \) is \( G_2 \). As seen in [19, Appendix A] or [20, p.217], the subspace \( K_0 \) consisting of pure imaginary elements in \( K \) forms a \( G_2 \)-\( R \)-module of dimension 7. Hence its complexification \( K_0^C \) becomes a \( G_2 \)-\( C \)-module. We write \( \rho_1 = [K_0^C] \in R(G_2) \), and put \( \rho_k = \lambda^k(\rho_1) \). Then \( R(G_2) \) equals the polynomial algebra \( \mathbb{Z}[\rho_1, \rho_2] \). Therefore, by the theorem of Hodgkin [8], \( K^*(G_2) = \Lambda_{\mathbb{Z}}(\beta(\rho_1), \beta(\rho_2)) \). On the other hand, \( H^*(G_2; \mathbb{Z})/\text{Tor.} = \Lambda_{\mathbb{Z}}(x_3, x_{11}) \), where \( x_i \in H^i(G_2; \mathbb{Z}) \). With this notation, [17, Theorem 7] tells us that

\[ ch(\beta(\rho_1)) = 2x_3 + \frac{1}{60}x_{11}, \]
\[ ch(\beta(\rho_2)) = 10x_3 - \frac{5}{12}x_{11}. \]
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By a calculation similar to the case of $SU(3)$ we have

**Theorem 3.** The action of $\psi^k$ on $\widetilde{K}^*(G_2)$ is given by:

(i) In $\widetilde{K}^{-1}(G_2) = \mathbb{Z}\{\beta(\rho_1), \beta(\rho_2)\}$,

$$\psi^k(\beta(\rho_1)) = \frac{k^2(k^4 + 5)}{6} \beta(\rho_1) - \frac{k^2(k^4 - 1)}{30} \beta(\rho_2),$$

$$\psi^k(\beta(\rho_2)) = -\frac{25k^2(k^4 - 1)}{6} \beta(\rho_1) + \frac{k^2(5k^4 + 1)}{6} \beta(\rho_2).$$

(ii) In $\widetilde{K}^{-2}(G_2) = \mathbb{Z}\{\beta(\rho_1)\beta(\rho_2)\}$,

$$\psi^k(\beta(\rho_1)\beta(\rho_2)) = k^8 \beta(\rho_1)\beta(\rho_2).$$

2. The Bott’s exact sequence. We will use the quaternionic representation ring functor $RSp( )$ and quaternionic $K$-theory $KSp^*( )$ as well as the real representation ring functor $RO( )$ and real $K$-theory $KO^*( )$. To do this we fix some notation. For details, see [1], [2], [4] and [9].

Let $c: RO(G) \rightarrow R(G)$ and $c: KO(X) \rightarrow K(X)$ be the complexifications; they preserve the $\lambda$-ring structures. Let $r: R(G) \rightarrow RO(G)$ and $r: K(X) \rightarrow KO(X)$ be the real restrictions; they preserve the additions only. Let $t: R(G) \rightarrow R(G)$ and $t: K(X) \rightarrow K(X)$ be the complex conjugations; they preserve the $\lambda$-ring structures. Finally, let $c': RSp(G) \rightarrow R(G)$ and $c': KSp(X) \rightarrow K(X)$ be the complex restrictions; they preserve the additions only. Then, among them, the following formulas hold:

\[(2.1a) \quad rc = 2, \quad cr = 1 + t, \quad tc = c, \quad tc' = c' \quad \text{and} \quad t^2 = 1.\]

\[(2.1b) \quad r(x \cdot c(z)) = r(x)z \quad \text{for all} \quad x \in K(X) \quad \text{and} \quad z \in KO(X).\]

The coefficient ring of reduced $\mathbb{Z}$-graded $KO$-theory is

$$\widetilde{KO}^*(S^0) = \mathbb{Z}[\eta, \nu, \sigma, \sigma^{-1}] / (2\eta, \eta^3, \eta \nu, \nu^2 - 4\sigma, \sigma \sigma^{-1} - 1),$$

where $\eta \in \widetilde{KO}^{-1}(S^0)$, $\nu \in \widetilde{KO}^{-4}(S^0)$ and $\sigma \in \widetilde{KO}^{-8}(S^0)$. When $\widetilde{K}^*(X)$ has been determined (our case is this), a basic tool for computing $\widetilde{KO}^*(X)$ is the Bott’s exact sequence

$$\cdots \rightarrow \widetilde{KO}^{1-q}(X) \overset{\eta}{\rightarrow} \widetilde{KO}^{-q}(X) \overset{c}{\rightarrow} \widetilde{K}^{-q}(X) \overset{\epsilon}{\rightarrow} \widetilde{KO}^{-q-2}(X) \rightarrow \cdots,$$
where \( \eta \) denotes multiplication by \( \eta \) and the map \( \delta \) is defined by

\[
\delta(x) = r(g^{-1}x) \quad \text{for} \quad x \in \bar{K}^{-q}(X).
\]

We have

\[
\begin{align*}
(2.3a) \quad c(1) &= 1, \quad c(\eta) = 0, \quad c(\eta^2) = 0, \quad c(\nu) = 2g^2 \quad \text{and} \quad c(\sigma) = g^4. \\
(2.3b) \quad r(1) &= 2, \quad r(g) = \eta^2, \quad r(g^2) = \nu, \quad r(g^3) = 0 \quad \text{and} \quad r(g^4) = 2\sigma.
\end{align*}
\]

From this and (2.1b), one can calculate \( r(g^i) \) for all \( i \in \mathbb{Z} \). The action of \( \psi^k \) on \( \bar{K}^*(S^0) \) is given by

\[
(2.4) \quad \psi^k(\eta) = k\eta, \quad \psi^k(\nu) = k^2\nu \quad \text{and} \quad \psi^k(\sigma) = k^4\sigma.
\]

The following two lemmas can be proved by using the Bott's exact sequence and are included in [16, Theorem 4.2]. So we omit the details of their proofs. By \( \bar{K}^*(S^0) \{x, \cdots\} \) we denote the free \( \bar{K}^*(S^0) \)-module generated by elements \( x, \cdots \).

**Lemma 4.** Suppose that, as a \( \mathbb{Z} \)-graded module with \( t \)-action, \( \bar{K}^*(X) \) has a direct summand \( T(x) \) which is a free \( \bar{K}^*(S^0) \)-module generated by two elements \( x, x' \in \bar{K}^n(X) \) such that \( t(x) = x' \) (and so \( t(x') = x \)), where \( x' \neq \pm x \). Then \( \bar{K}^*(X) \) contains the image \( r(\bar{K}^*(S^0)\{x\}) \) of \( \bar{K}^*(S^0)\{x\} \) under \( r \) as a direct summand. It is described by:

\[
\begin{align*}
\bar{K}^0(X) &\supset \mathbb{Z}\{r(x)\}, & \bar{K}^{n-1}(X) &\supset 0, \\
\bar{K}^{n-2}(X) &\supset \mathbb{Z}\{r(gx)\}, & \bar{K}^{n-3}(X) &\supset 0, \\
\bar{K}^{n-4}(X) &\supset \mathbb{Z}\{r(g^2x)\}, & \bar{K}^{n-5}(X) &\supset 0, \\
\bar{K}^{n-6}(X) &\supset \mathbb{Z}\{r(g^3x)\}, & \bar{K}^{n-7}(X) &\supset 0
\end{align*}
\]

where \( r(g^ix') = (-1)^ir(g^ix) \) for \( i \in \mathbb{Z} \).

**Proof.** We prove the last relation only.

\[
\begin{align*}
0 &= \delta c(r(g^{i+1}x)) \quad \text{by exactness of Bott's exact sequence} \\
&= \delta((1 + t)(g^ix)) \quad \text{since} \ c_r = 1 + t \text{ by (2.1a)} \\
&= \delta(g^{i+1}x + (-1)^{i+1}g^{i+1}x') \quad \text{since} \ t(g) = -g \text{ by (1.2)} \\
&= r(g^ix) + (-1)^{i+1}r(g^ix') \quad \text{by (2.2)}.
\end{align*}
\]
Lemma 5. Suppose that, as a $\mathbb{Z}$-graded module with $t$-action, $\widetilde{K}^*(X)$ has a direct summand $N(y)$ which is a free $\widetilde{K}^*(S^0)$-module generated by an element $y \in \widetilde{K}^n(X)$ such that $y = c(z)$ for some $z \in \widetilde{KO}^n(X)$. Then $\widetilde{KO}^*(X)$ contains the free $KO^*(S^0)$-module $\widetilde{KO}^*(S^0)\{z\}$ generated by $z$ as a direct summand. It is described by:

$$
\begin{align*}
\widetilde{KO}^n(X) & \supset \mathbb{Z}\{z\}, & \widetilde{KO}^{n-1}(X) & \supset \mathbb{Z}/(2)\{\eta z\}, \\
\widetilde{KO}^{n-2}(X) & \supset \mathbb{Z}/(2)\{\eta^2 z\}, & \widetilde{KO}^{n-3}(X) & \supset 0, \\
\widetilde{KO}^{n-4}(X) & \supset \mathbb{Z}\{\nu z\}, & \widetilde{KO}^{n-5}(X) & \supset 0, \\
\widetilde{KO}^{n-6}(X) & \supset 0, & \widetilde{KO}^{n-7}(X) & \supset 0.
\end{align*}
$$

Instead of Lemma 5, we show the following which we will often use.

Lemma 6. Suppose that $\widetilde{K}^*(X)$ is the free $\widetilde{K}^*(S^0)$-module generated by $m$ elements $b_1, b_2, \ldots, b_m$, $b_i \in \widetilde{K}^n_i(X)$, satisfying $t(b_i) = b_i$. Suppose further that there exist elements $a_1, a_2, \ldots, a_m$, $a_i \in \widetilde{KO}^{n_i}(X)$, satisfying $c(a_i) = b_i$. Then $\widetilde{KO}^*(X)$ is the free $\widetilde{KO}^*(S^0)$-module generated by $a_1, a_2, \ldots, a_m$.

Proof. (Note that, since $tc = c$ by (2.1a), $c(a_i) = b_i$ implies $t(b_i) = b_i.$) We use the machinery of exact couples (see [10]). An exact couple is an exact triangle of graded abelian groups.

$$
\begin{array}{ccc}
D & \overset{i}{\longrightarrow} & D \\
\downarrow{k} & & \downarrow{j} \\
E & \leftarrow & \langle i, j \rangle
\end{array}
$$

Then $d = jk: E \rightarrow E$ satisfies $d^2 = 0$, and there is another exact couple (the derived couple)

$$
\begin{array}{ccc}
D' & \overset{i'}{\longrightarrow} & D \\
\downarrow{k'} & & \downarrow{j'} \\
E' & \leftarrow & \langle i', j' \rangle
\end{array}
$$

where $D' = i(D)$, $E' = \text{Ker}d/\text{Im}d$, $i'$ is induced by $i$, $k'$ is induced by $k$, and $j'(i(a)) = [j(a)]$ for $a \in D$.

The Bott's exact sequence yields an exact couple by setting $D = \widetilde{KO}^*(X)$, $E = \widetilde{K}^*(X)$, $i = \eta: \widetilde{KO}^*(X) \rightarrow \widetilde{KO}^{*-1}(X)$, $j = c: \widetilde{KO}^*(X) \rightarrow \widetilde{K}^*(X)$, and $k = b: \widetilde{K}^*(X) \rightarrow \widetilde{KO}^{*-2}(X)$. (With this notation, since
\( \eta^3 = 0 \), it follows that \( D'' = 0 \) and \( E'' = 0 \). For any \( b \in \overline{K}^*(X) \)

\[
d(b) = c6(b) \quad \text{by the definition of } d
\]
\[
= cr(g^{-1}b) \quad \text{by (2.2)}
\]
\[
= (1 + t)(g^{-1}b) \quad \text{since } cr = 1 + t.
\]

By the first hypothesis,

(2.5) \[ E = Z \{g^kb_i \mid i = 1, 2, \ldots, m; k \in Z \}. \]

We have

\[
d(g^kb_i) = g^{k-1}b_i + (-1)^{k-1}g^{k-1} t(b_i)
\]

since \( t(g) = -g \) and \( t \) is a ring homomorphism
\[
= (1 + (-1)^{k-1})g^{k-1}b_i \quad \text{since } t(b_i) = b_i.
\]

Therefore

\[
\text{Ker } d = Z \{g^{2k}b_i \mid i = 1, 2, \ldots, m; k \in Z \}
\]

and

\[
\text{Im } d = Z \{2g^{2k}b_i \mid i = 1, 2, \ldots, m; k \in Z \}.
\]

Hence

(2.6) \[ E' = Z/(2)\{[g^{2k}b_i] \mid i = 1, 2, \ldots, m; k \in Z \}. \]

Using the second hypothesis, we prove that the multiples of \( a_i \) by \( \eta, \eta^2, \nu \) and \( \sigma \) are not zero. First of all, \( \sigma a_i \neq 0 \), since \( \sigma: \overline{KO}^*(X) \to \overline{KO}^{*-\delta}(X) \)

is an isomorphism. Let us verify that \( \nu a_i \neq 0 \). We have

\[
c(\nu a_i) = c(\nu)c(a_i) \quad \text{since } c \text{ is a ring homomorphism}
\]
\[
= 2g^2b_i \quad \text{since } c(\nu) = 2g^2 \text{ by (2.3a) and } c(a_i) = b_i.
\]

Assume that \( \nu a_i = 0 \). Then \( 2g^2b_i = 0 \). This contradicts (2.5), and proves the assertion. Let us next verify that \( \eta a_i \neq 0 \). Consider the homomorphism

\( \delta: E \to D \), where \( E \) is as in (2.5). For \( j = 1, 2, 3, 4 \) and for \( k \in Z \), we have

\[
\delta(g^{4k+j}b_i) = r(g^{4k+j-1}b_i)
\]
\[
= r(g^{j-1}c(\sigma^ka_i)) \quad \text{since } c(\sigma) = g^4 \text{ by (2.3a)}
\]
\[
= r(g^{j-1})\sigma^ka_i \quad \text{by (2.1b)}.
\]
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By this and (2.3b), we find that the elements which may belong to \( \text{Im} \delta \) are \( 2\sigma^k a_i, \eta^2 \sigma^k a_i \) (this may be zero at this point of time) and \( \nu \sigma^k a_i \). Assume that \( \eta a_i = 0 \). Then \( a_i \in \text{Ker} \eta = \text{Im} \delta \). This contradicts the above observation, and proves the assertion. Hence \( \eta \sigma^k a_i \) becomes a nonzero element of \( D' \). Let us finally verify that \( \eta^2 a_i \neq 0 \). Consider the homomorphism \( \delta' : E' \rightarrow D' \), where \( E' \) is as in (2.6). For \( k \in \mathbb{Z} \) we have

\[
\begin{align*}
\delta'([g^{4k}b_i]) &= \delta(g^{4k}b_i) = r(g^3)\sigma^{k-1}a_i = 0 \quad \text{and} \\
\delta'([g^{4k+2}b_i]) &= \delta(g^{4k+2}b_i) = r(g)\sigma^k a_i = \eta^2 \sigma^k a_i.
\end{align*}
\]

Assume that \( \eta^2 a_i = 0 \). Then, on the one hand, \( \eta a_i \in \text{Ker} \eta' = \text{Im} \delta' \) and on the other hand, the above calculation implies that \( \text{Im} \delta' = 0 \). This is a contradiction, and proves the assertion. Hence \( \eta^2 \sigma^k a_i \) becomes a nonzero element of \( D' \).

Since \( \text{Im} \epsilon' = \text{Ker} \delta' \), we have an exact sequence

\[
0 \rightarrow D'/\text{Ker} \epsilon' \xrightarrow{\epsilon'} E' \xrightarrow{\delta'} \text{Im} \delta' \rightarrow 0.
\]

Since \( \text{Ker} \epsilon' = \text{Im} \eta' \) and \( \text{Im} \delta' = \text{Ker} \eta' \), it can be rewritten in the form

\[
0 \rightarrow \text{Coker} \eta' \xrightarrow{\epsilon'} E' \xrightarrow{\delta'} \text{Ker} \eta' \rightarrow 0,
\]

where \( E' \) is as in (2.6),

\[
c'(\eta \sigma^k a_i) = [c(\sigma^k a_i)] = [c(\sigma)^k c(a_i)] = [g^{4k}b_i]
\]

and (2.7) holds. So we conclude that

\[
\text{Coker} \eta' = \mathbb{Z}/(2)\{\eta \sigma^k a_i | i = 1, 2, \ldots, m; k \in \mathbb{Z}\}
\]

and

\[
\text{Ker} \eta' = \mathbb{Z}/(2)\{\eta^2 \sigma^k a_i | i = 1, 2, \ldots, m; k \in \mathbb{Z}\}.
\]

Consider the exact sequence

\[
0 \rightarrow \text{Ker} \eta' \rightarrow D' \xrightarrow{\eta'} D' \rightarrow \text{Coker} \eta' \rightarrow 0.
\]

Then, since \( 2\eta = 0 \) and \( \eta^3 = 0 \), it follows that

\[
D' = \mathbb{Z}/(2)\{\eta \sigma^k a_i, \eta^2 \sigma^k a_i | i = 1, 2, \ldots, m; k \in \mathbb{Z}\}.
\]
There is a short exact sequence

$$0 \rightarrow \text{Ker} \, c \rightarrow D \overset{c}{\rightarrow} \text{Im} \, c \rightarrow 0.$$ 

Since $\text{Ker} \, c = \text{Im} \, \eta = D'$ and $\text{Im} \, c = \text{Ker} \, \delta$, it can be rewritten in the form

$$0 \rightarrow D' \rightarrow D \overset{c}{\rightarrow} \text{Ker} \, \delta \rightarrow 0.$$ 

From a description of the behavior of $\delta$ given in the preceding paragraph, we see that

$$\text{Ker} \, \delta = \mathbb{Z} \{g^{4k} b_i, 2g^{4k+2} b_i | i = 1, 2, \ldots, m; k \in \mathbb{Z}\}.$$ 

Therefore $D \cong D' \oplus \text{Ker} \, \delta$, and since $c(\sigma^k a_i) = g^{4k} b_i$ and $c(\nu \sigma^k a_i) = 2g^{4k} b_i$, it follows from (2.8) that

$$D = \mathbb{Z}/(2) \{\eta \sigma^k a_i, \eta^2 \sigma^k a_i | i = 1, 2, \ldots, m; k \in \mathbb{Z}\} \oplus \mathbb{Z} \{\sigma^k a_i, \nu \sigma^k a_i | i = 1, 2, \ldots, m; k \in \mathbb{Z}\}.$$ 

Thus the proof is completed.

Let $G$ be a compact connected Lie group. Recall from [1, Proposition 3.27] that $c: RO(G) \rightarrow R(G)$ and $c': RSp(G) \rightarrow R(G)$ are injective. A representation $\mu$ of $G$ is said to be self-conjugate if $t(\mu) = \mu$. Similarly, $\mu$ is said to be real if it lies in the image of $c: RO(G) \rightarrow R(G)$, and $\mu$ is said to be quaternionic if it lies in the image of $c': RSp(G) \rightarrow R(G)$. According to [1, Proposition 3.56], if $\mu$ is irreducible and self-conjugate, it is either real or quaternionic, but not both. The following is a collection of results from [1, Chapter 7] and [20, Chapter 5]:

**Proposition 7.** For $G = SU(3)$, $Sp(2)$ and $G_2$, the action of $t$ on $R(G)$ is given by:

1. In $R(SU(3)) = \mathbb{Z}[\lambda_1, \lambda_2]$,

$$t(\lambda_1) = \lambda_2 \quad \text{and} \quad t(\lambda_2) = \lambda_1.$$ 

2. In $R(Sp(2)) = \mathbb{Z}[\mu_1', \mu_2']$,

$$t(\mu_i') = \mu_i' \quad \text{for} \quad i = 1, 2$$

where $\mu_1'$ is quaternionic and $\mu_2'$ is real.
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(3) In \( R(G_2) = \mathbb{Z}[\rho_1, \rho_2] \),
\[
t(\rho_i) = \rho_i \quad \text{for} \quad i = 1, 2
\]
where both \( \rho_1 \) and \( \rho_2 \) are real.

Since \( \beta \) commutes with \( t \), from Proposition 7 we know the action of \( t \) on \( \tilde{K}^*(G) \), which can be deduced from Theorems 1 to 3 by taking \( k = -1 \), since \( t = \psi^{-1} \) (see [2]). These are essential data for computing \( KO^*(G) \).

There is a folkloric result which tells us how to represent the complexification map \( c: \tilde{K}O^*(X) \to \tilde{K}^*(X) \) as a (weak) map of \( \Omega \)-spectra \( c = \{ c_i; i \in \mathbb{Z} \} \): \( KO \to K \) and can be read off from [5] or [7]. To state it, we need some notation. A monomorphism \( iF: H \to G \) of (not necessarily compact) Lie groups induces the following maps

\[
G \xrightarrow{iF} G/H \xrightarrow{jF} BH \xrightarrow{\rho_F} BG.
\]

We denote by \( i_C \) the natural inclusions \( SO(n) \to SU(n), O(n) \to U(n) \) and their stable versions, by \( i_H \) the natural inclusions \( SU(n) \to Sp(n), U(n) \to Sp(n) \), etc., by \( i_R \) the standard monomorphism \( U(n) \to O(2n) \) etc., and by \( i_C \) the standard monomorphisms \( Sp(n) \to SU(2n), Sp(n) \to U(2n) \), etc., which arise from the correspondence

\[
H \ni \alpha + j \beta \mapsto \begin{pmatrix} \alpha & -\beta \\ \beta & \bar{\alpha} \end{pmatrix} \in M(2, \mathbb{C})
\]
where \( \alpha, \beta \in C \) and \( \bar{\alpha} \) is the complex conjugate of \( \alpha \) (see [11]).

**Proposition 8.** For each \( i \in \mathbb{Z} \), \( c_i \) is given as follows.

(i) If \( i \equiv 0 \pmod{8} \), \( c_i = \rho_C: BO \times \mathbb{Z} \to BU \times \mathbb{Z} \).
(ii) If \( i \equiv 1 \pmod{8} \), then \( c_i: U/O \to U \) is defined by

\[
c_i(xH) = x\sigma(x)^{-1} \quad \text{for} \quad xH \in G/H
\]
where \( (G, H) = (U, O) \) and \( \sigma = \sigma_{\infty}: U \to U \) is the limit of maps \( \sigma_n: U(n) \to U(n) \) defined by

\[
\sigma_n(A) = \bar{A}
\]
where \( \bar{A} \) is the complex conjugate of \( A \).

(iii) If \( i \equiv 2 \pmod{8} \), \( c_i = j_H: Sp/U \to BU \times \mathbb{Z} \).
(iv) If $i \equiv 3 \pmod{8}$, $c_i = i_{C'}: Sp \to U$.
(v) If $i \equiv 4 \pmod{8}$, $c_i = \rho_{C'}: BS\rightarrow \times \rightarrow BU \times Z$.
(vi) If $i \equiv 5 \pmod{8}$, then $c_i: U/Sp \to U$ is defined as in (2.9), where $(G, H) = (U, Sp)$ and $\sigma = \sigma'_{\infty}: U \to U$ is the limit of maps $\sigma'_n: U(2n) \to U(2n)$ defined by
\[
\sigma'_n(A) = J_n A J_n^{-1}
\]
where if $I_n$ denotes the unit matrix of degree $n$,
\[
J_n = \begin{pmatrix} O & -I_n \\ I_n & O \end{pmatrix}.
\]
(vii) If $i \equiv 6 \pmod{8}$, $c_i = j_R: O/U \to BU \times Z$.
(viii) If $i \equiv 7 \pmod{8}$, $c_i = i_{C}: O \to U$.

In the notation of Lemmas 4 and 5, Seymour [16, Theorem 5.6] showed that $\overline{K}^*(G)$ is a direct sum of $T\{x\}'s$ and $\tilde{N}(y)'s$, and then $\overline{KO}'(G)$ is a direct sum of $r(\overline{K}^*\{S^0\}\{x\})'s$ and $\overline{KO}'\{S^0\}\{x\}'s$ correspondingly.

Furthermore, we recall Seymour's comment [16, Lemma 5.3] on the summand $N(y)$. Let $G$ be a compact, 1-connected Lie group. Suppose that an irreducible, self-conjugate representation $\mu$ of $G$ is given. Then, by the theorem of Hodgkin [8], $\overline{K}^*(G)$ has a summand $\overline{K}^*\{S^0\}\{\beta(\mu)\}$ with $t(\beta(\mu)) = \beta(\mu)$. Here, two cases can occur. The first case is that $\mu$ is real and the second is that $\mu$ is quaternionic, as mentioned earlier. The beta-construction has the real and quaternionic analogues
\[
\beta_R: RO(G) \rightarrow \overline{KO}^{-1}(G),
\]
\[
\beta_H: RSp(G) \rightarrow \overline{KSp}^{-1}(G)
\]
which satisfy $c\beta_R = \beta c$ and $c'\beta_H = \beta c'$, respectively. In the first case, there exists a unique element $\hat{\mu} \in RO(G)$ such that $c(\hat{\mu}) = \mu$. Then $\beta_R(\hat{\mu}) \in \overline{KO}^{-1}(G)$ and $c(\beta_R(\hat{\mu})) = \beta(\mu)$. Thus $\overline{KO}'(G)$ has a summand $\overline{KO}'\{S^0\}\{\beta_R(\hat{\mu})\}$ corresponding to the summand $N(\beta(\mu))$ in $\overline{K}^*(G)$. In the second case, there exists a unique element $\hat{\mu} \in RSp(G)$ such that $c'(\hat{\mu}) = \mu$. Then $\beta_H(\hat{\mu}) \in \overline{KSp}^{-1}(G)$ and $c'(\beta_H(\hat{\mu})) = \beta(\mu)$. By Proposition 8(iv), since $\overline{KSp}^{-1}(X) = [X, Sp]$ (where $[ \ , \ ]$ denotes the set of homotopy classes of maps preserving base points) is identified with $\overline{K}^{-5}(X)$, it is restated as $c(\beta_H(\hat{\mu})) = g^{2}\beta(\mu)$, where we regard $\beta_H(\hat{\mu})$ as an element of $\overline{K}^{-5}(G)$. Thus $\overline{KO}'(G)$ has a summand $\overline{KO}'\{S^0\}\{\beta_H(\hat{\mu})\}$ corresponding to the summand $N\{g^{2}\beta(\mu)\}$ in $\overline{K}^*(G)$.
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After all, Seymour determined the ring structure of \( \widetilde{KO}^\ast(G) \) except the following point (see [16, Appendix]): what are the squares \( \beta_R(\widehat{\mu})^2 \) and \( \beta_H(\widehat{\mu})^2 \) in the above situation? The Bott’s exact sequence tells us only that they are in the image of \( \eta \). M. Crabb [6, p.67] and H. Minami [14, Proposition 2.2] answered these questions. They showed that

\[
\beta_R(\widehat{\mu})^2 = \eta \cdot \lambda^2(\beta_R(\widehat{\mu})) \quad \text{and} \quad \beta_H(\widehat{\mu})^2 = \eta \sigma \cdot \beta_R(\lambda^2(\mu))
\]

where in the first relation, \( \mu \in R(G) \) is real, and in the second relation, \( \mu \in R(G) \) is quaternionic and \( \lambda^2(\mu) \in R(G) \) is real (see [1, Remark 3.63]). So the problems reduce to determine \( \lambda^2(\beta_R(\widehat{\mu})) \) and \( \beta_R(\lambda^2(\mu)) \) in \( \widetilde{KO}^{-1}(G) \).

3. The rings \( KO^\ast(SU(3)), KO^\ast(Sp(2)) \) and \( KO^\ast(G_2) \). Using the notation of the previous section, we describe the \( KO^\ast(pt) \)-algebra structure of \( \widetilde{KO}^\ast(G) \), together with the action of \( \psi^k \) on it.

\( G = SU(3) \)

As seen in [16, Theorem 5.6], there exists an element \( \zeta_{1.2} \in \widetilde{KO}^0(SU(3)) \) such that \( c(\zeta_{1.2}) = g^{-1}\beta(\lambda_1)\beta(\lambda_2) \). It follows from this and Proposition 7(1) that

\[
\widetilde{K}^\ast(SU(3)) = T(\beta(\lambda_1)) \oplus N(g^{-1}\beta(\lambda_1)\beta(\lambda_2))
\]

**Theorem 9.** As a \( \mathbb{Z} \)-graded module,

\[
\widetilde{KO}^\ast(SU(3)) = r(\widetilde{K}^\ast(S^0)\{\beta(\lambda_1)\}) \oplus \widetilde{KO}^\ast(S^0)\{\zeta_{1.2}\}.
\]

More precisely,

\[
\begin{align*}
\widetilde{KO}^0(SU(3)) &= \mathbb{Z}\{\zeta_{1.2}\}, \\
\widetilde{KO}^{-1}(SU(3)) &= \mathbb{Z}/(2)\{\eta \zeta_{1.2}\} \oplus \mathbb{Z}\{r(\beta(\lambda_1))\}, \\
\widetilde{KO}^{-2}(SU(3)) &= \mathbb{Z}/(2)\{\eta^2 \zeta_{1.2}\}, \\
\widetilde{KO}^{-3}(SU(3)) &= \mathbb{Z}\{r(g\beta(\lambda_1))\}, \\
\widetilde{KO}^{-4}(SU(3)) &= \mathbb{Z}\{\nu \zeta_{1.2}\}, \\
\widetilde{KO}^{-5}(SU(3)) &= \mathbb{Z}\{r(g^2\beta(\lambda_1))\}, \\
\widetilde{KO}^{-6}(SU(3)) &= 0, \\
\widetilde{KO}^{-7}(SU(3)) &= \mathbb{Z}\{r(g^3\beta(\lambda_1))\}.
\end{align*}
\]
Its \( \overline{KO}^*(S^0) \)-module structure is given by

\[
\eta \cdot r(g^i\beta(\lambda_1)) = 0, \quad \nu \cdot r(g^i\beta(\lambda_1)) = 2r(g^{i+2}\beta(\lambda_1)) \quad \text{and} \\
\sigma \cdot r(g^i\beta(\lambda_1)) = r(g^{i+4}\beta(\lambda_1)).
\]

Its multiplicative structure is given by

\[
\begin{align*}
    r(g^i\beta(\lambda_1)) \cdot r(g^j\beta(\lambda_1)) &= (-1)^i r(g^{i+j+4})\zeta_{1,2}, \\
    r(g^i\beta(\lambda_1)) \cdot \zeta_{1,2} &= 0 \quad \text{and} \quad \zeta_{1,2}^2 = 0.
\end{align*}
\]

The action of \( \psi^k \) on \( \overline{KO}^*(SU(3)) \) is given by

\[
\psi^k(r(g^i\beta(\lambda_1))) = \begin{cases} 
    k^{i+2} r(g^i\beta(\lambda_1)) & \text{if } i \equiv 0 \pmod{2} \\
    k^{i+3} r(g^i\beta(\lambda_1)) & \text{if } i \equiv 1 \pmod{2}
\end{cases}
\]

and \( \psi^k(\zeta_{1,2}) = k^4 \zeta_{1,2} \).

Proof. The additive structure follows from (3.1) and Lemmas 4, 5. For the \( \overline{KO}^*(S^0) \)-module structure, we have

\[
\begin{align*}
    \eta \cdot r(g^i\beta(\lambda_1)) &= r(g^i\beta(\lambda_1))\eta \\
    &= r(g^i\beta(\lambda_1) \cdot c(\eta)) & \text{by (2.1b)} \\
    &= 0 & \text{since } c(\eta) = 0 \text{ by (2.3a)}
\end{align*}
\]

and the other equalities are obtained similarly.

For the multiplicative structure, we have

\[
\begin{align*}
    r(g^i\beta(\lambda_1)) \cdot r(g^j\beta(\lambda_1)) \\
    &= r(g^i\beta(\lambda_1) \cdot c r(g^j\beta(\lambda_1))) & \text{by (2.1b)} \\
    &= r(g^i\beta(\lambda_1) \cdot (1 + t)(g^j\beta(\lambda_1))) & \text{by (2.1a)} \\
    &= r(g^i\beta(\lambda_1) \cdot (g^j\beta(\lambda_1) + (-1)^i g^j\beta(\lambda_2))) & \text{since } t(\beta(\lambda_1)) = \beta(\lambda_2) \\
    &= r(g^{i+j}\beta(\lambda_1)^2 + (-1)^i r(g^{i+j}\beta(\lambda_1)\beta(\lambda_2))) \\
    &= (-1)^i r(g^{i+j+1}g^{-1}\beta(\lambda_1)\beta(\lambda_2)) & \text{since } \beta(\lambda_1)^2 = 0 \\
    &= (-1)^i r(g^{i+j+1})\zeta_{1,2} & \text{by (2.1b) and the definition of } \zeta_{1,2}
\end{align*}
\]

and the other equalities are obtained similarly.
For the action of $\psi^k$, we have

$$c(\psi^k r)(g^i \beta(\lambda_1)))$$

$$= \psi^k (cr(g^i \beta(\lambda_1)))$$

$$= \psi^k (g^i \beta(\lambda_1) + (-1)^i g^i \beta(\lambda_2))$$

$$= k^i g^i \left( \frac{k^2(k+1)}{2} \beta(\lambda_1) - \frac{k^2(k+1)}{2} \beta(\lambda_2) \right)$$

$$+ (-1)^i k^i g^i \left( \frac{k^2(k-1)}{2} \beta(\lambda_1) + \frac{k^2(k+1)}{2} \beta(\lambda_2) \right)$$

by (1.2) and Theorem 1(i)

$$= k^{i+2}(k+1 - (-1)^i k + (-1)^i) (g^i \beta(\lambda_1) + (-1)^i g^i \beta(\lambda_2))$$

$$= c \left( \frac{k^{i+2}(k+1 - (-1)^i k + (-1)^i)}{2} r(g^i \beta(\lambda_1)) \right).$$

By examining the behavior of $c: \widetilde{KO}^{-1-2i}(SU(3)) \to \widetilde{K}^{-1-2i}(SU(3))$, we see that this gives the first equality. The second equality follows similarly, and the proof is completed.

$G = Sp(2)$

It follows from Proposition 7(2) that

$$\widetilde{K}^*(Sp(2)) = N \langle g^2 \beta(\mu_1') \rangle \oplus N \langle \beta(\mu_2') \rangle \oplus N \langle g^2 \beta(\mu_1') \beta(\mu_2') \rangle.$$

**Theorem 10.** As a $KO^*(pt)$-module (but not as a ring),

$$KO^*(Sp(2)) = KO^*(pt) \otimes \Lambda_Z(\beta_H(\mu_1'), \beta_R(\mu_2')).$$

Its multiplicative structure is given by

$$\beta_H(\mu_1')^2 = \eta \sigma \cdot \beta_R(\mu_2') \quad \text{and} \quad \beta_R(\mu_2')^2 = 0.$$

The action of $\psi^k$ on $\widetilde{KO}^*(Sp(2))$ is given by

$$\psi^k(\beta_H(\mu_1')) = \frac{k^4(k^2+2)}{3} \beta_H(\mu_1') - \frac{k^4(k^2-1)}{12} \nu \beta_R(\mu_2'),$$

$$\psi^k(\beta_R(\mu_2')) = \frac{-2k^2(k^2-1)}{3} \nu \sigma^{-1} \beta_H(\mu_1') + \frac{k^2(2k^2+1)}{3} \beta_R(\mu_2'),$$

$$\psi^k(\beta_H(\mu_1') \beta_R(\mu_2')) = k^8 \beta_H(\mu_1') \beta_R(\mu_2').$$
Proof. It follows from (3.2) and Lemma 5 that, as a $\widetilde{KO}^*(S^0)$-module,
\begin{equation}
\widetilde{KO}^*(Sp(2)) = \widetilde{KO}^*(S^0)\{\beta_H(\mu'_1)\} \oplus \widetilde{KO}^*(S^0)\{\beta_R(\mu'_2)\} \\
\oplus \widetilde{KO}^*(S^0)\{\beta_H(\mu'_1)\beta_R(\mu'_2)\}.
\end{equation}

So the first statement follows.

For the multiplicative structure, the first equality is a consequence of the second relation of (2.12). It remains to prove the second equality. In view of the first relation of (2.12), we have to determine $\lambda^2(\beta_R(\mu'_2))$. For this purpose, since $c: \widetilde{KO}^{-1}(Sp(2)) \to \widetilde{K}^{-1}(Sp(2))$ is a monomorphism of $\lambda$-rings (compare (3.3) with (3.2)), it suffices to compute $\lambda^2(\beta(\mu'_2))$. We quote from Theorem 2(i) with $k = 2$ that
\[\psi^2(\beta(\mu'_2)) = -16\beta(\mu'_1) + 12\beta(\mu'_2).\]

Using the formula $\psi^2(x) = x^2 + 2\lambda^2(x) = 0$ for $x \in K(X)$ (see [2]) and the relation $\beta(\mu'_2)^2 = 0$, we have
\[\lambda^2(\beta(\mu'_2)) = 8\beta(\mu'_1) - 6\beta(\mu'_2).\]

By (2.12), since $2\eta = 0$, this gives the second equality.

For the action of $\psi^k$, we have
\[c\psi^k(\beta_H(\mu'_1))
\begin{align*}
&= \psi^k c(\beta_H(\mu'_1)) \\
&= \psi^k (g^2 \beta(\mu'_1)) \\
&= \psi^k (g^2 \psi^k (\beta(\mu'_1)) \\
&= (kg)^2 \left( \frac{k^2(k^2 + 2)}{3} \beta(\mu'_1) - \frac{k^2(k^2 - 1)}{6} \beta(\mu'_2) \right) \\
&\quad \text{by (1.2) and Theorem 2(i)} \\
&= \frac{k^4(k^2 + 2)}{3} g^2 \beta(\mu'_1) - \frac{k^4(k^2 - 1)}{6} g^2 \beta(\mu'_2) \\
&= c \left( \frac{k^4(k^2 + 2)}{3} \beta_H(\mu'_1) - \frac{k^4(k^2 - 1)}{12} \nu \beta_R(\mu'_2) \right) \text{ by (2.3a).}
\end{align*}

Since $c: \widetilde{KO}^{-5}(Sp(2)) \to \widetilde{K}^{-5}(Sp(2))$ is injective (compare (3.3) with (3.2)), this gives the first equality. The other equalities are obtained similarly, and the proof is completed.
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\[ G = G_2 \]

It follows from Proposition 7(3) that

\[ \widetilde{K}^* (G_2) = N (\beta (\rho_1)) \oplus N (\beta (\rho_2)) \oplus N (\beta (\rho_1) \beta (\rho_2)). \]

**Theorem 11.** As a \( KO^* (pt) \)-module (but not as a ring),

\[ KO^* (G_2) = KO^* (pt) \otimes L \mathbb{Z} (\beta_R (\rho_1), \beta_R (\rho_2)). \]

Its multiplicative structure is given by

\[ \beta_R (\rho_1)^2 = \eta \cdot \beta_R (\rho_1) + \eta \cdot \beta_R (\rho_2) = \beta_R (\rho_2)^2. \]

The action of \( \psi^k \) on \( \widetilde{KO}^* (G_2) \) is given by

\[
\psi^k (\beta_R (\rho_1)) = \frac{k^2 (k^4 + 5)}{6} \beta_R (\rho_1) - \frac{k^2 (k^4 - 1)}{30} \beta_R (\rho_2), \\
\psi^k (\beta_R (\rho_2)) = - \frac{25k^2 (k^4 - 1)}{6} \beta_R (\rho_1) + \frac{k^2 (5k^4 + 1)}{6} \beta_R (\rho_2), \\
\psi^k (\beta_R (\rho_1) \beta_R (\rho_2)) = k^8 \beta_R (\rho_1) \beta_R (\rho_2). 
\]

**Proof.** It follows from (3.4) and Lemma 5 that, as a \( \widetilde{KO}^* (S^0) \)-module,

\[ \widetilde{KO}^* (G_2) = \widetilde{KO}^* (S^0) \{ \beta_R (\rho_1) \} \oplus \widetilde{KO}^* (S^0) \{ \beta_R (\rho_2) \} \oplus \widetilde{KO}^* (S^0) \{ \beta_R (\rho_1) \beta_R (\rho_2) \}. \]

So the first statement follows.

For the multiplicative structure, in view of the first relation of (2.12), we have to determine \( \lambda^2 (\beta_R (\rho_i)) \) for \( i = 1, 2 \). For this purpose, since \( c: \widetilde{KO}^{-1} (G_2) \rightarrow \widetilde{K}^{-1} (G_2) \) is a monomorphism of \( \lambda \)-rings (compare (3.5) with (3.4)), it suffices to compute \( \lambda^2 (\beta (\rho_i)) \) for \( i = 1, 2 \). We quote from Theorem 3(i) with \( k = 2 \) that

\[
\psi^2 (\beta (\rho_1)) = 14 \beta (\rho_1) - 2 \beta (\rho_2), \\
\psi^2 (\beta (\rho_2)) = -250 \beta (\rho_1) + 54 \beta (\rho_2). 
\]

Using the formula \( \psi^2 (x) - x^2 + 2 \lambda^2 (x) = 0 \) and the relation \( \beta (\rho_i)^2 = 0 \), we have

\[
\lambda^2 (\beta (\rho_1)) = -7 \beta (\rho_1) + \beta (\rho_2), \\
\lambda^2 (\beta (\rho_2)) = 125 \beta (\rho_1) - 27 \beta (\rho_2). 
\]
By (2.12), since \(2\eta = 0\), these give the stated equalities.

The equalities describing the action of \(\psi^k\) are obtained in the same way as in the proof of Theorem 10.

4. The rings \(KO^*(SU(2n+1)/SO(2n+1))\) and \(KO^*(SU(2n)/Sp(n))\). Lemma 6 together with Proposition 8 can be used to compute the \(KO\)-theory of compact symmetric spaces \(SU(2n+1)/SO(2n+1)\) and \(SU(2n)/Sp(n)\). To begin with, the following result is in [1, Remark 3.63 and Theorems 7.3, 7.6, 7.7].

**Proposition 12.** For \(G = SU(n+1), SO(2n+1)\) and \(Sp(n)\), the action of \(t\) on \(R(G)\) is given by:

1. In \(R(SU(n+1)) = Z[\lambda_1, \ldots, \lambda_n] \) (where \(\lambda_1 = [C^{n+1}]\) and \(\lambda_k = \lambda^k(\lambda_1)\)),
   \[
   t(\lambda_k) = \lambda_{n+1-k} \quad \text{for} \quad k = 1, \ldots, n.
   \]

2. In \(R(SO(2n+1)) = Z[\mu_1, \ldots, \mu_n] \) (where \(\mu_1 = [(R^{2n+1})^C]\) and \(\mu_k = \lambda^k(\mu_1)\)),
   \[
   t(\mu_k) = \mu_k \quad \text{for} \quad k = 1, \ldots, n.
   \]

where \(\mu_k\) is real.

3. In \(R(Sp(n)) = Z[\mu'_1, \ldots, \mu'_n] \) (where \(\mu'_1 = [(H^n)^C]\) and \(\mu'_k = \lambda^k(\mu'_1)\)),
   \[
   t(\mu'_k) = \mu'_k \quad \text{for} \quad k = 1, \ldots, n.
   \]

where \(\mu'_{2l-1}\) is quaternionic and \(\mu'_{2l}\) is real.

The \(K\)-rings of \(SU(2n+1)/SO(2n+1)\) and \(SU(2n)/Sp(n)\) were determined by H. Minami [12]. We recall his result. Let \(G\) be a compact 1-connected Lie group. Suppose that there is an automorphism \(\sigma: G \to G\) such that \(\sigma^2 = 1_G\). Then the fixed point set \(G^\sigma = \{x \in G|\sigma(x) = x\}\) forms a closed connected subgroup of \(G\), and the coset space \(G/G^\sigma\) becomes a compact symmetric space (e.g., see [11, Chapter 3, §6]). Consider the induced homomorphism \(\sigma^*: R(G) \to R(G)\) and let \(\sigma^*(\lambda) = \lambda'\), where \(\lambda\) is a representation of \(G\). Then \(\dim \lambda = \dim \lambda' (= n)\) and \(\lambda|G^\sigma = \lambda'|G^\sigma\). So we have a map \(f_\lambda: G/G^\sigma \to U(n)\) defined by

\[
(4.1) \quad f_\lambda(xG^\sigma) = \lambda(x)\lambda'(x)^{-1} \quad \text{for} \quad xG^\sigma \in G/G^\sigma.
\]

Let \(\iota_n: U(n) \to U\) be the canonical injection. Then the composite \(\iota_nf_\lambda\) gives rise to a homotopy class \(\beta(\lambda - \lambda')\) in \([G/G^\sigma, U] = \tilde{K}^{-1}(G/G^\sigma)\).
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Let $\sigma = \sigma_{2n+1}: SU(2n+1) \to SU(2n+1)$ be the involution defined as in (2.10). Then, in the notation of Proposition 12,

\begin{equation}
\sigma^*(\lambda_k) = \lambda_{2n+1-k} \quad \text{and} \quad \mu_k = i_C^*(\lambda_{2n+1-k}) \quad \text{for} \quad k = 1, \ldots, n.
\end{equation}

Similarly, let $\sigma = \sigma'_n: SU(2n) \to SU(2n)$ be the involution defined as in (2.11). Then

\begin{equation}
\sigma^*(\lambda_k) = \lambda_{2n-k} \quad \text{and} \quad \mu'_k = i_C^{-1}(\lambda_{2n-k}) \quad \text{for} \quad k = 1, \ldots, n.
\end{equation}

**Proposition 13.** (1) ([12, Proposition 8.1]) As an algebra over $K^*(pt)$,

$$
K^*(SU(2n+1)/SO(2n+1)) = K^*(pt) \otimes \Lambda_{\mathbb{Z}}(\beta(\lambda_1 - \lambda_{2n}), \ldots, \beta(\lambda_n - \lambda_{n+1})).
$$

(2) ([12, Proposition 6.1]) As an algebra over $K^*(pt)$,

$$
K^*(SU(2n)/Sp(n)) = K^*(pt) \otimes \Lambda_{\mathbb{Z}}(\beta(\lambda_1 - \lambda_{2n-1}), \ldots, \beta(\lambda_{n-1} - \lambda_{n+1})).
$$

We can now deduce our main result.

**Theorem 14.** (1) As an algebra over $KO^*(pt)$,

$$
KO^*(SU(2n+1)/SO(2n+1)) = KO^*(pt) \otimes \Lambda_{\mathbb{Z}}(\lambda_{1,2n}, \ldots, \lambda_{n,n+1})
$$

where $\lambda_{k,2n+1-k} \in \overline{KO}^1(SU(2n+1)/SO(2n+1))$ is a unique element such that

$$
c(\lambda_{k,2n+1-k}) = g^{-1}\beta(\lambda_k - \lambda_{2n+1-k}).
$$

(2) As an algebra over $KO^*(pt)$,

$$
KO^*(SU(2n)/Sp(n)) = KO^*(pt) \otimes \Lambda_{\mathbb{Z}}(\lambda'_{1,2n-1}, \ldots, \lambda'_{n-1,n+1})
$$

where $\lambda'_{2l-1,2n-2l+1} \in \overline{KO}^3(SU(2n)/Sp(n))$ is a unique element such that

$$
c(\lambda'_{2l-1,2n-2l+1}) = g\beta(\lambda_{2l-1} - \lambda_{2n-2l+1}).
$$
and $\lambda_{2l,2n-2l} \in \widetilde{KO}^1(SU(2n)/Sp(n))$ is a unique element such that
\[ c(\lambda_{2l,2n-2l}) = g^{-1}(\lambda_{2l} - \lambda_{2n-2l}). \]

Proof. We first show (1). Consider $\lambda_k: SU(2n+1) \to U\left(\binom{2n+1}{k}\right)$ for $k = 1, \ldots, n$. By (4.2), $i_k^*\mu_k = \mu_k$ and by Proposition 12(2), $\mu_k \in R(SO(2n+1))$ is real, i.e., there is a (unique) $\mu_k \in RO(SO(2n+1))$ such that $c(\mu_k) = \mu_k$. Therefore, in the diagram
\[
\begin{array}{ccc}
SO(2n+1) & \xrightarrow{i_c} & SU(2n+1) & \xrightarrow{\pi_c} & SU(2n+1)/SO(2n+1) \\
O & \xrightarrow{i_c} & U & \xrightarrow{\pi_c} & U/O \\
\end{array}
\]
(4.4) \[
\kappa_{2n+1,k} \mu_k \quad \lambda_{k,2n+1-k} \quad \beta(\lambda_k - \lambda_{2n+1-k})
\]
(where $\kappa_{2n+1,k}: O\left(\binom{2n+1}{k}\right) \to O$ and $i_{2n+1,k}: U\left(\binom{2n+1}{k}\right) \to U$ are the canonical injections), the left square is commutative. So we have a map $\lambda_{k,2n+1-k}: SU(2n+1)/SO(2n+1) \to U/O$ which makes the middle square commute. Indeed, it is defined by
\[
(4.5) \quad \lambda_{k,2n+1-k}(xSO(2n+1)) = (i_{2n+1,k}\lambda_k)(x)O
\]
for $xSO(2n+1) \in SU(2n+1)/SO(2n+1)$. Since $\sigma_{2n+1}(\lambda_k) = \lambda_{2n+1-k} = t(\lambda_k)$ by (4.2) and Proposition 12(1), the diagram
\[
\begin{array}{ccc}
SU(2n+1) & \xrightarrow{\sigma_{2n+1}} & SU(2n+1) \\
\lambda_k \downarrow & & \lambda_k \downarrow \\
U\left(\binom{2n+1}{k}\right) & \xrightarrow{i_{n,k}} & U\left(\binom{2n+1}{k}\right) \\
\sigma_{2n+1,k} \downarrow & & \sigma_{2n+1,k} \downarrow \\
SU(2n+1) & \xrightarrow{\lambda_k} & U\left(\binom{2n+1}{k}\right) & \xrightarrow{i_{n,k}} & U
\end{array}
\]
(where $\sigma_{2n+1,k}$ is defined as in (2.10)) is commutative. So the right triangle in (4.4) is commutative:
\[
(c_1 \lambda_{k,2n+1-k})(xSO(2n+1)) \\
= (i_{2n+1,k} \lambda_k)(x)\sigma_{\infty}(i_{2n+1,k} \lambda_k)(x)^{-1} \quad \text{by (2.9) and (4.5)} \\
= (i_{2n+1,k} \lambda_k)(x)(i_{2n+1,k} \lambda_k \sigma_{2n+1})(x)^{-1} \\
= i_{2n+1,k}(\lambda_k(x)(\lambda_k \sigma_{2n+1})(x)^{-1}) \\
= \beta(\lambda_k - \lambda_{2n+1-k})(xSO(2n+1)) \quad \text{by (4.1) and (4.2)}.
\]
By Proposition 8(ii), this implies that $c(\lambda_{k,2n+1-k}) = g^{-1}\beta(\lambda_k - \lambda_{2n+1-k})$, where we regard $\lambda_{k,2n+1-k}$ as an element of $\widetilde{KO}^1(SU(2n+1)/SO(2n+1)) = \cdots$
[SU(2n + 1)/SO(2n + 1), U/O]. By this equality and Proposition 13(1), we can apply Lemma 6 to the case \( X = SU(2n + 1)/SO(2n + 1) \) and obtain the \( KO^*(pt) \)-module structure of \( KO^*(SU(2n + 1)/SO(2n + 1)) \). For the multiplicative structure, as is discussed at the end of section 2, whether \( \lambda_{k,2n+1-k}^2 \) is zero or not is a remaining question. Fortunately it says in Crabb [6, Example (6.6)] that \( \lambda_{k,2n+1-k} \) is zero since \( \lambda_{k,2n+1-k} \) has degree 1 and 1 \( \equiv -3 \) (mod 4). Hence (1) follows.

We next show (2). Consider \( \lambda_k: SU(2n) \to U\left(\binom{2n}{k}\right) \) for \( k = 1, \ldots, n - 1 \). By (4.3), \( i^*_C(\lambda_k) = \mu'_k \). From now on, our argument is divided into two cases.

Suppose that \( k \) is odd, i.e., \( k = 2l - 1 \) for some \( l \geq 1 \). Then, by Proposition 12(3), \( \mu'_k \in R(Sp(n)) \) is quaternionic, i.e., there is a (unique) \( \tilde{\mu}'_k \in RSp(Sp(n)) \) such that \( c'(\tilde{\mu}'_k) = \mu'_k \). Therefore, in the diagram

\[
\begin{array}{c}
Sp(n) \xrightarrow{i'_C} SU(2n) \xrightarrow{\pi'_C} SU(2n)/Sp(n) \\
\downarrow \xi_{2n,k} \tilde{\mu}'_k \downarrow \iota_{2n,k} \lambda_k \downarrow \lambda_{k,2n-k} \xrightarrow{c(\lambda_k - \lambda_{2n-k})} U/Sp \\
Sp \xrightarrow{i'_C} U \xrightarrow{\pi'_C} U/Sp \xrightarrow{c-3} U
\end{array}
\]

(where \( \xi_{2n,k}: Sp(\binom{2n}{k})/2 \to Sp \) and \( \iota_{2n,k}: U(\binom{2n}{k}) \to U \) are the canonical injections), the left square is commutative. So we have a map \( \lambda_{k,2n-k}: SU(2n)/Sp(n) \to U/Sp \) which makes the middle square commute. Since \( \sigma'_{n,k}(\lambda_k) = \lambda_{2n-k} = t(\lambda_k) \) by (4.3) and Proposition 12(1) and since \( j\alpha j^{-1} = \bar{\alpha} \) for \( \alpha \in C \), the diagram

\[
\begin{array}{c}
SU(2n) \xrightarrow{\lambda_k} U\left(\binom{2n}{k}\right) \xrightarrow{\iota_{n,k}} U \\
\downarrow \sigma'_{n,k} \downarrow \sigma'_{n,k} \downarrow \sigma'_{n,k} \\
SU(2n) \xrightarrow{\lambda_k} U\left(\binom{2n}{k}\right) \xrightarrow{\iota_{n,k}} U
\end{array}
\]

(where \( \sigma'_{n,k} \) is defined as in (2.11)) is commutative and so the right triangle in (4.6) is commutative. By Proposition 8(vi), this implies that \( c(\lambda_{k,2n-k}) = g\beta(\lambda_k - \lambda_{2n-k}) \), where \( \lambda'_{k,2n-k} \in \overline{KO}^{-3}(SU(2n)/Sp(n)) = [SU(2n)/Sp(n), U/Sp] \).

Suppose that \( k \) is even, i.e., \( k = 2l \) for some \( l \geq 1 \). Then, by Proposition 12 (3), \( \mu'_k \in R(Sp(n)) \) is real, i.e., there is a (unique) \( \mu'_k \in RO(Sp(n)) \)
such that \( \epsilon_k' \mu_k' = \mu_k' \). Therefore, in the diagram

\[
\begin{array}{ccccccccc}
S^p(n) \xrightarrow{ic'} & SU(2n) \xrightarrow{\pi c'} & SU(2n) / Sp(n) \\
\kappa_{2n,k} \mu_k' & \downarrow & \downarrow & \downarrow & \beta (\lambda_k - \lambda_{2n-k}) \\
O & \xrightarrow{ic} & U & \xrightarrow{\pi c} & U / O & \xrightarrow{c_1} & U \\
\end{array}
\]

(where \( \kappa_{2n,k} : O(\binom{2n}{k}) \to O \) is the canonical injection), the left square is commutative. So we have a map \( \lambda_{k,2n-k} : SU(2n) / Sp(n) \to U / O \) which makes the middle square commute. Since \( \sigma_1'(\lambda_k) = \lambda_{2n-k} = t(\lambda_k) \) by (4.3) and Proposition 12(1), the diagram

\[
\begin{array}{cccc}
SU(2n) & \xrightarrow{\lambda_k} & U(\binom{2n}{k}) & \xrightarrow{\iota_{n,k}} & U \\
\sigma_1' \downarrow & & \sigma_{2n,k} \downarrow & & \sigma_\infty \\
SU(2n) & \xrightarrow{\lambda_k} & U(\binom{2n}{k}) & \xrightarrow{\iota_{n,k}} & U \\
\end{array}
\]

(where \( \sigma_{2n,k} \) is defined as in (2.10)) is commutative and so the right triangle in (4.7) is commutative. By Proposition 8(ii), this implies that

\( c(\lambda_k') = g^{-1} \beta (\lambda_k - \lambda_{2n-k}) \), where \( \lambda_{k,2n-k} \in \widetilde{KO}'(SU(2n) / Sp(n)) \).

By these equalities and Proposition 13(2), we can apply Lemma 6 to the case \( X = SU(2n) / Sp(n) \). The rest is quite similar to the proof of (1), and (2) follows.

**Remark.** We have no good reasons to assert that, for example, \( \lambda_{k,2n+1-k} \) lies in \( \widetilde{KO}'(SU(2n+1) / SO(2n+1)) \) and does not lie in \( \widetilde{KO}^m(SU(2n+1) / SO(2n+1)) \) for some \( m \neq 0 \). But, since the CW-complex structure of \( SU(2n+1) / SO(2n+1) \) is known for small \( n \), one can compute \( \widetilde{KO}^*(SU(2n+1) / SO(2n+1)) \) by using cofibre sequences. Only such observation justifies our assertion.

**References**


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(Received April 24, 1994)