On the representations of the generalized symmetric group

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ON THE REPRESENTATIONS OF THE GENERALIZED SYMMETRIC GROUP

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Introduction. All permutations of the mn symbols commutative with

\[(1, 2, \ldots, m_1) (1, 2, \ldots, m_2) \cdots (1, 2, \ldots, m_n)\]

consistute a group of order \(n! m^n\). Let us denote this group by \(S(n, m)\). Obviously \(S(1, m)\) is the cyclic group with generator \(Q = (1, 2, \ldots, m)\). Since \(S(n, 1)\) is the symmetric group \(S_n\) on \(n\) symbols, \(S(n, m)\) will be called the generalized symmetric group [10]. \(S(n, 2)\) is the hyper-octahedral group of A. Young. The group \(S(n, m)\) was treated from other point of view by H. S. M. Coxeter [2]. We set \(Q_i = (1, 2, \ldots, m_i)\). The \(n\) cycles \(Q_i\) generate an invariant sub-group \(\mathcal{Q}\) of order \(m^n\) of \(S(n, m)\). The totality of permutations

\[W^* = \begin{pmatrix} (1, 2, \ldots, m_1) & (1, 2, \ldots, m_2) & \cdots & (1, 2, \ldots, m_n) \\ (1, 2, \ldots, m_1) & (1, 2, \ldots, m_2) & \cdots & (1, 2, \ldots, m_n) \end{pmatrix} \]

which transform the \(n\) cycles \(Q_i\) into each other, constitutes a subgroup \(S_{n^*}\) of \(S(n, m)\). \(S_{n^*}\) is isomorphic to \(S_n\) by the mapping

\[W = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix} \rightarrow W^*.
\]

We see easily that

\[S(n, m) = S_{n^*} \mathcal{Q}, \quad S_{n^*} \cap \mathcal{Q} = 1,
\]

so that \(S(n, m)/\mathcal{Q} \cong S_n\). Every element \(P\) of \(S(n, m)\) is expressed uniquely in the form \(P = W^* Q\), where \(W^* \in S_{n^*}\) and

\[Q = Q_{i_1}^{t_1} Q_{i_2}^{t_2} \cdots Q_{i_n}^{t_n} \quad (0 \leq t_i \leq m - 1).
\]

We have also

\[(W^*)^{-1} Q W^* = Q_{i_1}^{t_1} Q_{i_2}^{t_2} \cdots Q_{i_n}^{t_n}.
\]
In the present paper we shall first determine the irreducible representations of \( S(n, m) \) \([10, \text{Theorem 2}]\). For this purpose, we state in §1 some preliminary results for the induced representations of a group of finite order. As an application, the irreducible representations of \( S(n, m) \) will be determined in §2. In §3 some results in [11] and [12] are generalized for \( S(n, m) \). In particular, a generalization of the Murnaghan-Nakayama recursion formula plays an important role in the following section. Let \( p \) be a prime number. As was shown in [10], there exists the close relationship between the theory of the representations of \( S(b, p) \) and that of the modular representations of \( S_n \) for \( p \). In §4 we shall prove the theorems in [10] which were stated without proofs.

1. Preliminaries. Let \( \mathfrak{G} \) be a group of finite order. We consider the representations of \( \mathfrak{G} \) in an algebraically closed field of characteristic 0. Let \( \mathfrak{H} \) be an invariant subgroup of \( \mathfrak{G} \) and let \( \zeta_1, \zeta_2, \ldots, \zeta_n; \zeta_1', \zeta_2', \ldots, \zeta_m \) be the distinct irreducible characters of \( \mathfrak{G} \) and \( \mathfrak{H} \) respectively. As is well known, \( n \) is equal to the number of conjugate classes of \( \mathfrak{G} \). The characters \( \zeta \) of \( \mathfrak{G} \) are distributed in classes of characters which are associated with regard to \( \mathfrak{G} \); two characters \( \zeta_\mu \) and \( \zeta_\nu \) being associated if

\[
\zeta_\mu(H) = \zeta_\nu(G^{-1}HG),
\]

where \( H \) is a variable element of \( \mathfrak{H} \) and \( G \) is a fixed element of \( \mathfrak{G} \). The totality of elements \( G \in \mathfrak{G} \) which satisfy

\[
\zeta_\mu(H) = \zeta_\mu(G^{-1}HG) \quad \text{(for } H \in \mathfrak{H})
\]

constitutes a subgroup \( \mathfrak{H}_\mu \) of \( \mathfrak{G} \). Obviously \( \mathfrak{H} \subseteq \mathfrak{H}_\mu \). \( \mathfrak{H}_\mu \) is called the subgroup of \( \mathfrak{G} \) corresponding to \( \zeta_\mu \). Let \( \zeta_1, \zeta_2, \ldots, \zeta_r \) be the characters of \( \mathfrak{H} \) such that they all lie in different associated classes and every character \( \zeta \) is associated with one of them. Let \( (\mathfrak{G} : \mathfrak{H}_\mu) = s_\mu \) and

\[
\mathfrak{G} = \mathfrak{H}_\mu T_1 + \mathfrak{H}_\mu T_2 + \cdots + \mathfrak{H}_\mu T_{s_\mu}, \quad T_1 = 1.
\]

Then the number of characters \( \zeta \) associated with \( \zeta_\mu \) is \( s_\mu \). If we denote these characters by \( \zeta_\mu = \zeta_\mu^{(1)}, \zeta_\mu^{(2)}, \ldots, \zeta_\mu^{(s_\mu)} \), we may set

\[
\zeta_\mu^{(i)}(H) = \zeta_\mu(T_i^{-1}HT_i).
\]

We set

\[
\text{http://escholarship.lib.okayama-u.ac.jp/mjou/vol4/iss1/3}
\]
(1.4) \[ \theta_\mu(H) = \sum_i \zeta_\mu^{(i)}(H) = \sum_i \zeta_\mu(T^{-1}_iHT_i). \]

Every character \( \chi_i \), considered as a character of \( \mathcal{B} \), is expressed as

(1.5) \[ \chi_i(H) = a_i \theta_\mu(H) \quad \text{(for } H \in \mathcal{B}) \]

with a suitable \( \theta_\mu \). Here \( a_i \) is a positive integer. We shall say that \( \chi_i \) is the character of \( \mathcal{B} \) determined by \( \zeta_\mu \). Denote by \( \chi^{(1)}_\mu, \chi^{(2)}_\mu, \ldots, \chi^{(r)}_\mu \) the irreducible characters of \( \mathcal{B} \) determined by \( \zeta_\mu \). We then have

(1.6) \[ \sum_{\mu=1}^r t_\mu = n. \]

We consider a subgroup \( \mathcal{B}' \) of \( \mathcal{B} \). Let \( (\mathcal{B} : \mathcal{B}') = r \) and

\[ \mathcal{B} = \mathcal{B}'S_1 + \mathcal{B}'S_2 + \cdots + \mathcal{B}'S_r, \quad S_i = 1. \]

Let \( G' \rightarrow D(G') \) be a representation of \( \mathcal{B}' \). We set \( D(S_i^{-1}GS_j) = 0 \) if \( S_i^{-1}GS_j \) is not contained in \( \mathcal{B}' \). Then

(1.7) \[ G \rightarrow D^*(G) = (D(S_i^{-1}GS_j))_{ij}, \quad \text{(for } G \in \mathcal{B}) \]

forms a representation \( D^* \) of \( \mathcal{B} \) and is called the representation of \( \mathcal{B} \) induced by the representation \( D \) of \( \mathcal{B}' \). If \( \tilde{\xi} \) is the character of \( D \), we denote by \( \tilde{\xi} \) the character of \( D^* \). We define \( \tilde{\xi}(S_i^{-1}GS_j) = 0 \), if \( S_i^{-1}GS_j \) is not contained in \( \mathcal{B}' \). By (1.7) we then have

(1.8) \[ \tilde{\xi}(G) = \sum_{i=1}^r \tilde{\xi}(S_i^{-1}GS_i). \]

Let \( \mathcal{B} \) be an invariant subgroup of \( \mathcal{B} \) as before. The irreducible character \( \zeta_\mu \) of \( \mathcal{B} \) is not associated with any other \( \zeta \) with regard to \( \mathcal{B}_\mu \). Applying Frobenius' reciprocity theorem on induced characters, we obtain the following

**Theorem 1.** Let \( \zeta_\mu \) be any irreducible character of an invariant subgroup \( \mathcal{B}_\mu \) of \( \mathcal{B} \). Denote by \( \chi^{(1)}_\mu, \chi^{(2)}_\mu, \ldots, \chi^{(r)}_\mu \) the irreducible characters of \( \mathcal{B} \) determined by \( \zeta_\mu \) and by \( \xi^{(1)}_\mu, \xi^{(2)}_\mu, \ldots, \xi^{(r)}_\mu \) those of \( \mathcal{B}_\mu \). Then \( t_\mu = h_\mu \) and \( \tilde{\xi}^{(i)}_\mu = \chi^{(i)}_\mu \), if the notation is suitably chosen.

2. **The irreducible representations of \( S(n, m) \).** Any element \( Q \) of \( \mathcal{L} \) is expressed uniquely in the form \( Q = Q_1^{l_1} Q_2^{l_2} \cdots Q_n^{l_n} \) \((0 \leq l_i \leq m - 1)\). \( Q \) is called an element of type \((n_0, n_1, \ldots, n_{m-1})\), if the number of \( l_i \) such that \( l_i = k \) is \( n_k \).
Lemma 1. Two elements \( Q \) and \( Q' \) of \( \mathfrak{S} \) are conjugate in \( S(n,m) \) if and only if they are of same type.

In this section we assume that \( Q \) is an element of type \((n_0, n_1, \ldots, n_{m-1})\) such that \( l_1 = \ldots = l_{n_0} = 0, \ l_{n_0+1} = \ldots = l_{n_0+n_1} = 1 \), and so on.

Since the invariant subgroup \( \mathfrak{S} \) is a commutative group, every irreducible representation of \( \mathfrak{S} \) is of degree one. Denote by \( \omega \) a primitive \( m \)-th root of unity. Then

\[
Q_i \rightarrow \omega^{\alpha_i} \quad (0 \leq \alpha_i \leq m - 1), \quad i = 1, 2, \ldots, n
\]

forms an irreducible representation of \( \mathfrak{S} \). We denote by \( \zeta^{(\alpha)} \) the character of the representation defined above. The character \( \zeta^{(\alpha)} \) is called the character of type \((n_0, n_1, \ldots, n_{m-1})\), if the number of \( \alpha_i \) such that \( \alpha_i = k \) is \( n_k \). Two characters \( \zeta^{(\alpha)} \) and \( \zeta^{(\beta)} \) are associated with regard to \( S(n,m) \) if and only if they are of the same type. In what follows we assume that \( \zeta^{(\alpha)} \) is a character of type \((n_0, n_1, \ldots, n_{m-1})\) such that \( \alpha_1 = \ldots = \alpha_{n_0} = 0, \ \alpha_{n_0+1} = \ldots = \alpha_{n_0+n_1} = 1 \), and so on.

Lemma 2. Let \( \mathfrak{S}^{(\alpha)} \) be the subgroup of \( S(n,m) \) corresponding to the character \( \zeta^{(\alpha)} \). Then \( \mathfrak{S}^{(\alpha)} \) is the normalizer \( N(Q) \) of \( Q \) in \( S(n,m) \).

We have

\[
(2.1) \quad \mathfrak{S}^{(\alpha)} = S_{(\alpha)}^* \mathfrak{S}, \quad S_{(\alpha)}^* \cap \mathfrak{S} = 1,
\]

where \( S_{(\alpha)}^* \) is the subgroup of \( S_n^* \) and is the direct product of \( S_{n_i}^* \):

\[
S_{(\alpha)}^* = S_{n_0}^* \times S_{n_1}^* \times \ldots \times S_{n_{m-1}}^*.
\]

Hence

\[
(2.2) \quad (S(n,m) : \mathfrak{S}^{(\alpha)}) = (S_n^* : S_{(\alpha)}^*) = (S_n : S_{(\alpha)}).
\]

This implies that the number of irreducible characters \( \zeta \) of \( \mathfrak{S} \) associated with \( \zeta^{(\alpha)} \) with regard to \( S(n,m) \) is

\[
\frac{n!}{n_0! n_1! \ldots n_{m-1}!}.
\]

Let

\[
(2.3) \quad S_n = S_{(\alpha_1)} P_1 + S_{(\alpha_2)} P_2 + \ldots \ldots + S_{(\alpha_t)} P_t
\]

be the coset decomposition of \( S_n \) with respect to \( S_{(\alpha)} \). Then

\[
(2.4) \quad S(n,m) = \mathfrak{S}^{(\alpha_1)} P_1^* + \mathfrak{S}^{(\alpha_2)} P_2^* + \ldots \ldots + \mathfrak{S}^{(\alpha_t)} P_t^*,
\]

where \( P_i^* \) is the element of \( S_n^* \) corresponding to \( P_i \) of \( S_n \).
Let $U^* \to D(U^*)$, $U^* \in S_{(\alpha)}^*$, be an irreducible representation of degree $f$ of $S_{(\alpha)}^*$. Then $G = U^*Q \to \zeta^{(\alpha)}_Q D(U^*)$ is an irreducible representation of $\Omega^{(\alpha)}$, determined by $\zeta^{(\alpha)}_Q$. Conversely, if $G \to D'(G)$ is an irreducible representation of $\Omega^{(\alpha)}$ determined by $\zeta^{(\alpha)}_Q$, then $U^* \to D'(U^*)$ is an irreducible representation of $S_{(\alpha)}^*$. This implies that the number of irreducible representations of $\Omega^{(\alpha)}$ determined by $\zeta^{(\alpha)}_Q$ is equal to the number of irreducible representations of $S_{(\alpha)} = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_{m-1}}^*$.

We shall denote by $[\alpha]$ the irreducible representation of $S_n$ corresponding to a diagram $[\alpha]$ of $n$ nodes, and by $\chi^\alpha$ its character. The degree $\chi^\alpha(1)$ of $[\alpha]$ will be denoted by $f_\alpha$. Any irreducible representation of $S_{(\alpha)}$ is given by the Kronecker product representation

$$[\alpha_0] \times [\alpha_1] \times \cdots \times [\alpha_{n-1}],$$

where $[\alpha_i]$ is an irreducible representation of $S_{n_i}$.

Let us denote by $\xi^{(\alpha)}$ the character of the irreducible representation (2.5). As was shown previously, any irreducible character of $\Omega^{(\alpha)}$ determined by $\xi^{(\alpha)}$ is given by $\xi^{(\alpha)} \times \xi^{(\alpha)}$. Theorem 1 shows that the character of $S(n, m)$ induced by $\xi^{(\alpha)} \times \xi^{(\alpha)}$ is irreducible. Hence the irreducible characters of $S(n, m)$ determined by $\xi^{(\alpha)}$ are in (1-1) correspondance with star diagrams

$$[\alpha]_{m}^* = [\alpha_0] \cdot [\alpha_1] \cdot \cdots \cdot [\alpha_{m-1}]$$

of $n$ nodes such that the $i$-th component $[\alpha_i]$ is a diagram of $n_i$ nodes.

We shall denote by $(\alpha)^*$ the irreducible representation of $S(n, m)$ corresponding to $[\alpha]_{m}^*$, and by $\vartheta^*_\alpha$ its character. We see by (2.3) and (2.4) that

$$\vartheta^*_\alpha(W^*) = \sum_j \xi^{(\alpha)}_j (P_j - W J)$$

for $W^* \in S_{n}^*$,

where we set $\xi^{(\alpha)}_j (P_j - W J) = 0$ if $P_j - W J$ is not contained in $S_{(\alpha)}$.

$$\vartheta^*_\alpha(Q) = f_{\alpha_0} f_{\alpha_1} \cdots f_{\alpha_{m-1}} \sum_j \xi^{(\alpha)}_j ((P_j)^* - Q P_j^*)$$

for $Q \in \Omega$.

In particular, if $W^*$ in $S_n^*$ is not contained in $S_{(\alpha)}^*$, then

$$\vartheta^*_\alpha(W^*) = 0.$$
Let \( k(n) \) be the number of partitions of \( n \). The number of distinct irreducible representations of \( S_{(n)} \) is \( k(n) \; k(n) \; \ldots \; k(n_{m-1}) \). Hence, by (1.6) and Theorem 1, the number of irreducible representations of \( S(n, m) \) is given by

\[
I(n, m) = \sum_{n_0, n_1, \ldots, n_{m-1}} k(n_0) k(n_1) \; \ldots \; k(n_{m-1}),
\]

\((\sum n_i = n, \; 0 \leq n_i \leq n)\).

As in [12], we shall denote by \([\alpha]_m^* \) the reducible representation of \( S_n \) induced by the irreducible Kronecker product representation \([\alpha_1] \times [\alpha_2] \times \ldots \times [\alpha_m] \) of \( S_{(n)} \). The representation \([\alpha]_m^* \) is called the skew representation corresponding to the star diagram \([\alpha]_m^* \). We shall denote by \( \chi_{\alpha}^* \) the character of \([\alpha]_m^* \) and by \( f_{\alpha}^* \) its degree. (2.6) implies

\[
\varphi_{\alpha}^*(W^*) = \chi_{\alpha}^*(W).
\]

In particular, the degree of \((\alpha)^* \) is equal to

\[
f_{\alpha}^* = \frac{n!}{n_0! \; n_1! \; \ldots \; n_{m-1}!} f_{\alpha_0} f_{\alpha_1} \; \ldots \; f_{\alpha_{m-1}}.
\]

Thus we have proved the following

**Theorem 2.** The irreducible representations of \( S(n, m) \) are in (1.1) correspondence with star diagrams \([\alpha]_m^* \) of \( n \) nodes.

Let \( H_m \) be the hook product \([4; 4a] \) of a diagram \([\alpha] \) of \( n \) nodes. The degree \( f_{\alpha} \) of \([\alpha] \) is given by \( n! / H_{\alpha} \). We shall define the hook product \( H_{\alpha}^* \) of a star diagram \([\alpha]_m^* \) by

\[
H_{\alpha}^* = H_{\alpha_0} \cdot H_{\alpha_1} \; \ldots \; H_{\alpha_{m-1}}.
\]

**Theorem 3.** Let \((\alpha)^* \) be an irreducible representation of \( S(n, m) \) corresponding to \([\alpha]_m^* \). The degree of \((\alpha)^* \) is given by \( n! / H_{\alpha}^* \).

**Proof.** Our assertion follows immediately from \( f_{\alpha_i} = n_i! / H_{\alpha_i} \) and (2.11).

Let \( P \) be any element of \( S_n \) with \( b_1 \) 1-cycles, \( b_2 \) 2-cycles, \ldots, \( b_k \) \( k \)-cycles. The normalizer \( \mathcal{N}(P) \) of \( P \) in \( S_n \) is the direct product of \( S(b_1, \; 1) \times S(b_2, \; 2) \times \ldots \times S(b_k, \; k) \). Hence we can easily determine the irreducible representations of \( \mathcal{N}(P) \).
Let $A_n^*$ be the subgroup of $S_n^*$ corresponding to the alternating group $A_n$ of $S_n$. Evidently $A_n^* \subseteq$ is an invariant subgroup of $S(n, m)$. This will be denoted by $A(n, m)$ and will be called the generalized alternating group. We shall determine the irreducible representations of $A(n, m)$. If the rows and columns of a diagram $[\alpha]$ are interchanged, the resulting diagram $[\bar{\alpha}]$ is said to be conjugate to $[\alpha]$. If $[\alpha] = [\bar{\alpha}]$, then $[\alpha]$ is called self-conjugate. For a star diagram, we shall say that $[\bar{\alpha}]^* = [\bar{\alpha}_1^*][\bar{\alpha}_1^*] \cdots [\bar{\alpha}_{m-1}^*]$ is conjugate to $[\alpha]^*$. A star diagram $[\alpha]^*$ is called self-conjugate, if $[\alpha]^* = [\bar{\alpha}]^*$.

**Theorem 4.** Let $(\alpha)^*$ be an irreducible representation of $S(n, m)$ corresponding to a star diagram $[\alpha]^*$. If $[\alpha]^*$ is self-conjugate, then $(\alpha)^*$ breaks up into two irreducible conjugate parts of equal degree as a representation of $A(n, m)$. If $[\alpha]^*$ is not self-conjugate, then $(\alpha)^*$ remains irreducible as a representation of $A(n, m)$. Moreover two representations $(\alpha)^*$ and $(\bar{\alpha})^*$ of $A(n, m)$ are equivalent.

We shall study the modular representations of $S(n, m)$ in a forthcoming paper.

3. A generalization of the Murnaghan-Nakayama recursion formula. We first consider the conjugate classes of $S(n, m)$. We see easily that if two elements $W^*$ and $U^*$ of $S_n^*$ are conjugate in $S(n, m)$, then they are conjugate in $S_n^*$. Generally we have

**Lemma 3.** If two elements $W^*Q$ and $U^*Q'$ are conjugate in $S(n, m)$, then $W^*$ and $U^*$ are conjugate in $S_n^*$.

Let $C^*$ be an element of $S_n^*$ corresponding to a $b$-cycle $C = (i_1 i_2 \cdots i_b)$ of $S_n$:

\[
(3.1) \quad C^* = (1_{i_1} 1_{i_2} \cdots 1_{i_b}) (2_{i_1} 2_{i_2} \cdots 2_{i_b}) \cdots (m_{i_1} m_{i_2} \cdots m_{i_b}).
\]

$C^* Q_{i_\alpha}^{-1} (1 \leq l \leq m - 1, 1 \leq \alpha \leq b)$ is the cycle of length $mb$. We shall say that $C^* Q_{i_\alpha}^{-1}$ is a permutation of type $(b, l)$ and denote it by $P(b, l)$. Of course, $P(b, 0) = C^*$. If $i \neq j$, then $P(b, i)$ and $P(b, j)$ are not conjugate in $S(n, m)$. We consider a permutation $P$ of $S(n, m)$ such that

\[
P = P(a_{i_{(0)}}^{(0)}, 0) P(a_{i_{(0)}}^{(m)} , 0) \cdots P(a_{i_{(m-1)}}^{(m-1)}, m - 1),
\]

where no two of $P(a_{i_{(0)}}, k)$ have common symbols. For a fixed $i$, we may assume that $a_{i_{(0)}}^{(0)} \geq a_{i_{(1)}}^{(0)} \geq \cdots \geq a_{i_{(m-1)}}^{(0)} \geq 0$. We set

\[
a_{i_{(0)}}^{(0)} + a_{i_{(1)}}^{(0)} + \cdots + a_{i_{(m-1)}}^{(0)} = b_i.
\]
Then
\[ b_0 + b_1 + \cdots + b_{m-1} = n \quad (0 \leq b_i \leq n). \]

We set \([\alpha] = [a_1^{(\alpha)}, a_2^{(\alpha)}, \ldots, a_t^{(\alpha)}] \) and associate \(P\) with a star diagram \([\alpha] = [a_0], [\alpha_1], \ldots, [\alpha_{m-1}]\) of \(n\) nodes. We then have

**Lemma 4.** Let \(S\) and \(T\) be two elements of \(S(n, m)\) corresponding to the star diagrams \([\alpha]_n^*\) and \([\beta]_m^*\) of \(n\) nodes respectively. \(S\) and \(T\) are conjugate in \(S(n, m)\) if and only if \([\alpha]_n^* = [\beta]_m^*\).

Since there exists an element of \(S(n, m)\) corresponding to an arbitrary star diagram of \(n\) nodes, Lemma 4 implies that there exist at least the \(l(n, m)\) elements which are not mutually conjugate in \(S(n, m)\). On the other hand, Theorem 2 shows that the number of conjugate classes of \(S(n, m)\) is \(l(n, m)\). Thus, if we denote by \(P^*_\alpha\) the element of \(S(n, m)\) corresponding to \([\alpha]_n^*\), then the \(l(n, m)\) elements \(P^*_\alpha\) form a complete system of representatives for the conjugate classes of \(S(n, m)\). Hence we have obtained the following

**Theorem 5.** The conjugate classes of \(S(n, m)\) are in (1, 1) correspondence with star diagrams \([\alpha]_n^*\) of \(n\) nodes.

We shall summarize some results of G. de B. Robinson \([11; 12]\) on the skew representations of the symmetric group which are significant hereafter. Let \([\alpha] - [\beta]\) be a skew diagram \([11]\) of \(l\) nodes. \([\alpha] - [\beta]\) determines a reducible representation of \(S_l\). This is called a skew representation of \(S_l\) and is denoted by \([\alpha] - [\beta]\). We shall denote by \(x^\alpha_\beta\) the character of \([\alpha] - [\beta]\). The irreducible representation \([\alpha]_n\) of \(S_n\) is reducible considered as a representation of a subgroup \(S_k \times S_l\). Let \([\alpha] = \sum \gamma g_{\gamma} [\beta] \times [\gamma]\). Then \([\alpha] - [\beta] = \sum_\gamma g_{\gamma} [\gamma], so that

\[
[\alpha] = \sum_\beta [\beta] \times ([\alpha] - [\beta]).
\]

Hence we have for \(S = S^{(\alpha)} S^{(\beta)} \in S_k \times S_l\)

\[
x^\alpha_\beta(S) = \sum_\beta x^\alpha_\beta(S^{(\alpha)}) x^\beta_\beta(S^{(\beta)}).
\]

If \(C\) is a cycle of length \(l\) in \(S_l\), then

\[
x^\alpha_\beta(C) = (-1)^r \quad \text{or} \quad 0,
\]

according as \([\alpha] - [\beta]\) is a skew hook equivalent to the right hook \(H_r = [n - r, 1^r]\) or not. We can prove, as in \([11]\), the Murnaghan-Nakayama recursion formula \([5; 7]\) by (3.3) and (3.4).
We shall prove, by the analogous method, a generalization of the Murnaghan-Nakayama recursion formula for $S(n, m)$. Let $(\alpha)^*$ be an irreducible representation of $S(n, m)$ corresponding to a star diagram $[\alpha]_*$. Let $[\alpha_i] - [\beta_i]$ be a skew diagram of $l_i$ nodes. A diagram which has $[\alpha_i] - [\beta_i]$ as its $i$-th component will be called a skew star diagram and will be denoted by $[\alpha]^* - [\beta]^*$:

$$[\alpha]^* - [\beta]^* = [\alpha_{i_1}] - [\beta_{i_1}]; [\alpha_{i_2}] - [\beta_{i_2}]; \cdots; [\alpha_{i_{m-1}}] - [\beta_{i_{m-1}}].$$

We set $\sum l_i = l$. Then $[\alpha]^* - [\beta]^*$ corresponds to a reducible representation of $S(l, m)$, which will be denoted by $(\alpha)^* - (\beta)^*$, where $(\beta)^*$ denotes the irreducible representation of $S(n - l, m)$ corresponding to $[\beta]^* = [\beta_{i_1}]; [\beta_{i_2}]; \cdots; [\beta_{i_{m-1}}]$. The representation $(\alpha)^*$ is reducible considered as a representation of a subgroup $S(n - l, m) \times S(l, m)$. Let

$$(\alpha)^* = \sum h_{\alpha \beta^*}(\beta)^* \times (\tau)^*$$

as a representation of $S(n - l, m) \times S(l, m)$.

**Theorem 6.** Let $[\alpha_i] - [\beta_i] = \sum g_{a_i b_i \gamma_i} [\tau_i]$. Then

$$(\alpha)^* - (\beta)^* = \sum h_{\alpha \beta^*}(\tau)^*,$$

where $h_{\alpha \beta^*} = \prod g_{\alpha_i b_i \gamma_i}$ and $(\tau)^*$ is an irreducible representation of $S(l, m)$ corresponding to $[\tau]^* = [\gamma_{i_1}] [\gamma_{i_2}] \cdots [\gamma_{i_{m-1}}]$.

If $[\alpha_i] = [\beta_i]$, we must set $g_{a_i b_i \gamma_i} = 1$ in Theorem 6. We obtain by Theorem 6 and (3.5)

$$(\alpha)^* = \sum (\beta)^* \times ((\alpha)^* - (\beta)^*).$$

We shall denote by $\vartheta_{\alpha \beta^*}$ the character of $(\alpha)^* - (\beta)^*$. By (3.6) we have for $T = T^{(\alpha)} T^{(\beta)} \in S(n - l, m) \times S(l, m)$

$$\vartheta_{\alpha \beta^*}(T) = \sum \vartheta_{\alpha \beta^*}(T^{(\alpha)}) \vartheta_{\alpha \beta^*}(T^{(\beta)}).$$

In particular, if $T^{(\alpha)} = U^*$ is an element of the subgroup $S_i^*$ of $S(l, m)$, then

$$\vartheta_{\alpha \beta^*}(U^*) = \sum h_{\alpha \beta^*} \chi_{i^*}(U),$$

where $U$ is an element of $S_i$ corresponding to $U^*$ of $S_i^*$. Let $C^*$ be an element of type $(l, 0)$, that is, an element of $S_i^*$ corresponding to an $l$-cycle $C$ of $S_i$. We shall determine the value of $\chi_{i^*}(C)$. Let $l_i < l$ for every $i$. Since $C$ is not contained in a subgroup $S_i \times S_{i_0},$
× ... × S_{m-1} of S_m, we have \( \chi_*(C) = 0 \) by (2.8). Next we consider the case when one of \( l_i \), say \( l_0 \), is equal to \( l \) and \( l_i = 0 \) \((0 < i)\). We see by (3.4) that \( \chi_*(C) = \chi_{a_0}^{(n)}(C) = (-1)^r \) or 0, according as \([a_0] - [\beta_0]\) is a skew hook equivalent to the right hook \( H_r = [l - r, 1^r] \) or not. In this case we have \( g_{a_i a_i r_i} = 1 \) for every \( i > 0 \). Hence we can conclude that

\[
\vartheta_{a_i}^{(n)}(C) = (-1)^r \text{ or } 0,
\]

according as \([a_i]^* - [\beta]^*\) is a skew hook of some component \([a_i]\) equivalent to the right hook \( H_r = [l - r, 1^r] \) or not. (3.7), combined with (3.9), yields a generalization of the Murnaghan-Nakayama recurrence formula for \( S(n, m) \).

**Theorem 7.** Let \( H_1, H_2, \ldots \) be the totality of hooks of length \( l \) in the star diagram \( T^* = [\alpha]^* \), and let \( \vartheta^*(T^*) \) be the character of \((\alpha)^* \) of \( S(n, m) \) corresponding to \( T^* \). Then

\[
\vartheta^*(T^*; P) = \sum_i (-1)^{r_i} \vartheta^*(T^* - H_i; \bar{P}),
\]

where \( P \) is any permutation of \( S(n, m) \) which contains a permutation \( C^* \) of \( S^*_m \) corresponding to a cycle \( C \) of length \( l \) and \( \bar{P} \) is the permutation of \( S(n - l, m) \) obtained by removing \( C^* \) from \( P \). If \( T^* \) has no hook of length \( l \), then \( \vartheta^*(T^*; P) = 0 \).

As a special case of Theorem 7, we obtain

**Corollary.** Let \( H_1, H_2, \ldots \) be the totality of hooks of length \( l \) in the star diagram \( T^* = [\alpha]^* \), and let \( \chi^*(T^*) \) be the character of the skew representation \([\alpha]^* \) of \( S_n \). Then

\[
\chi^*(T^*; P) = \sum_i (-1)^{r_i} \chi^*(T^* - H_i; \bar{P}),
\]

where \( P \) is any permutation of \( S_n \) which contains a cycle \( C \) of length \( l \) and \( \bar{P} \) is the permutation on \( n - l \) symbols obtained by removing \( C \) from \( P \). If \( T^* \) has no hook of length \( l \), then \( \chi^*(T^*; P) = 0 \).

In what follows we shall denote by \([\alpha]^*\) the irreducible representation of \( S(n, m) \) corresponding to a star diagram \([\alpha]^*\) in place of \((\alpha)^*\) and by \( \chi_{\alpha^*} \) its character.

**4. The decomposition numbers of \( S_n \).** Let \( p \) be a fixed prime number. If \( b \) \( p \)-hooks are removable from \([\alpha]\) of \( n \) nodes, we shall say that \([\alpha]\) is of weight \( b \) and residue \([\alpha^{(p)}]\) of \( n - bp \) nodes is called the \( p \)-core of \([\alpha]\). The \( p \)-hook structure of \([\alpha]\) is completely repre-
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presented by the star diagram $[\alpha]_p^* = [\alpha_0] \cdot [\alpha_1] \cdot \cdots \cdot [\alpha_{p-1}]$ of $b$ nodes [12; also 8, 13]. Namely, each node of $[\alpha]_p^*$ represents a $p$-hook of $[\alpha]$ and each $r$-hook of $[\alpha]_p^*$ represents an $rp$-hook of $[\alpha]$. Let $H = [g-r, 1^r]$ be a $g$-hook of $[\alpha]$. $(-1)^r$ is called the parity of $H$ and is denoted by $\sigma(H)$. Let us consider a $cp$-hook $H = [cp-r, 1^r]$ of $[\alpha]$ and suppose that its representative in $[\alpha]_p^*$ is $H^* = [c-s, 1^r]$. If we denote by $H_i$ the $i$-th of the $c$ component $p$-hooks of $H$, then we have [11]

\begin{equation}
\sigma(H) = \sigma(H^*) \prod_i \sigma(H_i).
\end{equation}

Let $[\beta]$ be a diagram obtained by removing successively $b, p$-hook $H_1$, $b, p$-hook $H_2$, $\cdots$, $b, p$-hook $H_b$ from $[\alpha]$. We set $\sigma'(\alpha, \beta) = \prod \sigma(H_i)$. Suppose that the representatives of $H_i$ in $[\alpha]_p^*$ are $H_i^*$. We set $\sigma'(\alpha^*, \beta^*) = \prod \sigma(H_i^*)$. Let $b = \sum_i b_i$. Since $[\beta]$ is obtained by removing successively $b$ $p$-hooks from $[\alpha]$, we shall denote by $\sigma(\alpha, \beta)$ the product of parities of these $b$ $p$-hooks. Then it follows from (4.1) that

\begin{equation}
\sigma'(\alpha, \beta) = \sigma(\alpha^*, \beta^*) \sigma(\alpha, \beta).
\end{equation}

Let $P \in S_n$ be the product of $a_1, p$-cycle $Q_1$, $a_2, p$-cycle $Q_2$, $\cdots$, $a_r, p$-cycle $Q_r$, where $a_1 \geq a_2 \geq \cdots \geq a_r \geq 1$. $P$ is called an element of type $(a_1, a_2, \cdots, a_r)$ and of weight $a = \sum a_i$ [10]. We shall associate $P$ with the diagram $[\mu] = [a_1, a_2, \cdots, a_r]$ and $P$ will be denoted by $P_\mu$. The number of elements of weight $a$ such that they all lie in different conjugate classes of $S_n$ is $k(a)$, where $k(a)$ denotes, as before, the number of diagrams of $a$ nodes. We set $n = n' + tp$ ($0 \leq n' < p$) and $\sum_{a=1}^r k(a) = r$. We then have $r$ elements $P_\mu$ of $S_n$, where $[\mu]$ ranges over $r$ diagrams of $a$ nodes ($0 \leq a \leq t$). Every conjugate class contains an element of the form $VP_\mu$, where $[\mu]$ is uniquely determined by the class and where $V$ is a $p$-regular element of $S_{n-s-p}$, if $[\mu]$ is a diagram of $a$ nodes. In what follows we shall denote by $n_\mu$ the number of nodes of $[\mu]$. Let $[\alpha^{(0)}]$ be a $p$-core with $m$ nodes and $n = m + bp$, and let $B$ be the $p$-block of $S_n$ with $p$-core $[\alpha^{(0)}]$. We denote by $\chi^{(\infty)}_\beta$ the character of the irreducible representation $[\beta]$ of $S_{n-s-p}$. Let $P_\mu$ be an element of type $[\mu] = [a_1, a_2, \cdots, a_r]$. Applying the Murnaghan-Nakayama recursion formula iterated $s$ times to $[\alpha] \subset B$, we obtain
(4.3) \[ \chi_{a}(VP_{\mu}) = \begin{cases} \sum_{\beta} \sigma'(\alpha, \beta) h^{(\omega)}(\alpha, \beta) \chi_{\mu}^{(\omega)}(V), & [\beta] \subseteq B^{(\omega)}, \\ 0 & (\text{for } n_{\mu} \leq b), \\ & (\text{for } b < n_{\mu}), \end{cases} \]

where the \( h^{(\omega)}(\alpha, \beta) \) are rational integers \( \geq 0 \), and \( B^{(\omega)} \) denotes the block of \( S_{n-n_{\mu}} \) with \( p \)-core \([\alpha^{(\omega)}]\). Let \( \varphi_{\lambda}^{(\omega)} \) be the character of \( S_{n-n_{\mu}} \) in the modular irreducible representation \( \lambda \). We then have

(4.4) \[ \chi_{\mu}^{(\omega)}(V) = \sum_{\lambda} d_{\mu \lambda}^{(\omega)} \varphi_{\lambda}^{(\omega)}(V) \quad (V \text{ in } S_{n-n_{\mu}} \text{, } p\text{-regular}), \]

where the \( d_{\mu \lambda}^{(\omega)} \) are the decomposition numbers of \( S_{n-n_{\mu}} \). Hence (4.3), combined with (4.4), yields

(4.5) \[ \chi_{a}(VP_{\mu}) = \sum_{\lambda} u_{a \lambda}^{(\omega)} \varphi_{\lambda}^{(\omega)}(V), \]

where

(4.6) \[ u_{a \lambda}^{(\omega)} = \sum_{\beta} \sigma'(\alpha, \beta) h^{(\omega)}(\alpha, \beta) d_{\mu \lambda}^{(\omega)}. \]

The \( u_{a \lambda}^{(\omega)} \) will be called the \( u \)-numbers of \( S_{n} \). Let \( D = (d_{a \lambda}) \) be the decomposition matrix of \( S_{n} \). For \( P_{0} = 1 \), we have

(4.7) \[ u_{a \lambda}^{(\omega)} = d_{a \lambda}. \]

In [10] we have proved the orthogonality relations for the \( u \)-numbers \( u_{a \lambda}^{(\omega)} \):

(4.8) \[ \sum_{a} u_{a \lambda}^{(\omega)} u_{a \nu}^{(\omega)} = 0 \quad [\alpha] \subseteq B, \quad \text{if } [\mu] \neq [\nu], \]

(4.9) \[ \sum_{a} u_{a \lambda}^{(\omega)} u_{a \nu}^{(\omega)} = c_{a \lambda}^{(\omega)} II_{i}(k_{i}! (i\bar{p})^{e_{i}}) \quad [\alpha] \subseteq B, \]

where the \( c_{a \lambda}^{(\omega)} \) denote the Cartan invariants of \( S_{n-n_{\mu}} \) and \([\mu] = (1^{x_{1}}, 2^{x_{2}}, \ldots, n^{x_{n}})\). In particular, by (4.7) and (4.8)

(4.10) \[ \sum_{a} d_{a \lambda} u_{a \lambda}^{(\omega)} = 0 \quad [\alpha] \subseteq B, \quad \text{if } [\mu] \neq [0]. \]

Let \( P_{a}^{*} \) be, as before, a complete system of representatives for the conjugate classes of \( S(b, \bar{p}) \). \( P_{a}^{*} \) is contained in \( S_{n} \) if and only if the first component \([\alpha_{0}]\) of \([\alpha]_{P}^{*}\) is a diagram of \( b \) nodes and \([\alpha_{0}] = [0] \) for \( 0 < i \). On the other hand, \( P_{a}^{*} \) is contained in \( \emptyset \) if and only if \([\alpha_{i}] = [1^{x_{i}}] \) or \([0] \) for every \( i \). We associate \( P_{a}^{*} \) with a diagram \([\mu] \), if \([\alpha_{0}] = [\mu] \). The number of \( P_{a}^{*} \) associated with a fixed \([\mu] \) is \( l^{*}(b-n_{\mu}) \). Here \( l^{*}(a) \) is defined by
\[(4.11) \quad l^{*}(a) = \sum_{b_1, b_2, \ldots, b_{p-1}} k(b_1) k(b_2) \cdots k(b_{p-1}), \quad (\sum b_i = a, \ 0 \leq b_i \leq a).\]

We have proved [9; also 6, 3, 10] that the number of modular irreducible representations in a $p$-block of weight $a$ is $l^{*}(a)$. Let $P_{\mu}^{*}$ be any element of $S(b, p)$ associated with $[\mu]$. Then $P_{\mu}^{*}$ is expressed in the form $T_{i}^{(\nu)} R_{\mu}^{*} = R_{\mu}^{*} T_{i}^{(\nu)}$, where $R_{\mu}^{*}$ is an element of $S_{\mu}^{*}$ corresponding to $[\mu] \cdot [0] \cdots \cdot [0]$, considered as an element of $S(n, p)$, and $T_{i}^{(\nu)}$ is an element corresponding to $[0] \cdot [\alpha_i] \cdots \cdot [\alpha_{p-1}]$, considered as an element of $S(b - n, p)$. Hence the $l(b, p)$ elements

\[T_{i}^{(\nu)} R_{\mu}^{*} \quad (i = 1, 2, \ldots, l^{*}(b - n))\]

form a complete system of representatives for the conjugate classes of $S(b, p)$, if $[\mu]$ ranges over all diagrams of $a$ nodes $(0 \leq a \leq b)$. In particular, the $T_{i}^{(\nu)} (i = 1, 2, \ldots, l^{*}(b))$ are the elements of $S(b, p)$ corresponding to $[\alpha]^{*}$ such that $[\alpha] = [0]$.

We consider a diagram $[\alpha]$ with $p$-core $[\alpha^{(0)}]$ belonging to a $p$-block $B$ of weight $b$. Let $[\alpha]^{*}$ be the irreducible representation of $S(b, p)$ corresponding to the star diagram $[\alpha]^{*}$ of $[\alpha]$ and let $[\mu] = [\alpha_1, \alpha_2, \ldots, \alpha_s]$. Applying the Murnaghan-Nakayama recursion formula (Theorem 7) iterated $s$ times to $[\alpha]^{*}$, we obtain

\[(4.12) \quad \chi_{\alpha}^{*}(T_{i}^{(\nu)} R_{\mu}^{*}) = \sum_{[\beta]} \sigma^{*}(\alpha^{*}, \beta^{*}) h^{(\nu)}(\alpha^{*}, \beta^{*}) \chi_{\beta}^{*}(T_{i}^{(\nu)}),\]

where $[\beta]$ ranges over all star diagrams of $S(b - n, p)$. Moreover we see that $h^{(\nu)}(\alpha^{*}, \beta^{*})$ is equal to $h^{(\nu)}(\alpha, \beta)$ in (4.3):

\[(4.13) \quad h^{(\nu)}(\alpha^{*}, \beta^{*}) = h^{(\nu)}(\alpha, \beta).\]

For any $R_{\mu}^{*}$ of $S_{\mu}$ corresponding to $[\mu] \cdot [0] \cdots \cdot [0]$, we have

\[\chi_{\alpha}^{*}(R_{\mu}^{*}) = \sigma^{*}(\alpha^{*}, 0) h^{(\nu)}(\alpha^{*}, 0) = \sigma^{*}(\alpha^{*}, 0) h^{(\nu)}(\alpha, \alpha^{(0)}).\]

Let $VP_{\mu}$ be an element of $S_{n}$ such that $[\mu]$ is a diagram of $b$ nodes and $V$ is any $p$-regular element on the fixed symbols of $P_{\mu}$. We have by (4.2) and (4.3)

\[\chi_{\alpha}(VP_{\mu}) = \sigma'_{\alpha}(\alpha, \alpha^{(0)}) h^{(\nu)}(\alpha, \alpha^{(0)}) \chi_{\alpha}^{(0)}(V) = \sigma^{*}(\alpha^{*}, 0) \sigma(\alpha, \alpha^{(0)}) h^{(\nu)}(\alpha, \alpha^{(0)}) \chi_{\alpha}^{(0)}(V) = \sigma_{\alpha} \chi_{\alpha}^{*}(R_{\mu}^{*}) \chi_{\alpha}^{(0)}(V),\]
where \( \sigma_a = \sigma(\alpha, \alpha^{(0)}) \). This result was first obtained by R. M. Thrall and G. de B. Robinson [14]. Since \([\alpha^{(0)}] \) is the \( p \)-core, \( \chi_a^{(0)} \) is irreducible as a modular character of \( S_{n-bp} \). If we set \( \chi_a^{(0)} = \varphi_a^{(0)} \), we have

\[
(4.14) \quad u_{\alpha}^{(0)} = \sigma_a \chi_r^{(0)}(R_\mu^*) \quad \text{(for \([\mu]\) of \( b \) nodes).}
\]

(4.14) combined with (4.10), yields

\[
(4.15) \quad \sum \sigma_a d_{\alpha \lambda} \chi_a^{*}(R_\mu^*) = 0 \quad \text{(for \([\mu]\) of \( b \) nodes),}
\]

where \([\alpha] \) ranges over all diagrams in a \( p \)-block \( B \) of weight \( b \).

Generally, by (4.8) and (4.13), we have [10, Theorem 3] for any \([\mu]\) of \( b \) nodes and \([\nu]\) of \( a \) nodes with \( a \neq b \)

\[
(4.16) \quad \sum \chi_a(VP_\nu) \chi_r^{*}(R_\mu^*) = 0 \quad [\alpha] \subset B.
\]

We shall consider the special case when \( b = 1 \). Since \( S(1, p) \) is the cyclic group of order \( p \) with generator \( Q = (1 \ 2 \ \ldots \ \ p) \), the number of irreducible characters of \( S(1, p) \) is \( p \). Let \( \omega \) be a primitive \( p \)-th root of unity. The irreducible character \( \chi_a^{*} \) of the representation \( Q \to \omega^i (0 \leq i \leq p - 1) \) corresponds to the star diagram \([\alpha]^*\) of one node with \( i \)-th component \([\alpha_i] = \{1\}\). Also \( Q^i \) corresponds to the same star diagram. Let \((d_{\alpha \lambda})\) be the decomposition matrix of a \( p \)-block \( B \) of weight \( 1 \). As was shown previously, \((d_{\alpha \lambda})\) is a matrix of type \((p, p - 1)\). Hence each column of \((\sigma_a d_{\alpha \lambda})\) can be written as a linear combination of the columns of \((\chi_a^{*}(Q^i))\):

\[
\sigma_a d_{\alpha \lambda} = \sum_{i=0}^{p-1} m_{i \lambda} \chi_a^{*}(Q^i) \quad [\alpha] \subset B.
\]

By the orthogonality relations for group characters of \( S(1, p) \), we have

\[
m_{i \lambda} = \frac{1}{p} \sum_a \sigma_a d_{\alpha \lambda} \chi_a^{*}(Q^{-i}).
\]

According to (4.14), we obtain

\[
\sum_a \sigma_a d_{\alpha \lambda} \chi_a^{*}(1) = \sum_a \sigma_a d_{\alpha \lambda} = 0,
\]

whence \( m_{i \lambda} = 0 \). This implies that

\[
(4.17) \quad \sigma_a d_{\alpha \lambda} = (\chi_a^{*}(Q^i)) M_i \quad l = 1, 2, \ldots, p - 1.
\]

Here \( M_i = (m_{i \lambda}) \) with \( l \) (\( 1 \leq l \leq p - 1 \)) as row index and \( i \) as column index. We see easily that \( M_i \) is non-singular.
Now we shall prove the following theorem [10, Theorem 5].

**Theorem 8.** Let \( D = (d_{\alpha \lambda}) \) be the decomposition matrix of a \( p \)-block \( B \) of weight \( b \). Let \( T_i^{(\mu)} (i = 1, 2, \ldots, l^*(b)) \) be the elements of \( S(b, p) \) associated with \([\mu] \neq [0]\). There exists a non-singular matrix \( M_b \) of degree \( l^*(b) \) which satisfy

\[
(\sigma_\alpha d_{\alpha \lambda}) = (\chi_\alpha^* (T_i^{(\mu)})) M_b.
\]

**Proof.** \( D \) is a matrix of type \((l(b), l^*(b))\). (Since \( p \) is a fixed prime number, we shall denote \( l(b, p) \) simply by \( l(b) \).) It follows from (4.12) and (4.13) that

\[
(\chi_\alpha^* (T_i^{(\mu)} R_{\mu^0})) = (\sigma^*(\alpha^*, \beta^*) h^{(\alpha)}(\alpha, \beta)) (\chi_{\beta^0}^* (T_i^{(\mu^0)})
\]

for a fixed diagram \([\mu] \neq [0]\). As was shown before, the theorem is true for \( b = 1 \). We shall assume it to be true for all \( p \)-blocks of weight less than \( b > 1 \). By our inductive assumption, we have

\[
(\sigma_\beta d_{\beta \lambda}^{(\mu^0)}) = (\chi_{\beta^0}^* (T_i^{(\mu^0)})) M_{b-n_{\mu^0}}.
\]

Observe that \( T_i^{(\mu)} \) corresponds to the star diagram of \( b-n_{\mu^0} \) nodes with the first component \([0]\), considered as the element of \( S(b - n_{\mu^0}, p) \). We have by (4.2)

\[
\sigma_\alpha = \sigma_\beta \sigma(\alpha, \beta) = \sigma_\beta \sigma(\alpha, \beta) \sigma^*(\alpha^*, \beta^*),
\]

where we set \( \sigma_\beta = \sigma(\beta, \alpha^0) \). Hence it follows from (4.18), (4.19) and (4.6) that

\[
(\chi_\alpha^* (T_i^{(\mu^0)} R_{\mu^0})) = (\sigma^*(\alpha^*, \beta^*) h^{(\alpha)}(\alpha, \beta)) (\sigma_\beta d_{\beta \lambda}^{(\mu^0)}) M_{b-n_{\mu^0}}^{-1}
\]

\[
= (\sum_\beta \sigma_\alpha \sigma(\alpha, \beta) h^{(\alpha)}(\alpha, \beta) d_{\beta \lambda}^{(\mu^0)}) M_{b-n_{\mu^0}}^{-1}
\]

\[
= (\sigma_\alpha u_{\alpha}^{(\mu)}) M_{b-n_{\mu^0}}^{-1}.
\]

This, combined with (4.10), yields

\[
\sum_\alpha \sigma_\alpha d_{\alpha \lambda} \chi_\alpha^* (T_i^{(\mu)} R_{\mu^0}) = 0 \quad [\alpha] \subset B,
\]

for any \([\mu] \neq [0]\). By the orthogonality relations for group characters of \( S(b, p) \), each column of \( (\sigma_\alpha d_{\alpha \lambda}) \) can be written as a linear combination of the columns of \( (\chi_\alpha^* (T_i^{(\mu)})) (i = 1, 2, \ldots, l^*(b)) \). Thus we have

\[
(\sigma_\alpha d_{\alpha \lambda}) = (\chi_\alpha^* (T_i^{(\mu)})) M_b,
\]

where \( M_b \) is non-singular.
(4.21) yields

\[ \sum_{\alpha} \sigma_{\alpha}(V) \chi_{\alpha}(T_{i}^{(\alpha\lambda \lambda')} R_{\mu}^\ast) = 0 \quad [\alpha] C B \]

for any \( p \)-regular element \( V \) of \( S_n \) and any \( [\mu] \neq [0] \) [10, Theorem 4].

Generally we have by (4.8) and (4.20)

\[ \sum_{\alpha} \sigma_{\alpha}(VP_{\nu}) \chi_{\alpha}(T_{i}^{(\alpha\lambda \lambda')} R_{\mu}^\ast) = 0 \quad [\alpha] C B, \quad \text{if} \quad [\nu] \neq [\mu]. \]

As an application of Theorem 8, we shall prove the following theorem [10, Corollary to Theorem 5].

**Theorem 9.** Let \((d_{\alpha \lambda})\) and \((\bar{d}_{\alpha \lambda})\) be the decomposition matrices of \( p \)-blocks \( B \) and \( \bar{B} \) of same weight \( b \) respectively, and let \([\alpha] \) and \([\alpha']\) have the same star diagram \([\alpha']^\ast\). Then

\[ (\sigma_{\alpha}, \bar{d}_{\alpha \lambda}) = (\sigma_{\alpha}, d_{\alpha \lambda})(w_{\lambda \lambda'}), \]

where the \( w_{\lambda \lambda'} \) are rational integers and \( |w_{\lambda \lambda'}| = \pm 1. \)

**Proof.** We have by Theorem 8

\[ (\sigma_{\alpha}, \bar{d}_{\alpha \lambda}) = (\chi_{\alpha}(T_{i}^{(\alpha)})) \bar{M}_{b}. \]

Hence

\[ (\sigma_{\alpha}, \bar{d}_{\alpha \lambda}) = (\sigma_{\alpha}, d_{\alpha \lambda}) M_{b}^{-1} \bar{M}_{b}. \]

If we set \( M_{b}^{-1} \bar{M}_{b} = W_{b} = (w_{\lambda \lambda'}), \) then we see by Theorem 14 [1] that each column of \((w_{\lambda \lambda'})\) can be written as a linear combination \( \sum_{\alpha} s_{\alpha'}(\sigma_{\alpha}, \bar{d}_{\alpha \lambda'}) \), where the \( s_{\alpha'} \) are rational integers which do not depend on \( \lambda \). This shows that the \( w_{\lambda \lambda'} \) are rational integers. Then, applying again Theorem 14 [1] to \((\sigma_{\alpha}, \bar{d}_{\alpha \lambda'})\), we can conclude that \( |W_{b}| = \pm 1. \)

It follows from (4.24) that

\[ (\bar{d}_{\alpha \lambda'}) = W_{b}'(c_{\alpha \lambda}) W_{b}, \]

where \( W_{b}' \) denotes the transpose of \( W_{b} \) and where \((c_{\alpha \lambda}), (\bar{c}_{\alpha \lambda'})\) are the matrices of Cartan invariants corresponding to \( B \) and \( \bar{B} \) respectively. (4.25), combined with \( |W_{b}| = \pm 1, \) yields the following theorem [10, Theorem 6].

**Theorem 10.** Two matrices of Cartan invariants corresponding to the \( p \)-blocks of same weight have the same elementary divisors.

Let \( U = (u_{\alpha \lambda}^{(\omega)}) \) be the matrix of \( u \)-numbers corresponding to a
$p$-block $B$ of weight $b$ [10]. $U$ is a square matrix of degree $l(b)$ and is non-singular. We have by (4.20)

$$(4.26) \quad (\sigma_u u_{\alpha_b}^{(\omega)}) = (\chi_u^{(\alpha)} (T_i^{(\omega)} R_{u_i}^{(\ast)})) M,$$

where

$$M = \begin{pmatrix} M_b & 0 \\ M_{b-1} & \cdots \\ 0 & M_0 \end{pmatrix}, \quad M_0 = I,$$

if the rows and columns are arranged suitably.

**Theorem 11.** Let $(u_{\alpha_b}^{(\omega)})$ and $(\tilde{u}_{\alpha_{b,b}}^{(\omega)})$ be the matrices of $u$-numbers corresponding to the $p$-blocks $B$ and $\tilde{B}$ of same weight respectively, and let $[\alpha]$ and $[\alpha']$ have the same star diagram $[\alpha]^*$. Then $(\sigma_r u_{\alpha_b}^{(\omega)})$ and $(\sigma_r \tilde{u}_{\alpha_{b,b}}^{(\omega)})$ have the same elementary divisors.

**Proof.** We have by (4.26)

$$(\sigma_r \tilde{u}_{\alpha_{b,b}}^{(\omega)}) = (\sigma_r u_{\alpha_b}^{(\omega)}) W,$$

where

$$W = \begin{pmatrix} W_b & 0 \\ W_{b-1} & \cdots \\ 0 & W_0 \end{pmatrix}.$$

Since $|W| = \pm 1$, our assertion follows immediately.

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