Theory of connections and a theorem of E. Cartan on holonomy groups I

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THEORY OF CONNECTIONS AND A THEOREM OF E. CARTAN ON HOLOMONY GROUPS I

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E. Cartan [1] proved locally a fundamental theorem on holonomy groups of spaces with generalized connections as follows:

**Theorem.** Let $H$ be the holonomy group of a space with a connection of structure group $G$, then the space is equivalent to a space with a connection of structure group $H$.

The proof of E. Cartan holds good for the space whose underlying manifold is an $n$-cell. In this paper, we shall investigate the theorem in the large by means of fibre bundles. For fibre bundles, we shall utilize the notations in [2]. In §§2-5, we will give an elementary explanation on the relation between the concept of infinitesimal connections in fibre bundles introduced by C. Ehresmann [3] and the classical one of E. Cartan [1].

§1. We consider a fibre bundle $\mathcal{B} = \{B, p, X, Y, G\}$. For the purpose of differential geometry the following assumptions will be made:

1) The bundle space $B$, the base space $X$, the fibre $Y$ are connected, differentiable$^2$ manifolds;

2) the group $G$ of the bundle is a Lie group which acts differentiably and effectively on $Y$;

3) the projection $p$ of $B$ onto $X$ is differentiable.

We assume that a differentiable family of tangent subspaces to $B$ which are transversal to the fibres is given. For any curve $\mathcal{C}$ of class $C^r (r \geq 2)$ in $X$ from $x_0$ to $x_1$, and any point $b_0 \in p^{-1}(x_0)$, we have an uniquely determined curve $\zeta$ in $B$ from $b_0$ to a point $b_1 \in p^{-1}(x_1)$ such that $p(\zeta) = \mathcal{C}$ and at any point $b \in \zeta$, $\zeta$ is tangent to the tangent subspace at $b$ of the family. Then, corresponding $b_1$ to $b_0$, we get a homeomorphism

$$\rho(\mathcal{C}) : p^{-1}(x_1) = Y_{x_1} \rightarrow p^{-1}(x_0) = Y_{x_0}.$$

Furthermore, we assume that $\rho(\mathcal{C})$ is a bundle mapping. Then, according to C. Ehresmann [3], we will say an infinitesimal connection

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1) Numbers enclosed in brackets refer to the bibliography.

2) In the following, we suppose that all the manifolds $B, X, Y, \ldots$ are of class $C^r (r \geq 2)$ and the differentiabilities of mappings are of suitable orders respectively.
$\mathcal{I}$ is given in $\mathcal{B}$. Then the group $G$ is called the **structure group** of the connection.

Let us put

$$Q_{x_0, x_1} = \text{the set of curves of class } D^r \text{ in } X \text{ from } x_0 \text{ to } x_1,$$

and

$$Q = \bigcup_{x_0, x_1 \in X} Q_{x_0, x_1}.$$

The above-mentioned $\rho(\mathcal{C})$ can be also defined for any curve of class $D^r$ by combining the homeomorphisms corresponding to subarcs of class $C^r$. Then, by the definition, we have

$$\rho(\mathcal{C}_1 \mathcal{C}_2) = \rho(\mathcal{C}_1) \rho(\mathcal{C}_2), \quad \mathcal{C}_1 \in Q_{x_0, x_1}, \quad \mathcal{C}_2 \in Q_{x_1, x_2}.$$  \hspace{1cm} (1)

Let $Q_x = Q_{x, x}, \chi_x = \rho \mid Q_x$, then by (1) the transformation $\chi_x : Q_x \to \chi_x(Q_x) = \phi_x$ is a homomorphism of the group $Q_x$ of closed paths at $x$ and a group of bundle mappings of $Y_x$ on itself. Let $\xi$ be any admissible map at $x \in X$, then $H_x = \xi^{-1} \phi_x \xi$ is a subgroup of $G$.  \hspace{1cm} (2)

We call $H_x$ the **holonomy group** at $x$ of the bundle $\mathcal{B}$ with the infinitesimal connection $\mathcal{I}$.

Let be given another fibre bundle $\mathcal{B}' = \{B', p', X, Y, G\}$ with an infinitesimal connection $\mathcal{I}'$ as $\mathcal{B}$. Let $\rho', \chi', \phi', H_x'$ be the maps and the groups defined for $\mathcal{B}'$ as analogous to $\rho, \chi, \phi, H_x$.

If for a point $x \in X$, we can take two admissible mappings $\xi : Y \to Y_x, \xi' : Y \to Y'_x$ such that $\xi^{-1} \chi_x(\xi) \xi = \xi'^{-1} \chi'_x(\xi') \xi'$ for any $\xi \in Q_x$, which we denote simply by $\xi^{-1} \chi_x \xi = \xi'^{-1} \chi'_x \xi'$, we denote this by $\chi_x \simeq \chi'_x$.

We shall prove the following lemma.

**Lemma 1.** **Fibre bundles** $\mathcal{B}, \mathcal{B}'$ **with infinitesimal connections**, the same base space, fibre and group are equivalent in $G$ (G-equivalent) as fibre bundles if $\chi_x \simeq \chi'_x$ at a point $x_0 \in X$.

**Proof.** By the assumption of this theorem, let us put

$$\xi^{-1} \chi_{x_0} \xi = \xi'^{-1} \chi'_{x_0} \xi', \hspace{1cm} (2)$$

where $\xi, \xi'$ are admissible mappings of $\mathcal{B}, \mathcal{B}'$ at $x_0$.

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1) A curve in $X$ is said to be of class $D^r, r > 0$, if it is defined by a continuous mapping of a closed interval into $X$, and if the interval can be divided into a finite set of subintervals on the closure of each of which the mapping is of class $C^r$.

2) $\xi^{-1} \phi \xi$ is an abstract subgroup of $G$ and may not be a closed subgroup of $G$. 

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For any point $x \in X$, let $\mathcal{C}$ be a curve of $\mathcal{Q}_{\alpha, \varepsilon}$ and define $\hat{h}_x : Y_x \to Y'_x$ by

\begin{equation}
\hat{h}_x = \rho'(\mathcal{C}^{-1})\xi' \xi^{-1} \rho(\mathcal{C}).
\end{equation}

If $\mathcal{C}_1$ is another curve of $\mathcal{Q}_{\alpha, \varepsilon}$ and $h_{1, x}$ is the corresponding mapping, then we have by (1), (2)

\begin{align*}
\hat{h}_x^{-1}h_{1, x} &= [\rho(\mathcal{C}^{-1})\xi \xi'^{-1} \rho'(\mathcal{C})][\rho'(\mathcal{C}^{-1})\xi' \xi^{-1} \rho(\mathcal{C})] \\
&= [\rho(\mathcal{C}^{-1})\xi][\xi^{-1}\rho'(\mathcal{C})\xi']\xi'^{-1} \rho(\mathcal{C})] \\
&= [\rho(\mathcal{C}^{-1})\xi][\xi'^{-1}x_x(\mathcal{C} \xi^{-1})\xi][\xi'^{-1} \rho(\mathcal{C})] \\
&= \rho(\mathcal{C}^{-1})\rho(\mathcal{C}^{-1})\rho(\mathcal{C}) = 1,
\end{align*}

that is $h_x = h_{1, x}$.

Then, we define an one-to-one transformation $h : B \to B'$ by $h | Y_x = h_x$. For a fixed point $x_1 \in X$, let $U$ be a coordinate neighborhood of $x_1$ which is simply covered by a differentiable family of curves issuing from $x_1$. For $x \in U$, let $\mathcal{C}_x$ be the curve from $x_1$ to $x$ of the family. Then, since $\Gamma$ is differentiable, $\rho(\mathcal{C}_x)(b), b \in \Gamma^{-1}(x)$, is a differentiable mapping of $\Gamma^{-1}(U)$ onto $Y_x$, and $\rho(\mathcal{C}_x^{-1})(b), b \in Y_x$, is a differentiable homeomorphism of $Y_x \times U$ onto $\Gamma^{-1}(U)$. $\rho'(\mathcal{C}_x)$ has the same property as $\rho(\mathcal{C}_x)$. Let $\mathcal{C}_1$ be a curve of $\mathcal{Q}_{\alpha, \varepsilon}$, then we have

\begin{equation}
\hat{h}_x = \rho'(\mathcal{C}_1^{-1}\mathcal{C}_x^{-1})\xi' \xi \rho(\mathcal{C}_x) = \rho'(\mathcal{C}_1^{-1})h_x \rho(\mathcal{C}_x).
\end{equation}

This relation shows that $h$ is continuous at $x_1$, furthermore, $h$ is a differentiable homeomorphism.

Let $\{U_a\}$ be a system of admissible coordinate neighborhoods as above which is a covering of $X$ and

\begin{align*}
\phi_a : U_a \times Y &\longrightarrow \Gamma^{-1}(U_a), \\
\phi'_a : U_a \times Y &\longrightarrow \Gamma^{-1}(U_a)
\end{align*}

be the coordinate functions of $\mathcal{B}$ and $\mathcal{B}'$ respectively. Define

\begin{align*}
\rho_a : \Gamma^{-1}(U_a) &\longrightarrow Y, \\
\rho'_a : \Gamma^{-1}(U_a) &\longrightarrow Y
\end{align*}

by $\rho_a | Y_x = \phi_a | Y_x^{-1}, \rho'_a | Y'_x = \phi'_a | Y'_x^{-1}$. If $U_a \cap U_b \neq \emptyset$, let
be the coordinate transformations of $\mathcal{B}$, $\mathcal{B}'$ respectively.

These mappings have the property as

\begin{equation}
\begin{align*}
\Phi_{\alpha\beta}(x) \Phi_{\beta\gamma}(x) &= \Phi_{\alpha\gamma}(x), \\
\Phi_{\alpha\beta}'(x) \Phi_{\beta\gamma}'(x) &= \Phi_{\alpha\gamma}'(x), \\
x \in U_\alpha \cap U_\beta \cap U_\gamma.
\end{align*}
\end{equation}

If the point $x_i \in U_\alpha \cap U_\gamma$, $\mathcal{E}_s \subset U_\alpha \cap U_\gamma$, then we have

\[ \overline{E}_{\gamma \beta}(x_i) = \Phi_{\gamma \beta}(x) \Phi_{\gamma \alpha}(x_i) \Phi_{\alpha \beta}(x), \]

\[ x \in U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta. \]

Therefore, $h$ is a differentiable bundle mapping. $\mathcal{B}$, $\mathcal{B}'$ are equivalent in $G$.

\section*{§2.}

Let $\mathcal{B} = \{B, \beta, X, Y, G\}$ be a fibre bundle with an infinitesimal connection $\tilde{\tau}$ as in §1, then we can give an infinitesimal connection $\tilde{\tau}$ for the associated principal bundle$^1$ $\tilde{\mathcal{B}} = \{\tilde{B}, \tilde{\beta}, X, G, G\}$ of $\mathcal{B}$ such that for any point $x \in X$ and any curve $\mathcal{E} \in \mathcal{B}$ such that for any point $x_0 = x_1$, $x \in X$ and any curve $\mathcal{E} \in \mathcal{B}$ such that for any point $x_0 = x_1$, $x_1 \in \mathcal{B}$, $\mathcal{E} = G$, since $\rho(\mathcal{E})$ is a bundle mapping. Denoting the right translation corresponding to $g \in G$ by $r(g)$, we get from (6)

\begin{equation}
\begin{align*}
(\tilde{\rho}(\mathcal{E}) r(g))(\xi_{x_1}) &= \tilde{\rho}(\mathcal{E}) (\xi_{x_1} g) \\
&= \rho(\mathcal{E}) (\xi_{x_1} g) \\
&= (\rho(\mathcal{E}) \xi_{x_1}) g \\
&= r(g)(\tilde{\rho}(\mathcal{E}) (\xi_{x_1})).
\end{align*}
\end{equation}

$^1$ See [2], §8.
hence

\[ \tilde{\rho}(\mathcal{C}) r(g) = r(g) \tilde{\rho}(\mathcal{C}). \]

This shows that \( \mathcal{F} \) is invariant under right translations.

Conversely, if we have a differentiable family of tangent subspaces to \( \mathcal{B} \) which are transversal to the fibres and are invariant under right translations, there exists an infinitesimal connection \( \Gamma \) in \( \mathcal{B} \) such that (6) holds good.

By virtue of the above argument, in the following, we may consider only principal fibre bundles.

Let \( \mathcal{B} = \{ B, \rho, X, G, G \} \) be a differentiable principal fibre bundle as in §1 and let \( \Gamma \) be a differentiable family of tangent subspaces \( \Gamma_b \subset T_b(B), \ b \in B \), which are transversal to the fibres \( G_{\rho(b)} \) and are invariant under right translations, that is

\[ \begin{aligned}
    \rho_*(\Gamma_b) &= T_{\rho(b)}(X), \\
    r(g)_* \Gamma_b &= \Gamma_{\rho(g)(b)},
\end{aligned} \]

\[ b \in B, \ g \in G \]

where \( \rho_*, r(g)_* \) denote the differential mappings of \( \rho, r(g) \).

The decomposition of \( T_b(B) \) into the direct sum

\[ T_b(B) = \Gamma_b + T_b(G_{\rho(b)}) \]

define the projection \( \mu_b : T_b(B) \to T_b(G_{\rho(b)}) \). Let \( \mu \) be the mapping \( T(B) \to T(B) \) by \( \mu(v) = \mu_b(v) \) for any \( v \in T_b(B) \). Let \( \epsilon_* \) be the imbedding mapping of \( G_\ast \) into \( B \), then, by the definition of \( \mu_b \), we get

\[ \mu \epsilon_* \ast = \epsilon_* \ast. \]

For any \( v \in T_b(B), \ g \in G \), by (8) and the relation

\[ r(g)_*(v) = r(g)_*(v - \mu_b(v)) = r(g)_*(v - \mu_b(v)) + r(g)_* \mu_b(v) \]

we get

\[ r(g)_* \mu_b = \mu_{r(g)(b)} r(g)_* \]

or

1) For a differentiable manifold \( X \), we denote the tangent space at \( x \in X \) by \( T_x(X) \) and the bundle space of the tangent bundle of \( X \) by \( T(X) \).

2) Let \( X, Y \) be any differentiable manifolds and let \( f \) be a differentiable mapping \( X \to Y \). Then we denote by \( f_* : T(X) \to T(Y) \) the differential mapping of \( f \). If \( f : X \to Y, \ h : Y \to Z, \) then \( (fh)_* = f_* h_* \). See [4] or [5].
\[ r(g)_* \mu = \mu r(g)_*. \]

We denote by the same notation \( b \) the mapping of \( G \) onto \( G \) that \( b(e) = b \) and define a linear transformation \( \pi_b : T_b(B) \to T_b(G) \) by
\[ \pi_b = (b_*)^{-1}\mu_b \]
where \( e \) denotes the identity element of \( G \). Thus, we obtain a set of linear differential forms on \( B \) with values in the Lie algebra \( L(G) \cong T_e(G) \) (as vector space). \(^1\)

Since \( r(g)(b) = bg = bl(g), \quad br(g) = r(g)b \), where \( l(g) : G \to G \) denotes the left translation corresponding to \( g \), for any \( v \in T_b(B) \), we get by (10), (11)
\[
(r(g)^* \pi)(b) = \pi(r(g)_* bv) = \pi_{v_0}(r(g)_* b)v
= ((bg)_*)^{-1}\mu_{v_0}r(g)_* bv
= ((bg)_*)^{-1}r(g)_* \mu_b v
= ((bl(g))_*)^{-1}r(g)_* \mu_b v
= l(g^{-1})_* r(g)_* r(g)_{-1}b_{-1}r(g)_* \mu_b v
= l(g^{-1})_* r(g)_* b_{-1}v
= l(g^{-1})_* r(g)_* \pi(b).
\]

Putting \( ad(g) = (l(g)r(g^{-1}))_* \) which is the differential mapping of the adjoint mapping \( A(g) : G \to G \) by \( A(g)(y) = gyg^{-1}, \ y \in G \), the above relation is written as
\[ (12) \quad r(g)^* \pi = ad(g^{-1}) \pi. \]

For \( v \in T_b(G), \ b \in \mathcal{P}^{-1}(x) \), we have
\[
(t_* b)^* \pi(v) = \pi((t_* b)_* b)
= (bg)_*^{-1}\mu_{v_0}(t_* b)_* b
= l(g^{-1})_* b_{-1}b_{-1}v = l(g^{-1})_*(v).
\]

If we define
\[ (13) \quad (t_* b)^* \pi = \omega, \]

1) We denote by \( T^*(X, L(G)) \) the bundle space of the fibre bundle over \( X \) whose fibre at \( x \in X \) is \( \mathcal{L}(T_x(X); L(G)) \). Let \( f \) be a differentiable mapping \( X \to Y \), then we denote by \( f^* : T^*(Y, L(G)) \to T^*(X, L(G)) \) the dual mapping of \( f_* \). It \( f : X \to Y \), \( h : Y \to Z \), then \( (hf)^* = f^* h^* \).

2) By the natural isomorphism \( l(g)_* : T_e(G) \to T_e(G), \ T_e(G) \cong T_e(G) \).

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the above relation is written as
\[ \omega(v) = l(g^{-1})_* v, \quad v \in T_g(G), \quad g \in G. \]

From this relation, we obtain
\[
\begin{align*}
(14) \quad \begin{cases} \quad l(g)^* \omega = \omega, & g \in G, \\
\quad \omega(v) = v, & v \in T_g(B). \end{cases}
\end{align*}
\]

This shows that the \( L(G) \)-valued linear differential form \( \omega \) on \( G \) is independent of \( b \in B \).

Conversely, we can define a differentiable family of tangent subspaces satisfying (8) from a \( L(G) \)-valued linear differential from \( \pi \) on \( B \) satisfying (12), (13).

\section*{§3.} Now, let \( \iota_a \) be the imbedding mapping \( p^{-1}(U_a) \to B \) and define a mapping \( \rho_a : U_a \to U_a \times G \) by
\[ \rho_a(x) = x \times e, \quad x \in U_a. \]

Define a \( L(G) \)-valued linear differential form \( \theta_a \) on \( U_a \) by
\[ \theta_a = (\iota_a \phi_a \rho_a)^* \pi. \]

Since \( b = \tau(p_a(b)) \iota_a \phi_a \rho_a(b) \), \( b \in p^{-1}(U_a) \), for any \( v \in T_b(B) \), we have
\[ v = (\tau(g) \iota_a \phi_a \rho_a p)_* v + (\iota_a \phi_a(x, e))_* p_a^* v, \quad x = p(b), \quad g = p_a(b). \]

Hence, we get by (12), (13), (14), (15)
\[
\begin{align*}
\pi_b &= p^* \rho_a^* \phi_a^* \iota_a^* \tau(g)^* \pi_b + p_a^* \phi(x, e)^* \iota_a^* \pi_b \\
&= p^* (\iota_a \phi_a \rho_a)^* (\text{ad} (g^{-1}) \pi_b) + p_a^* \phi(x, g)^* \iota_a^* \pi_b \\
&= \text{ad} (g^{-1}) p^* \theta_a, + p_a^* \pi_b \\
&= \text{ad} (g^{-1}) p^* \theta_a, + p_a^* \omega_p,
\end{align*}
\]

that is
\[
\begin{align*}
(16) \quad \pi_b &= \text{ad} (g^{-1}) p^* \theta_a, + p_a^* \omega_p, \quad p(b) = x, \quad p_a(b) = g.
\end{align*}
\]

If \( b \in p^{-1}(U_a \cap U_b) \), then \( p_b(b) = g_{ba}(p(b))p_a(b) \). Hence, at \( b \), we have the relation
\[
\begin{align*}
\pi_b &= (l(g_{ba}(p(b)))_* p_a^* + \tau(p_a(b))_* g_{ba}^* p_a^* \pi_a, \\
p_b^* \omega &= p_a^* l(g_{ba}(p(b)))^* \omega + p_a^* g_{ba}^* \tau(p_a(b))^* \omega \\
&= p_a^* \omega + p_a^* g_{ba}^* (\text{ad} (p_a(b)^{-1}) \omega).
\end{align*}
\]
By the relations above and the equation
\[ \text{ad}(p_\alpha(b^{-1})) \theta_{\beta,\alpha} + p_\alpha^* \omega_{\beta,\alpha} = \text{ad}(p_\beta(b^{-1})) \theta_{\beta,\alpha} + p_\beta^* \omega_{\beta,\alpha}, \]
we get
\[ \theta_{\alpha,\alpha} = p_\alpha^* \{ \text{ad}(g_{\beta,\alpha}(x^{-1})) \theta_{\beta,\alpha} + g_{\beta,\alpha}^* \omega_{\beta,\alpha}(x) \}, \]
from which we obtain
\[ \theta_{\alpha,\alpha} = \text{ad}(g_{\beta,\alpha}(x^{-1})) \theta_{\beta,\alpha} + g_{\beta,\alpha}^* \omega_{\beta,\alpha}(x), \quad (17) \]
or simply
\[ \theta_{\alpha} = \text{ad}(g_{\beta,\alpha}^{-1}) \theta_{\beta} + g_{\beta,\alpha}^* \omega, \quad (17') \]
since \( p \) is onto.

Conversely, on each \( U_\alpha \), let be given a system of \( L(G) \)-valued linear differential forms \( \theta_\alpha \) satisfying (17), then we can obtain a \( L(G) \)-valued linear differential form \( \pi \) satisfying (12), (13) by (16).

Thus we see that an infinitesimal connection \( \Gamma \) as in §1 is given in \( \mathcal{B} \) is equivalent to that on each coordinate neighborhood \( U_\alpha \), a \( L(G) \)-valued linear differential form satisfying (17') is given. The components of \( \theta_\alpha \) are the parameters of the connection in the classical sense and (17') is the transformation equation of the parameters for coordinate transformations.

§4. In \( U_1 \), let be given a differentiable family of curves \( \mathcal{S}(x, x) \in U_1 \) which covers simply over \( U_1 \) except \( x_1 \). Then, \( \rho(\mathcal{S}(x, x)) : G_\pi \to G_{\pi_1} \) define a differentiable mapping
\[ F : p^{-1}(U_1) \to G_{\pi_1} \quad \text{by} \quad F(b) = \rho(\mathcal{S}(x, p(b)) \, b). \]
Since \( F | G_\pi \) is a bundle mapping, we can define a differentiable mapping \( \gamma : U_1 \to G \) by
\[ \gamma(x, g) = f(x, g). \quad (18) \]

Let \( \tau_1 : U_1 \times G \to U_1 \), \( \tau_2 : U_1 \times G \to G \) be the natural projections, then for any \( v \in T_{\pi_1}(U_1 \times G) \), we get by (14), (16)
\[ f_\pi v = (\pi \tau_1)_* v + \pi \tau_2_*, \]
\[ (\pi_1 F \phi_1)_* v = p_1^* \mu_{\phi_1} \phi_1_*, \quad b_1^{-1} \mu_{\phi_1} \phi_1_* v = \pi_1 (\phi_1_* v) \]

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\[\gamma(x) = e, \quad p_i(b_i) = e, \quad \tau_1 = p_i \phi_i, \quad \tau_2 = p_i \phi_i. \]

Hence, from (18) and the above relation we obtain

\[\gamma_*(\tau_1 \times v) = \theta_1(\tau_1 \times v)\]

or

\[\gamma^* \omega = \theta_1, \pi_1.\]

This equation will imply the following result which is in connection with the development of a curve in \(X\) on a tangent space to \(X\) at a point of the curve, in the classical differential geometry.

For any curve \(\mathcal{C}\) of class \(C^1\) from \(x_0\) to \(x_1: x = \psi(t), 0 \leq t \leq 1,\)

let \(\mathcal{C} \subset U_{\lambda}, \lambda = 1, 2, \ldots, m,\) be the subarc of \(\mathcal{C}\) corresponding to the interval \(t_{\lambda-1} \leq t \leq t_{\lambda}, 0 = t_0 < t_1 < \ldots < t_m = 1.\) Then, we can determine mappings

\[\gamma_\lambda: [t_{\lambda-1}, t_\lambda] \rightarrow G,\]

so that

\[\begin{align*}
\gamma_\lambda^* \omega &= \psi_\lambda^* \theta_{\alpha_\lambda}^* \\
\gamma_\lambda(t_{\lambda-1}) &= \gamma_{\lambda-1}(t_{\lambda-1}) g_{\alpha_{\lambda-1} a_\lambda}^\lambda(\psi(t_{\lambda-1}))
\end{align*}\]

where \(\psi_\lambda = \psi | [t_{\lambda-1}, t_\lambda].\) This is to integrate some system of ordinary differential equations in each coordinate neighborhood under certain conditions. If we extend each solution \(\gamma_\lambda(t)\) for \([t_{\lambda-1}, t_\lambda]\) to both sides of the interval, then in \(U_{\alpha_{\lambda-1}} \cap U_{\alpha_\lambda}\), by means of (17') we have

\[\gamma_\lambda(t) = \gamma_{\lambda-1}(t) g_{\alpha_{\lambda-1} a_\lambda}^\lambda(\psi(t)).\]

We define an element of \(G\) by

\[k_{a_0 a_m}(\mathcal{C}) = \gamma_1(0)^{-1} \gamma_m(1),\]

and for any curve \(\mathcal{C} \in \mathcal{D}_{a_0, a_m},\) we define likewise \(k_{a_0 a_m}(\mathcal{C}).\) Since \(\omega\) is left-invariant, \(k_{a_0 a_m}(\mathcal{C})\) is independent of the choice of the initial point \(\gamma_1(0).\) Furthermore, we get easily the relation
(22) \[ k_{a_1a_2}(\xi', \xi) = k_{a_1a_2}(\xi) k_{a_2a_3}(\xi), \]

\[ \xi' \in \mathcal{O}_{x_1}, x_1 \in U_{a_1}, \xi \in \mathcal{O}_{x_2}, x_2 \in U_{a_2}, \]

By means of (19), between \( \rho \) and \( k \), there exists the following relation

(23) \[ \rho(\xi) \cdot \phi_{a_1a_2} = \phi_{a_1a_2} \cdot k_{a_1a_2}(\xi), \]

\[ \xi \in \mathcal{O}_{x_1}, x_1 \in U_1, x_2 \in U_2. \]

§5. Now, in each coordinate neighborhood \( U_a \), we take a differentiable mapping \( f_a : U_a \to G \) and define a \( L(G) \)-valued linear differential form by

(24) \[ \dot{\theta}_a = \text{ad}(f_a)\theta_a + (f_a^{-1})^*\omega, \]

then we get

(25) \[ \dot{\hat{\theta}}_\beta = \text{ad}(f_\beta g_{\alpha\beta} f_\beta^{-1}) \dot{\theta}_a + (f_\beta g_{\alpha\beta} f_\beta^{-1})^*\omega, \]

where we put \( f_a^{-1}(x) = (f_a(x))^{-1} \). If we take, in each neighborhood \( U_a \), a coordinate function

(26) \[ \hat{g}_{\alpha\beta}(x) = \phi_{x, a} f_a(x)^{-1}, \]

then we get the coordinate transformation of the bundle

Then, the fibre bundle \( \hat{\mathcal{B}} = \{ B, p, X, Y, G, \hat{\theta}_a \} \) with the infinitesimal connection \( \{ \hat{\theta}_a \} \) is \( G \)-equivalent to the fibre bundle \( \mathcal{B} = \{ B, p, X, Y, G, \phi_a \} \) with the infinitesimal connection \( \{ \theta_a \} \), that is, \( \{ \theta_a \} \) is obtained from \( \{ \hat{\theta}_a \} \) by transformations of frames. In both \( \mathcal{B} \) and \( \hat{\mathcal{B}} \), \( B \) has the same family of tangent subspaces to \( B \) which are transversal to the fibres. For \( \hat{k} \) in \( \hat{\mathcal{B}} \) and \( k \) in \( \mathcal{B} \), from (23), (25) we get easily the relation

(27) \[ \hat{k}_{\alpha\beta} = f_a k_{a\beta} f_\beta^{-1}. \]

Now, we take a coordinate neighborhood \( U \) such that if \( U \ni x = (x^1, \ldots, x^n) \), then \( U \ni (tx^1, \ldots, tx^n), 0 \leq t \leq 1 \). Let \( \theta \) be the \( L(G) \)-valued linear differential form in \( U \). Let \( o \) be the origin of the coordinate system and \( \hat{o} x \) be the image of the segment joining \( o \) and \( x \) in the coordinates. Define a mapping \( f : U \to G \) by
The mapping \( f \) is differentiable. For any point \( x \in U \), we define the mapping \( a_s : 0 \leq t \leq 1 \rightarrow U \) by \( a_s(t) = (tx') \). Then, we have by (20), (28), (24)

\[
\begin{align*}
    a_s^* f^* \omega &= a_s^* \theta, \\
    a_s^* \theta &= a_s^* (\text{ad}(f^{-1}) \hat{\theta} + f^* \omega).
\end{align*}
\]

Hence we obtain

\[ a_s^* \hat{\theta} = 0. \tag{29} \]

Now, let \( X \) be an \( n \)-cell. \( U = X \) be an coordinate neighborhood as above. Then, we get from (29)

\[ \hat{k}(\bar{o}x) = e. \]

Hence, by (23), (27), for any \( \mathcal{C} \in \Omega_{s_1} \), we have

\[
\begin{align*}
    \hat{\phi}_{u,0}^{-1} \zeta_0 (\bar{o}x \mathcal{C} \bar{o}x_1^{-1}) \hat{\phi}_{u,0} &= \hat{k}(\bar{o}x \mathcal{C} \bar{o}x_1^{-1}) = \hat{k}(\mathcal{C}) \\
    &= k(\bar{o}x \mathcal{C} \bar{o}x_1^{-1}) \in H_n
\end{align*}
\]

since \( f(\omega) = e \). From this and (19), \( \hat{\theta} \) is a \( L(H_0) \)-valued linear differential form. In other words, if \( X \) is an \( n \)-cell, we can take a \( L(H) \)-valued linear differential form \( \theta \) by a suitable transformation of coordinate functions (that is, by a suitable choice of frames).

\section{Lemma 2.}

Let \( X, Y, G \) be differentiable manifolds, a Lie group as stated in Section 1. For a point \( x_0 \in X \), let be given a transformation \( \gamma_0 : \Omega_{x_0} \rightarrow G \) with the properties as follows:

i) \( \gamma_0(\mathcal{C}_1 \mathcal{C}_2) = \gamma_0(\mathcal{C}_1) \gamma_0(\mathcal{C}_2), \mathcal{C}_1, \mathcal{C}_2 \in \Omega_{x_0} \);

ii) \( \gamma_0(D \mathcal{P}) = \gamma_0(D \mathcal{P} \mathcal{P}^{-1} \mathcal{P}), D, \mathcal{P} \in \Omega, \mathcal{P} \in \Omega_{x_0} \);

iii) \( \gamma_0 \) is differentiable.

Then there exists a fibre bundle \( B = \{ B, p, X, Y, G \} \) with an infinitesimal connection \( \Gamma \) such that \( \chi_{x_0} \simeq \chi_0 \).

In the lemma, the differentiability of \( \chi_0 \) is in the sense as follows. For any points \( x_1, x \in X \), let \( \mathcal{D}(x_1, x), \mathcal{D}(x, x'), \mathcal{D}(x_2, x') \) be differentiable families of curves, \( x \in \) a coordinate neighborhood \( U, \)

\[ \chi_0 \simeq \chi_0 \]
$x' \in a \text{ coordinate neighborhood } V$, then

$$x_0(\mathcal{E}_1 \mathcal{B}(x_1, x) \mathcal{B}(x, x') \mathcal{B}(x_2, x')^{-1} \mathcal{E}_2^{-1}) \in G,$$

is differentiable with respect to $x, x'$.

**Proof.** Let $\{U_a\}$ be a covering system of coordinate neighborhoods such that if $U_a \ni x = (x^1, \ldots, x^n)$, then $U \ni (tx^1, \ldots, tx^n)$, $0 < t < 1$. Let $x_a$ be the point whose coordinates in $U_a$ are $(0, \ldots, 0)$, and for $x \in U_a$, let $x_a(x)$ be the curve which is the locus of points whose coordinates are $(tx^1, \ldots, tx^n)$, $0 < t < 1$, in $U_a$. For each point $x_a$, we take a fixed curve $x_a \in \mathcal{L}_{x_0, x_a}$.

In $U_a \cap U_\beta \neq \emptyset$, define $g_{fa} : U_a \cap U_\beta \to G$ by

$$g_{fa}(x) = x_0(\mathcal{E}_1 \mathcal{E}_2 (x_\beta, x) \mathcal{E}(x_a, x)^{-1} \mathcal{E}_a^{-1}), \quad x \in U_a \cap U_\beta.$$  

By iii), $g_{fa}$ is differentiable. For any point $x \in U_a \cap U_\beta \cap U_\gamma$, we get by i), ii)

$$g_{fa}(x)g_{fa}(x) = x_0(\mathcal{E}_1 \mathcal{E}_2 (x_\beta, x) \mathcal{E}(x_\gamma, x)^{-1} \mathcal{E}_\gamma^{-1} \mathcal{E}_\beta^{-1} \mathcal{E}_a^{-1}) = x_0(\mathcal{E}_1 \mathcal{E}_2 (x_\gamma, x) \mathcal{E}(x_a, x)^{-1} \mathcal{E}_\gamma^{-1} \mathcal{E}_\beta^{-1} \mathcal{E}_a^{-1}) = x_0(\mathcal{E}_1 \mathcal{E}_2 (x_\gamma, x) \mathcal{E}(x_a, x)^{-1} \mathcal{E}_\gamma^{-1} \mathcal{E}_\beta^{-1} \mathcal{E}_a^{-1}) = g_{fa}(x),$$

that is

$$g_{fa}(x)g_{fa}(x) = g_{fa}(x).$$

Hence, there exists a fibre bundle $\mathcal{B} = \{B, p, X, Y, G\}$ with fibre $Y$, group of bundle $G$ whose coordinate transformations are $g_{fa}(x)$ with respect to the covering $\{U_a\}$.

In the next place, for any curve $\mathcal{B}(x, x') \subset U_a$, $\mathcal{B}(x, x') \in \mathcal{L}_{x, x'}$, define $g_a$ by

$$g_a(\mathcal{B}(x, x')) = x_0(\mathcal{E}_a \mathcal{E}(x_a, x) \mathcal{B}(x, x') \mathcal{E}(x_a, x')^{-1} \mathcal{E}_a^{-1})$$

and define $\rho(\mathcal{B}(x, x')) : Y_{x'} \to Y_x$ by

$$\rho(\mathcal{B}(x, x')) = \phi_{a, x}g_a(\mathcal{B}(x, x')) \phi_{a, x'}^{-1}.$$
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\[ \phi_{\beta', \gamma} \mathcal{D}(x, x') \hat{p}_{\beta', \gamma} = \phi_{\alpha, \beta} \mathcal{D}(x, x') \hat{p}_{\alpha, \beta} \]

\[ = \phi_{\alpha, \beta} x_0(\mathcal{C}_\alpha \mathcal{C}(x_\alpha, x) \mathcal{D}(x_\beta, x') \mathcal{C}(x_\beta, x')^{-1} \mathcal{C}_\beta^{-1}) \]

\[ x_0(\mathcal{C}_\alpha \mathcal{C}(x_\alpha, x) \mathcal{D}(x_\beta, x') \mathcal{C}(x_\beta, x')^{-1} \mathcal{C}_\beta^{-1}) \]

\[ = \phi_{\alpha, \beta} x_0(\mathcal{C}_\alpha \mathcal{C}(x_\alpha, x) \mathcal{D}(x_\beta, x') \mathcal{C}(x_\beta, x')^{-1} \mathcal{C}_\beta^{-1}) \hat{p}_{\alpha, \beta} \]

This shows that \( \rho(\mathcal{D}(x, x')) \) is independent of \( U_a \supset \mathcal{D}(x, x') \).

Now, we will show that \( \rho(\mathcal{D}(x, x')) \) commutes with right translations of \( \mathcal{B} \).

Let \( \mathcal{B} = \{ \mathcal{B}, \mathcal{P}, X, G, G \} \) be the associated principal fibre bundle of \( \mathcal{B} \) and by means of (32), define \( \bar{\rho}(\mathcal{D}(x, x')) : G \rightarrow G \) by

\[ \bar{\rho}(\mathcal{D}(x, x'))(\phi_{\alpha, x} g) = \phi_{\alpha, x} \mathcal{D}(x, x') \hat{p}_{\alpha, x} \phi_{\alpha, x} g \]

\[ = \phi_{\alpha, x} \mathcal{D}(x, x') \hat{p}_{\alpha, x} g \in G. \]

This shows that

\[ \bar{\rho}(\mathcal{D}(x, x')) \Gamma(g_0) = \Gamma(g_0) \bar{\rho}(\mathcal{D}(x, x')) \]

If \( \mathcal{D}(x, x') \) is a differentiable family of curves, then \( \mathcal{D}(x, x') \) is differentiable with respect to \( x, x' \) by iii). Hence, we can obtain an infinitesimal connection \( \Gamma \) in \( \mathcal{B} \) such that the holonomy map \( \rho \) with respect to \( \Gamma \) coincides with the transformation as above for \( \mathcal{D}(x, x') \subset U_a \).

It follows that for \( \mathcal{C} \in \mathcal{D}_x \) such that

\[ \mathcal{C} = \mathcal{D}_0 \mathcal{D}_1 \cdots \mathcal{D}_m, \quad \mathcal{D}_\lambda \subset U_{a_\lambda}, \quad \lambda = 0, 1, \ldots, m, \]

\[ \rho(\mathcal{C}) = \rho(\mathcal{D}_0) \rho(\mathcal{D}_1) \cdots \rho(\mathcal{D}_m). \]

Lastly, we will prove \( x_\alpha \approx x_{\alpha'} \). For any points \( x, x' \in X \), let \( \mathcal{D} \in \mathcal{D}_{x, x'} \) and

\[ \mathcal{D} = \mathcal{D}_1 \mathcal{D}_2 \cdots \mathcal{D}_m, \quad \mathcal{D}_\alpha \subset U_{a_\alpha}, \quad \mathcal{D}_\alpha \in \mathcal{D}_{x_{\alpha-1} x_\alpha}, \quad x = x_0, x' = x_m. \]

By (32), we get

\[ \rho(\mathcal{D}_a) = \phi_{\alpha, \beta} (\mathcal{D}_a) \hat{p}_{\alpha, \beta} \]

\[ \rho(\mathcal{D}_a) \rho(\mathcal{D}_a + 1) = \phi_{\alpha, \beta} (\mathcal{D}_a) \mathcal{D}_a, (x_\alpha) \mathcal{D}_a (x_{\alpha+1}) \hat{p}_{\alpha+1, \beta} \]

and

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\[
\rho(D_1) \rho(D_2) \cdots \rho(D_m) = \phi_{1, \ast_0} g_1(D_1) g_2(D_2) \cdots g_m(D_m) p_{m, \ast_m}.
\]

By i), ii), (30), (31), we get

\[
g_1(D_1) g_2(D_2) g_3(D_3) g_m(D_m) = x_0(\xi_1(x_1, x_0) D_1 \xi(x_1, x_1^{-1} \xi_1) x_0(\xi_2(x_2, x_1) D_2 \xi(x_2, x_2^{-1} \xi_2)) \cdots x_0(\xi_m(x_m, x_m) D_m \xi(x_m, x_m^{-1} \xi_m))
\]

Accordingly, we get the relation

\[
(35) \quad \rho(D_1) \rho(D_2) \cdots \rho(D_m) = \rho(D) = \phi_{1, \ast_0} x_0(\xi(x_1, x) D \xi(x_m, x^{-1} \xi_m)) p_{m, \ast_m}.
\]

Especially, if we put \( x = x' = x_0, x_0 \in U_1 \), then

\[
x_0(\xi) = \phi_{1, \ast_0} x_0(\xi) p_{1, \ast_0},
\]

that is

\[
x_{\ast_0} \approx x_0.
\]

Q.E.D.

§7. Lemma 3. Let \( \mathcal{B} = \{B, p, X, Y, G\} \) be a differentiable fibre bundle with an infinitesimal connection \( \Gamma \) whose structure group is \( G \) and let \( H \) be the holonomy group of \( \Gamma \) at \( x_0 \in X \). Then \( \mathcal{B} \) with \( \Gamma \) is \( G \)-equivalent to another fibre bundle \( \mathcal{B}' = \{B', p', X, Y, H\} \) with an infinitesimal connection \( \Gamma' \) whose structure group is \( G \).

Proof. We will use the same notations as before. Using Lemme 2, we can obtain a differentiable fibre bundle \( \mathcal{B}' = \{B', p', X, Y, H\} \) with an infinitesimal connection \( \Gamma'' \) whose structure group is \( H \), and whose holonomy map \( x_0'' \approx x_0 \) of \( \Gamma \). By means of Lemma 1, \( \mathcal{B} \) and \( \mathcal{B}' \) is \( G \)-equivalent as fibre bundles. Let \( h : B \to B' \) be the differentiable bundle mapping satisfying the condition \( p' h = p \). Then, we can obtain a differentiable family \( \Gamma' \) of tangent subspaces to \( B' \) by \( \Gamma' = h_\ast \Gamma \). Since \( h \) is a bundle mapping, \( \Gamma' \) define an infinitesimal connection in \( \mathcal{B}' \). For any points \( x, x' \in X \) and any curve \( \xi \in \mathcal{O}_{x, x'} \), the mapping \( p'((\xi)) : Y'_{x'} \to Y'_{x} \) is clearly given by \( p'((\xi)) = h p((\xi)) h^{-1} \), where \( Y'_{x} \) denotes the fibre of \( \mathcal{B}' \) at \( x \) and \( p' \) is the map defined for
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the fibre bundle with the infinitesimal connection $\Gamma'$ as in $\mathfrak{B}$ (see §1). Thus, $\mathfrak{H}$ with the infinitesimal connection $\Gamma'$ is $G$-equivalent to $\mathfrak{H}'$ with the infinitesimal connection $\Gamma'$ whose structure group is $G$. Q.E.D.

Now, we shall deal with the theorem of E. Cartan stated in Introduction. Let $\mathfrak{B} = \{B, \rho, X, Y, G\}$ be a differentiable fibre bundle with an infinitesimal connection $\Gamma$ whose structure group is $G$. Let $\{U_a\}$ be a system of coordinate neighborhoods which is an open covering of $X$, and let $\theta_a$ be the $L(G)$-valued linear differential form in $U_a$ derived from $\Gamma$ as in §§2-4. For each $U_a$, let $x_a$ be the origin of the coordinate neighborhood. Then $H_a = H_{x_a} = k_{x_a}(\Omega_{\tau_a}, x_a)$ is the holonomy group of $\Gamma$ at $x_a$. For any curve $\gamma \in \Omega_{\tau_a, x_a}$, we have by means of (22) the relation

\begin{equation}
H_a = k_{x_a}(\gamma) H_{\rho} k_{x_a}(\gamma)^{-1}.
\end{equation}

This shows that $H_a$ are homologous each other. Let $K$ be the minimal invariant subgroup of $G$ which contains $H_a$. We may suppose that each $U_a$ is a coordinate neighborhood as $U$ in §5. Let $\mathfrak{B}_a$ be the portion of $\mathfrak{B}$ over $U_a$ and $\Gamma_a$ be the subfamily of $\Gamma$ on $B \cap \rho^{-1}(U_a)$, then the holonomy group of $\Gamma_a$ at $x_a$ is clearly a subgroup of $H_a$. Hence, by virtue of the consideration in §5, for each $U_a$, we can obtain a mapping $f_a : U_a \rightarrow G$ such that $\hat{\theta}_a = \text{Ad}(f_a) \theta_a + (f_a)^{-1} \omega$ is a $L(H_a)$-valued linear differential form and $f_a(x_a) = e.$ If $U_a \cap U_b = \phi$, we have

\begin{equation}
\hat{\theta}_{ab} = \text{Ad}(\hat{g}_{ab}) \hat{\theta}_a + (\hat{g}_{ab})^{*} \omega,
\end{equation}

where

\begin{equation}
\hat{g}_{ab}(x) = f_a(x) g_{ab}(x) f_b(x)^{-1}, \quad x \in U_a \cap U_b.
\end{equation}

Now, it may be suppose that $g_{ab} : U_a \cap U_b \rightarrow H$, by means of Lemma 3, and that if $U_a \cap U_b \neq \phi$, then $U_a \cap U_b$ is connected. Then, the above relations imply that $\hat{g}_{ab}$ can be written as

\begin{equation}
\hat{g}_{ab}(x) = \lambda_{ab} h_{ab}(x), \quad h_{ab}(x) \in K, \lambda_{ab} \in G, \quad x \in U_a \cap U_b.
\end{equation}

For each $U_a$, define a mapping $h_a : U_a \rightarrow G$ by $h_a(x) = \tau_x f_a(x)$, where $\tau_x$ is a fixed element of $G$, and define a $L(G)$-valued linear differential form from $\hat{\theta}_a$ by

\begin{equation}
\hat{\theta}_a = \text{Ad}(h_a) \theta_a + (h_a^{-1})^{*} \omega.
\end{equation}
Since $h_a^{-1} = r(r_a^{-1})f_a^{-1}$, we have
\[
\tilde{\theta}_a = \text{ad}(r_a) \text{ad}(f_a) \theta_a + (f_a^{-1})^* r(r_a^{-1})^* \omega
= \text{ad}(r_a) \text{ad}(f_a) \theta_a + (f_a^{-1})^* (\text{ad}(r_a) \omega)
= \text{ad}(r_a) \{ \text{ad}(f_a) \theta_a + (f_a^{-1})^* \omega \}
= \text{ad}(r_a) \tilde{\theta}_a.
\]

Hence, $\tilde{\theta}_a$ is a $L(K)$-valued linear differential form. By this change of coordinate functions, the coordinate transformation $g_{ab}$ on $U_a \cap U_b$ is replaced by
\[
\tilde{g}_{ab}(x) = h_a(x) g_{ab}(x) h_b(x)^{-1}
= r_a \tilde{g}_{ab}(x) r_b^{-1}
= r_a^{-1} \lambda_{ab} h_a(x) r_b^{-1}
= r_a^{-1} \lambda_{ab} \tau_{b^{-1}}(r_b h_a(x) r_{b^{-1}}).
\]

Accordingly, if we can choose $\{r_a\}$ so that
\begin{equation}
(38) \quad r_a \lambda_{ab} r_{b^{-1}} \in K, \quad \text{as} \quad U_a \cap U_b = \phi,
\end{equation}
then $\tilde{g}_{ab}$ maps $U_a \cap U_b$ into $K$.

On the other hand, if $U_a \cap U_b \cap U_\gamma = \phi$, we have
\[
e = \tilde{g}_{ab}(x) \tilde{g}_{b\gamma}(x) \tilde{g}_{\gamma a}(x)
= \lambda_{ab} h_a(x) \lambda_{b\gamma} h_{b\gamma}(x) \lambda_{\gamma a} h_\gamma(x)
= \lambda_{ab} \lambda_{b\gamma} \tau_{b^{-1}}(r_b h_a(x) r_{b^{-1}})^{-1} h_{b\gamma}(x) \lambda_{\gamma a} h_\gamma(x),
\]
from which we obtain the relation
\begin{equation}
(39) \quad \lambda_{ab} \lambda_{b\gamma} \lambda_{\gamma a} \in K, \quad \text{as} \quad U_a \cap U_b \cap U_\gamma = \phi,
\end{equation}
since $K$ is an invariant subgroup of $G$.

Since $X$ is differentiable manifold, there exists a differentiable simplicial triangulation of $X$. Let $A_\alpha$, $\alpha = 1, 2, \ldots$, be the vertices of this complex $\mathfrak{R}$ and let $U_\alpha$ be the open set defined by the star of $A_\alpha$ of $\mathfrak{R}$. Then, the system $\{U_\alpha\}$ has all the properties above-mentioned. Thus, the above problem is written as follows:

For each oriented 1-simplex $A_\alpha A_\beta$ of $\mathfrak{R}$, let be given an element $\lambda_{ab} \in G$ such that
\[
\lambda_{ab} \lambda_{b\gamma} \lambda_{\gamma a} \in K, \quad \text{for any 2-simplex } A_\alpha A_\beta A_\gamma \text{ of } \mathfrak{R}.
\]
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Then, can we choose \( \tau_a \in G, \alpha = 1, 2, \ldots \), so that

\[
\tau_a \lambda_{a \beta} \tau_{\beta}^{-1} \in K, \quad \text{for each } A_a A_\beta \in \mathfrak{g}.
\]

If \( X \) is simply connected, we can easily prove that there exists a system of \( \{ \theta_a \} \), \( \{ \mathfrak{g}_{a\beta} \} \), we can obtain a fibre bundle \( \mathfrak{B} = \{ \mathfrak{B}, \tilde{p}, X, Y, K \} \) with an infinitesimal connection \( \tilde{\Gamma} \) whose structure group is \( K \), the \( L(K) \)-valued linear differential form on \( U_a \) is \( \tilde{\theta}_a \) and the coordinate transformations are \( \mathfrak{g}_{a\beta} \). \( \mathfrak{B} \) with \( \tilde{\Gamma} \) is clearly \( G \)-equivalent to \( \mathfrak{B} \) with \( \Gamma \). For the holonomy groups of \( \tilde{\Gamma} \), we have by (27)

\[
\tilde{H}_{a} = h_a(x_a) H_j a h_a(x_a)^{-1} = \tau_a H_j a \tau_a^{-1}.
\]

Since we can put \( \tau = e \), we have \( \tilde{H} = H \). Accordingly, by virtue of Lemma 3, \( \mathfrak{B} \) with \( \tilde{\Gamma} \) is \( K \)-equivalent to a fibre bundle \( \mathfrak{B}' = \{ \mathfrak{B}', \tilde{p}', X, Y, H \} \) with an infinitesimal connection \( \Gamma' \) whose structure group is \( K \).

Thus, we obtained a following theorem.

**Theorem 1.** Let \( \mathfrak{B} = \{ B, p, X, Y, G \} \) be a differentiable fibre bundle with an infinitesimal connection \( \Gamma \) whose structure group is \( G \). Let \( H \) be the holonomy group of \( \Gamma \) at a point \( x_a \in X \), and \( K \) be the minimal invariant subgroup of \( G \) which contains \( H \). Then \( \mathfrak{B} \) with \( \Gamma \) is \( G \)-equivalent to another fibre bundle \( \mathfrak{B}' = \{ B', p', X, Y, H \} \) with an infinitesimal connection \( \Gamma' \) whose structure group \( L \), where

i) if \( X \) is an \( n \)-cell, then \( L = H \);

ii) if \( X \) is simply connected, then \( L = K \);

iii) otherwise, \( L = G \).

From this theorem, we see that the theorem of E. Cartan on holonomy groups holds good, in the large, at least in the following cases:

i) \( X \) is an \( n \)-cell.

ii) \( X \) is simply connected and \( H \) is an invariant subgroup of \( G \).

**Bibliography**


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(Received June 30, 1954)