Notes on blocks of group characters

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NOTES ON BLOCKS OF GROUP CHARACTERS

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Introduction. We consider a group $G$ of finite order $g = p^k g'$, where $p$ is a prime number and $(g', p) = 1$. Let $\mathcal{O} = \mathcal{O}(G)$ denote the corresponding group ring formed with regard to an algebraic number field $\mathcal{O}$ which contains the $g$-th roots of unity. Let $K_1, K_2, \ldots, K_m$ be the classes of conjugate elements of $G$. Then $\Gamma$ splits into a direct sum of $m$ simple ideals $\Gamma_i$:

(1) \hspace{1cm} \Gamma = \Gamma_1 \oplus \Gamma_2 \oplus \ldots \oplus \Gamma_m.

Denote the center of $\Gamma$ by $A = A(\mathcal{O})$. Corresponding to the decomposition (1) we have

(2) \hspace{1cm} A = A_1 \oplus A_2 \oplus \ldots \oplus A_m,

where each $A_i$ is isomorphic to $\mathcal{O}.$

Let $\mathcal{O}$ be the ring of all integers of $\mathcal{O}$ and let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}$ dividing $p$. We denote by $\mathcal{O}^*$ the ring of all $\mathfrak{p}$-integers of $\mathcal{O}$, i.e., of all $a/b$, where $a, b$ lie in $\mathcal{O}$ and $(b, \mathfrak{p}) = 1$. The ideal $\mathfrak{p}$ generates an ideal of $\mathcal{O}^*$ which will be denoted by $\mathfrak{p}^*$. We then have

$$\Omega^* = \frac{\mathcal{O}^*}{\mathfrak{p}^*} \cong \mathcal{O}/\mathfrak{p}$$

for the residue class field. Let $\Gamma^* = \Gamma^*(\mathcal{O})$ be the modular group ring of $\mathcal{O}$ over $\mathcal{O}^*$ and let $A^* = A^*(\mathcal{O})$ be its center.

In the present paper we study the structure of the center $A^*$ and derive some results [1], [2] stated by R. Brauer without proofs. Some new results are also obtained. In section 1 certain ideals of $A^*$ are defined. We determine the primitive idempotent elements of $A^*$ in section 2. Let

$$A^* = A_1^* \oplus A_2^* \oplus \ldots \oplus A_s^*$$

be the decomposition of $A^*$ into indecomposable ideals $A_i^*$. The ordinary irreducible characters $\chi_i$ of $\mathcal{O}$ and the modular irreducible characters $\varphi_k$ of $\mathcal{O}$ (for $p$) are distributed into $s$ blocks $B_1, B_2, \ldots, B_s$, each $\chi_i$ and $\varphi_k$ belonging to exactly one block $B_s$. In section 3 we investigate the properties of the defect group of a block $B_s$.

1) The same result has been obtained by H. Nagao independently.

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Section 4 deals with the elementary divisors of the Cartan matrix $C_\sigma$ of $B_\sigma$.

1. The classes of conjugate elements $K_1, K_2, \ldots, K_m$ of $\mathfrak{g}$ form a basis of $\Lambda$. Here each class $K_\alpha$ is interpreted as the sum of all elements in $K_\alpha$. We then have

$$K_\alpha K_\beta = \sum_{\gamma} a_{\alpha\beta\gamma} K_\gamma,$$

where the $a_{\alpha\beta\gamma}$ are rational integers, $a_{\alpha\beta\gamma} \geq 0$. Evidently $a_{\alpha\beta\gamma} = a_{\beta\alpha\gamma}$. Further we see easily that $\sum_{\alpha} a_{\alpha\beta\gamma} = g_\beta$, where $g_\beta$ denotes the number of elements in $K_\beta$. The order of the normalizer $\mathfrak{N}(G_\alpha)$ of $G_\alpha$ in $\mathfrak{g}$ is given by $n_\alpha = g_\alpha / g_\beta$ for every element $G_\alpha$ in $K_\alpha$. Let $K_\alpha^*$ denote the class which contains the elements reciprocal to those of $K_\alpha$.

**Lemma 1.** $a_{\alpha\beta\gamma} = a_{\alpha^*\beta^*\gamma} n_\alpha / n_\beta$.

**Proof.** Let $G_\alpha^{(i)} (i = 1, 2, \ldots, k_\alpha)$ be the elements in $K_\alpha$ and let $G_\beta$ be a fixed element in $K_\beta$. The number of elements $G_\alpha^{(i)} G_\beta$ which lie in $K_\gamma$ is equal to $a_{\alpha^*\beta^*\gamma}$. Hence

$$n_\beta K_\alpha K_\beta = K_\alpha \left( \sum_{G \in \mathfrak{g}} G^{-1} G_\beta G \right) = \sum_{G \in \mathfrak{g}} G^{-1} K_\alpha G_\beta G$$

$$= \sum_{G \in \mathfrak{g}} \left( \sum_{j=1}^{k_\alpha} G_\alpha^{(j)} G_\beta \right) G = \sum_{\gamma} a_{\alpha^*\beta^*\gamma} n_\alpha K_\gamma.$$

On the other hand, it follows from (3) that

$$n_\beta K_\alpha K_\beta = \sum_{\gamma} a_{\beta^*\gamma^*} n_\beta K_\gamma.$$

This proves our assertion.

We shall say that a group $\mathfrak{g}_\alpha$ of order $p^{h_\alpha}$ is the defect group [2] of a class $K_\alpha$ if $\mathfrak{g}_\alpha$ is a $p$-Sylow-subgroup of the normalizer of suitable elements in $K_\alpha$. The exponent $h_\alpha$ is called the defect of $K_\alpha$. If we consider conjugate subgroups of $\mathfrak{g}$ as not essentially different, then $\mathfrak{g}_\alpha$ is uniquely determined by $K_\alpha$.

**Lemma 2.** Let $\rho$ be a fixed rational integer such that $0 \leq \rho \leq \alpha$. The classes $K_\beta$ with $h_\beta \leq \rho$ form a basis of an ideal $\mathfrak{g}_\alpha$ of the center $\Lambda^*$ of the modular group ring $\Gamma^*$.1)

**Proof.** If $h_\beta < h_\gamma$, then $a_{\beta^*\gamma^*} \equiv 0 \pmod{p}$ by Lemma 1, whence for any class $K_\alpha$.

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1) See [7], §4.
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\[ K_\alpha K_\beta \equiv \sum_{\gamma} a_{\alpha\beta\gamma} K_\gamma \pmod{p}, \]

where the sum extends over all \( K_\gamma \) with \( h_\gamma \leq h_\beta \).

We have by Lemma 2 the following series:

(4) \[ A^* = 3_a = 3_{a_0} \supset 3_{a_1} \supset \cdots \supset 3_{a_k} \supset 0 \quad (0 \leq a_\ell). \]

**Lemma 3.** If no element of \( K_\beta \) lies in the centralizer \( C(\mathcal{D}_\gamma) \) of \( \mathcal{D}_\gamma \) in \( \mathfrak{S} \), then \( a_{\alpha\beta\gamma} \equiv 0 \pmod{p} \).\(^1\)

Assume that \( a_{\alpha\beta\gamma} \equiv 0 \pmod{p} \) in (3). It follows from Lemma 3 that there exists an element in \( K_\beta \) which commutes with all elements of \( \mathcal{D}_\gamma \), and hence \( \mathcal{D}_\gamma \subseteq \mathcal{D}_\beta \).\(^2\) We then have

\[ K_\alpha K_\beta \equiv \sum_{\gamma} a_{\alpha\beta\gamma} K_\gamma \pmod{p}, \]

where the sum extends over all \( K_\gamma \) with \( \mathcal{D}_\gamma \subseteq \mathcal{D}_\beta \). Thus we obtain the

**Lemma 4.** Let \( K_\alpha \) be a given class with the defect group \( G_\alpha \). The classes \( K_\beta \) with \( \mathcal{D}_\gamma \subseteq \mathcal{D}_\beta \) form a basis of an ideal \( \mathfrak{Z}(\mathcal{D}_\beta) \) of \( A^* \).

Let \( \mathcal{D}_1^{(a)}, \mathcal{D}_2^{(a)}, \cdots, \mathcal{D}_n^{(a)} \) be a system of defect groups of order \( p^i \) such that every defect group of order \( p^i \) is conjugate to exactly one \( \mathcal{D}_i^{(a)} \). We then see that

\[ 3_{a_i} = 3(\mathcal{D}_1^{(a)}) + 3(\mathcal{D}_2^{(a)}) + \cdots + 3(\mathcal{D}_n^{(a)}) + 3_{a_{i+1}}. \]

If we set

\[ 3_{a_i}^{(a)} = 3(\mathcal{D}_1^{(a)}) + \cdots + 3(\mathcal{D}_n^{(a)}) + 3_{a_{i+1}}, \]

then every \( 3_{a_i}^{(a)} \) is an ideal of \( A^* \) and

(5) \[ 3_{a_i} = 3_{a_i}^{(a)} \supset 3_{a_i}^{(a-1)} \supset \cdots \supset 3_{a_i}^{(a-1)} \supset 3_{a_{i+1}}. \]

Further if we set \( (3(\mathcal{D}_1^{(a)}) + 3_{3a_{i+1}})/3_{a_{i+1}} = \mathfrak{M}_i \), then

\[ 3_{a_i}/3_{a_{i+1}} = \mathfrak{M}_i \oplus \mathfrak{M}_2 \oplus \cdots \oplus \mathfrak{M}_i. \]

2. Every ordinary irreducible character \( z_i \) of \( \mathfrak{S} \) determines a character \( \omega_i \) of \( A \) which is given by

(6) \[ \omega_i(K_\alpha) = g_{\alpha} z_i(G_{\alpha})/z_i, \]

where \( G_{\alpha} \) is an element in \( K_\alpha \) and \( z_i \) is the degree of \( z_i \). The

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1) See [1], p.112.
2) Strictly speaking, \( \mathcal{D}_\gamma \) is conjugate in \( \mathfrak{S} \) to a subgroup of \( \mathcal{D}_\beta \).
modular characters $\omega^*$ of $\Lambda^*$ are obtained by considering the different $\omega_i \pmod{p}$. As was shown in [6], two characters $\chi_i$ and $\chi_j$ belong to the same block if and only if for every class $K_a$

$$\omega_i(K_a) \equiv \omega_j(K_a) \pmod{p}.$$

As is well known, the primitive idempotent element $e_i$ of $\Lambda$ corresponding to the character $\chi_i$ is expressed in the form

$$e_i = \frac{1}{g} \sum_{a=1}^{m} \chi_i(G^{-1}_a) K_a.$$

We set

$$E_a = \sum_{i} e_i = \frac{1}{g} \sum_{a=1}^{m} (\sum_{i} \chi_i(G^{-1}_a)) K_a,$$

where the sum extends over those $i$ for which the $\chi_i$ belong to a block $B_e$. If we set

$$b_a = \frac{1}{g} \sum_{i} \chi_i(G^{-1}_a),$$

then $b_a = 0$ for any $p$-singular class $K_a$ [5]. We may assume that $K_1, K_2, \ldots, K_m$ are the $p$-regular classes of $\mathfrak{S}$. We then have

$$E_a = \sum_{a=1}^{m} b_a K_a = \frac{1}{g} \sum_{a=1}^{m} (\sum_{i} \chi_i(V^{-1}_a)) K_a,$$

where $V_a$ is an element in $K_a$ ($\alpha = 1, 2, \ldots, m^*$). Denote by $\psi_\kappa$ the character of the indecomposable constituent of the regular representation of $\mathfrak{S}$ corresponding to $\rho_\kappa$ and by $\kappa$ its degree. Since

$$\sum_{i} \chi_i(V^{-1}_a) = \sum_{\kappa} \kappa \psi_\kappa(V^{-1}_a),$$

we see that the $b_a$ ($\alpha = 1, 2, \ldots, m^*$) are $\mathfrak{p}$-integers of $\mathfrak{S}$. Observe that $\psi_\kappa \equiv 0 \pmod{p^s}$ for every $\kappa$. Since $\omega_i(E_a) = \sum_{a=1}^{m^*} b_a \omega_i(K_a) = 1$ for any character $\chi_i$ in $B_\kappa$, we have

$$\sum_{a=1}^{m^*} b_a^* \omega_i^*(K_a) = 1,$$

where $b_a^* = b_a \pmod{p}$. This implies that there exists a coefficient $b_a^*$ such that $b_a^* \equiv 0$. If we set $E_a^* = E_a \pmod{p}$, then we see by the above discussion that $E_a^* \equiv 0$. Evidently

$$E_a^* = (E_a^*)^2, \quad E_a^* E_{a^*} = 0 \quad (\sigma \neq \tau)$$
and hence \( s \) primitive idempotent elements of \( \Lambda^* \) are given by \( E^*_\sigma \) (\( \sigma = 1, 2, \ldots, s \)).

**Theorem 1.** Every block \( B_\sigma \) contains an indecomposable character \( \gamma_\sigma \) of degree \( u_\sigma \equiv 0 \) (mod \( p^{(s+1)} \)).

**Proof.** Suppose that \( u_\sigma \equiv 0 \) (mod \( p^{(s+1)} \)) for all \( \gamma_\sigma \) in \( B_\sigma \). Then \( b_\sigma \equiv 0 \) (mod \( \mathfrak{p} \)) for \( \sigma = 1, 2, \ldots, m^* \). This gives a contradiction.

We have for any \( \chi_j \) outside of \( B_\sigma \)

\[
\omega_j(E_\sigma) = \sum_{\sigma=1}^{m^*} b_\sigma \omega_j(K_\sigma) \equiv 0 \quad \text{(mod } \mathfrak{p} \text{)}.
\]

We then obtain by (9)

**Theorem 2.** Two characters \( \chi_i \) and \( \chi_j \) belong to the same block if and only if \( \omega_i(K_\sigma) \equiv \omega_j(K_\sigma) \) (mod \( \mathfrak{p} \)) for all \( p \)-regular classes \( K_\sigma \).

**Lemma 5.** Let \( V \) be a fixed \( p \)-regular element of \( G \). If \( \chi_i(V) \equiv 0 \) (mod \( \mathfrak{p} \)) for all \( \chi_i \) in \( B_\sigma \), then \( \varphi_i(V) \equiv 0 \) (mod \( \mathfrak{p} \)) for all \( \varphi_i \) in \( B_\sigma \).

**Proof.** Denote by \( y_\sigma \) the number of modular characters \( \varphi_i \) in \( B_\sigma \). Our assertion follows immediately from the fact that the decomposition matrix \( D_\sigma \) of \( B_\sigma \) has the rank \( y_\sigma \) when it is considered mod \( \mathfrak{p} \) [6].

Let \( p^t \) be the highest power of \( p \) dividing one of the number \( g/z_i \) with \( \chi_i \) in \( B_\sigma \). The exponent \( d \) is called the defect of \( B_\sigma \). In the following we consider a block \( B_\sigma \) of defect \( d \). Since \( \omega_i(K_\sigma) = g_{\chi_i}(V_\sigma)/z_i = g_{\chi_i}(V_\sigma)/n_{\chi_i}z_i \) are algebraic integers, we have for all \( p \)-regular classes \( K_\sigma \) with \( h_\sigma > d \) and for all \( \chi_i \) in \( B_\sigma \)

\[
\chi_i(V_\sigma) \equiv 0 \quad \text{(mod } \mathfrak{p} \text{)}.
\]

Hence it follows from Lemma 5 that \( b_\sigma^* = 0 \) for all \( p \)-regular classes \( K_\sigma \) with \( h_\sigma > d \). On the other hand, we have \( \omega_i(K_\sigma) \equiv 0 \) (mod \( \mathfrak{p} \)) for all \( p \)-regular classes \( K_\sigma \) with \( h_\sigma < d \) and for all \( \chi_i \) in \( B_\sigma \). Consequently we have for any \( \chi_i \) in \( B_\sigma \)

\[
\sum_{a} b_\sigma^* \omega_i^*(K_\sigma) = 1,
\]

where the sum extends over all \( p \)-regular classes \( K_\sigma \) of defect \( d \). This implies that two characters \( \chi_i \) and \( \chi_j \) belonging to blocks of defect \( d \) appear in the same block if and only if \( \omega_i(K_\sigma) \equiv \omega_j(K_\sigma) \) (mod \( \mathfrak{p} \)) for all \( p \)-regular classes of defect \( d \).

It follows from (11) that there exists a \( p \)-regular class \( K_\gamma \) of defect \( d \) such that

\[
b_\gamma^* \equiv 0, \quad \omega_j^*(K_\gamma) \equiv 0
\]
for any \( \chi_i \) in \( B_\sigma \). We have by (12) the

**Lemma 6.** A character \( \chi_i \) belongs to a block of defect \( d \) if and only if \( \omega_i(K_a) \equiv 0 \pmod{p} \) for all \( p \)-regular classes \( K_a \) with \( h_a < d \) and \( \omega_i(K_p) \equiv 0 \pmod{p} \) for at least one \( p \)-regular class \( K_p \) of defect \( d \).

We see further that if \( \omega_i(K_a) \equiv 0 \pmod{p} \) for a \( p \)-regular class \( K_a \), then \( \chi_i \) belongs to a block of defect \( d \leq h_a \).

We consider a character \( \chi_i \) of degree \( z_i \equiv 0 \pmod{p^a} \). We set

\[
e_i = \frac{1}{g} \sum_{a=1}^{m} z_i \chi_i(G_a) K_a = \sum_{a=1}^{m} a_a K_a.
\]

Then \( a_a \equiv 0 \pmod{p} \) for all \( K_a \) with \( h_a > 0 \) since \( z_i(G_a^{-1}) \equiv 0 \pmod{p} \) for \( G_a \) in these classes. Hence

\[
\sum_{a} a_a \omega_i(K_a) \equiv 1 \pmod{p},
\]

where the sum extends over all \( K_a \) of defect 0. Thus we see that there exists a class \( K'_a \) of defect 0 such that \( \omega_i(K'_a) \equiv 0 \pmod{p} \). Since any class of defect 0 is \( p \)-regular, we have by Lemma 6 the following

**Lemma 7.** A character \( \chi_i \) of degree \( z_i \equiv 0 \pmod{p^a} \) belongs to a block of defect 0.

**Theorem 3.** Let \( B \) be a set of ordinary characters of \( \mathcal{G} \) such that

\[
\sum_{\chi_i \text{ in } B} \chi_i(V) \chi_i(S) = 0
\]

for any \( p \)-regular element \( V \) and for any \( p \)-singular element \( S \). Then \( B \) is a collection of blocks of \( \mathcal{G} \).

**Proof.** Denote by \( B'_\sigma \) the set of characters \( \chi_i \) which lie in both \( B \) and \( B_\sigma \). We then have [6, Theorem 6]

\[
\sum' \chi_i(V) \chi_i(S) = 0,
\]

where the sum extends over all \( \chi_i \) in \( B'_\sigma \). We shall prove that if \( B'_\sigma \) is not empty, then \( B'_\sigma = B \), namely, \( B \) contains all \( \chi_i \) in \( B_\sigma \). For a fixed \( p \)-regular element \( V \), we consider a generalized character

\[
\theta_r(G) = \sum' \chi_i(V) \chi_i(G),
\]

where the sum extends over all \( \chi_i \) in \( B'_r \). Applying Theorem 17 [4] to \( \theta_r(G) \), we have \( \theta_r(G) = \sum s_a(V) \gamma_a(G) \). Since the \( \chi_i(V) \) are algebraic integers, the \( s_a(V) \) are also algebraic integers.\(^2\) This implies that

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1) The converse of the theorem is also true. See Theorem VIII [5].
2) Cf. the proof of second half of Theorem 17 [4].
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\[
\frac{1}{g} \theta_r(1) = \frac{1}{g} \sum' z_i \chi_i(V) = \frac{1}{g} \sum' \mu \chi_i(V)
\]

is a \( p \)-integer for any \( p \)-regular element \( V \) and so

\[
E'_r = \frac{1}{g} \sum_{a=1}^{m^*} (\sum' z_i \chi_i(V_{a^{-1}})) K_a
\]

is an idempotent element of \( A^* \). Since \( \omega_r(E'_r) = 1 \) for \( \chi_j \) in \( B'_r \), \( E'_r \equiv 0 \) (mod \( p \)). Suppose that a character \( \chi_k \) in \( B'_r \) does not appear in \( B'_r \). Then \( \omega_k(E'_r) = 0 \). On the other hand, we have \( \omega_k(E'_r) \equiv 1 \) (mod \( p \)) since \( \omega_k(K_a) \equiv \omega_j(K_a) \) (mod \( p \)). This gives a contradiction. Hence if \( B \) contains a character \( \chi_i \) in \( B'_r \), then all characters in \( B'_r \) appear in \( B \).

3. Let \( \mathfrak{H} \) be any subgroup of \( \mathfrak{S} \) and let its order be \( p^h \), \( h \geq 0 \). Denote by \( \mathfrak{C}(\mathfrak{H}) \) the centralizer of \( \mathfrak{H} \) in \( \mathfrak{S} \) and by \( \mathfrak{N}(\mathfrak{H}) \) the normalizer of \( \mathfrak{H} \) in \( \mathfrak{S} \). Let \( \mathfrak{R} \) be a subgroup such that

\[
\mathfrak{HC}(\mathfrak{H}) \subseteq \mathfrak{R} \subseteq \mathfrak{N}(\mathfrak{H}).
\]

If \( K^0_a \) is the part of \( K \) which lies in \( \mathfrak{C}(\mathfrak{H}) \), then either \( K^0_a = 0 \) or \( K^0_a \) is a sum of complete classes of \( \mathfrak{R} \). As was shown in [2], we have from (3)

\[
K^0_a K^0_b \equiv \sum \gamma a_{ab\gamma} K^0_\gamma \quad \text{(mod } p).\]

Hence the classes \( K_a \) with \( K^0_a = 0 \) form a basis of an ideal \( T^* \) of \( A^* \). On the other hand, the \( K^0_a \equiv 0 \) can be considered as the basis of a subring \( R^* \) of the center \( A^*(\mathfrak{H}) \) of the modular group ring \( \Gamma^*(\mathfrak{H}) \) of \( \mathfrak{R} \). (15) implies

\[
R^* \cong A^*/T^*.
\]

Let \( E^*_r \) be a primitive idempotent element of \( A^* \) corresponding to \( B'_r \). Suppose that \( E^*_r \notin T^* \) and let \( \tilde{E}^*_r \) be the element of \( R^* \) corresponding to \( E^*_r \) (mod \( T^* \)) in (16). Then \( \tilde{E}^*_r \) is a sum of primitive idempotent elements of \( A^*(\mathfrak{H}) \). We denote by \( \tilde{B}^*(\mathfrak{H}) \) the family of blocks of \( A^*(\mathfrak{H}) \) determined by \( \tilde{E}^*_r \). If a block \( \tilde{B}_r \) of \( A^*(\mathfrak{H}) \) is contained in \( \tilde{B}^*(\mathfrak{H}) \), then we say that \( \tilde{B}_r \) determines the block \( B_r \) of \( A^* \). We have for \( \chi_i \) in \( B_r \) and \( \tilde{\chi}_i \) in \( \tilde{B}_r \)

\[
\omega_i(K_a) \equiv \sum \tilde{\omega}_i(\tilde{K}_a) \quad \text{(mod } p),
\]

where \( \tilde{K}_a \) ranges over all classes of \( \mathfrak{R} \) which lie in \( K_a \) and whose
elements belong to the centralizer $C(\mathbb{S})$ of $\mathbb{S}$. If $K_\alpha$ belongs to $T^*$, then

$$\omega_i(K_\alpha) \equiv 0 \pmod{p}.$$  

**Lemma 8.** If $E_\sigma^* \in T^*$, then there is a class $K_\alpha$ in $T^*$ such that $\omega_i(K_\alpha) \equiv 0 \pmod{p}$ for $\chi_i$ in $B_\sigma$, and conversely.

As was shown in section 2, there is a $p$-regular class $K_\gamma$ of defect $d$ such that $b_\gamma^* \neq 0$ and $\omega_i(K_\gamma) \equiv 0$ for $\chi_i$ in $B_\sigma$ of defect $d$. Let $\mathbb{S}$ be a subgroup of $\mathbb{S}$ which is not conjugate to a subgroup of the defect group $\mathbb{S}_\gamma$ of $K_\gamma$ and let $p^d$ be its order. Choose $\mathbb{R}$ in (14) as the normalizer $\mathbb{R}(\mathbb{S})$ of $\mathbb{S}$. Our assumption implies that $K_\sigma^* = 0$ and hence $K_\gamma$ lies in $T^*$. Since $\omega_i(K_\gamma) \equiv 0 \pmod{p}$, it follows from Lemma 8 that $E_\sigma^* \in T^*$. Consequently $b_\sigma^* = 0$ for any class $K_\alpha$ outside of $T^*$. We then have

**Theorem 4.** Let $E_\sigma^* = \sum_{\alpha=1}^n b_\sigma^* K_\alpha$ be a primitive idempotent element of $A^*$ corresponding to a block $B_\sigma$ and let $b_\gamma^* \neq 0$, $\omega_i(K_\gamma) \equiv 0$ for $\chi_i$ in $B_\sigma$. If $b_\sigma^* \neq 0$, then $\mathbb{S}_\sigma \subseteq \mathbb{S}_\gamma$.

The defect group $\mathbb{S}_\gamma$ of $K_\gamma$ in Theorem 4 is called the defect group of the block $B_\sigma$. Theorem 4 implies that the defect group of $B_\sigma$ is uniquely determined by $B_\sigma$ if we consider conjugate subgroups of $\mathbb{S}$ as not essentially different. The defect group of $B_\sigma$ will be denoted by $\mathbb{D}_\sigma$. It follows that $A^* = A^*E_\sigma^* \subseteq \mathbb{S}(\mathbb{D}_\sigma)$.

**Corollary 1.** Let $B_\sigma$ be a block of defect $d$ with the defect group $\mathbb{D}$. Then $\sum b_\sigma \omega_i(K_\alpha) \equiv 1 \pmod{p}$ for $\chi_i$ in $B_\sigma$, where the sum extends over all $p$-regular classes $K_\alpha$ with $\mathbb{S}_\alpha = \mathbb{D}$.

**Corollary 2.** Two characters $\chi_i$ and $\chi_j$ belonging to blocks with the defect group $\mathbb{D}$ appear in the same block if and only if $\omega_i(K_\alpha) \equiv \omega_j(K_\alpha) \pmod{p}$ for all $p$-regular classes $K_\alpha$ with $\mathbb{S}_\alpha = \mathbb{D}$.

It follows from (18) that if $\omega_i(K_\alpha) \equiv 0 \pmod{p}$ for $\chi_i$ in $B_\sigma$, with the defect group $\mathbb{D}$, then $\mathbb{S}_\sigma \subseteq \mathbb{S}_\alpha$.

**Lemma 9.** If $\mathbb{S}$ contains a normal subgroup $\mathbb{S}_\gamma$ of order $p^d$, $d > 0$, then all blocks of $\mathbb{S}$ have at least the defect $d$.

**Proof.** Since every block $B_\sigma$ of $\mathbb{S}$ contains at least one character of $\mathbb{S} / \mathbb{S}_\sigma$, our assertion is proved readily.

**Theorem 5.** The defect group $\mathbb{D}$ of a block $B_\sigma$ is a maximal normal $p$-subgroup of the normalizer $\mathbb{R}(\mathbb{D})$ of $\mathbb{D}$ in $\mathbb{S}$.

**Proof.** Choose $\mathbb{R}$ in (14) as the normalizer $\mathbb{R}(\mathbb{D})$. Since there exists a $p$-regular class $K_\gamma$ with the defect group $\mathbb{S}_\gamma = \mathbb{D}$ such that
\( \omega_i(K_d) \equiv 0 \pmod{p} \) for \( \chi \) in \( B_\chi \) and since \( K_d \) contains only one class \( \tilde{K}_d \) of \( \mathcal{R}(\Sigma) \) which consists of elements of \( \mathcal{C}(\Sigma) \), we have by (17)

\[
\omega_i(K_d) \equiv \tilde{\omega}_d(\tilde{K}_d) \equiv 0 \pmod{p}
\]

for any \( \tilde{z}_d \) in a block \( \tilde{B}_d \) of \( \mathcal{R}(\Sigma) \) corresponding to \( B_\chi \). Hence it follows from Lemmas 6 and 9 that the defect group of \( \tilde{B}_d \) is \( \tilde{\Sigma} \). We then see by Lemma 9 that \( \tilde{\Sigma} \) is a maximal normal \( p \)-subgroup of \( \mathcal{R}(\Sigma) \).

Let \( \mathcal{S} \) be a normal subgroup of \( \mathcal{G} \) and let its order be \( p^h \), \( h > 0 \). We choose \( \mathcal{R} \) in (14) now as the normalizer \( \mathcal{N}(\mathcal{S}) = \mathcal{G} \). Since \( \mathcal{C}(\mathcal{S}) \) is a normal subgroup of \( \mathcal{G} \), if \( K_d^* \neq 0 \), then \( K_d^* = K_d \). The classes \( K_d \) such that \( K_d^* \neq 0 \) form a basis of a subring \( R^* \) of \( A^* \). We then have

\[
A^* = R^* + T^*, \quad R^* \cong A^*/T^*.
\]

Since the defect group of every block \( B_\chi \) of \( \mathcal{G} \) contains \( \mathcal{S} \), no \( E_{\chi}^* \) lies in \( T^* \) and hence \( E_{\chi}^* \in R^* \). Consequently \( T^* \) is contained in the radical of \( A^* \). This, combined with Theorem 4, yields the

**Lemma 10.** Let \( \mathcal{S} \) be a normal \( p \)-subgroup of \( \mathcal{G} \) and let \( B_\chi \) be a block of \( \mathcal{G} \) with the defect group \( \mathcal{S} \). Then

\[
E_{\chi}^* = \sum_a b_{\chi}^a K_a,
\]

where the sum extends over the \( p \)-regular classes \( K_a \) with \( \mathcal{S}_a = \mathcal{S} \).

Now we can prove the following

**Theorem 6.** \( \mathcal{G} \) possesses \( r \) blocks of defect \( d \) with the defect group \( \mathcal{S} \) if and only if \( \mathcal{R}(\mathcal{S}) \) possesses \( r \) blocks of defect \( d \) (with the defect group \( \mathcal{S} \)).

**Theorem 7.** If \( \mathcal{G} \) contains a normal \( p \)-subgroup \( \mathcal{S} \) and if the centralizer \( \mathcal{C}(\mathcal{S}) \) of \( \mathcal{S} \) in \( \mathcal{G} \) is also a \( p \)-group, then \( \mathcal{G} \) possesses only one block.\(^3\)

**Proof.** The subring \( R^* \) of \( A^* \) in (19) can be considered as the subring of the center \( A^*(\mathcal{C}(\mathcal{S})) \) of \( \mathcal{R}^*(\mathcal{C}(\mathcal{S})) \). Hence, by our hypothesis, \( R^* \) contains only one primitive idempotent element. Since any primitive idempotent element of \( A^* \) is contained in \( R^* \), we see that \( A^* \) is completely primary.

4. We arrange \( \varphi_\chi(V_a), \eta_\chi(V_a) \) in matrix form

---

1) See Lemma 1 [3].
2) This is an improvement of Lemma 2 [3].
\[ \phi = (\varphi_{\kappa}(V_{\alpha})), \quad H = (\varphi_{\alpha}(V_{\kappa})) \]

(\kappa \text{ row index, } \alpha \text{ column index; } \kappa, \alpha = 1, 2, \ldots, m^n). \text{ We have by [6]} \]

\[ | \phi | \equiv 0 \pmod{p}. \]

We denote by \( \tilde{\phi}' \) the transpose of \( \tilde{\phi} = (\varphi_{\alpha}(V_{\kappa}^{-1})) \). Then

\[ \tilde{\phi}'H = (n_{\alpha} \delta_{\alpha \beta}) = T. \]

We set \( Y := HT^{-1} = (\varphi_{\alpha}(V_{\kappa})/n_{\alpha}) \), where the \( \varphi_{\alpha}(V_{\kappa})/n_{\alpha} \) are \( p \)-integers [5, Theorem V]. Since \( \tilde{\phi}'Y = I \), we have by (20)

\[ | Y | \equiv 0 \pmod{p}. \]

If the block \( B_{\sigma} \) contains \( \varphi_{\sigma} \) modular characters \( \varphi_{\sigma} \), then we can choose a minor \( | \phi_{\sigma} | \) of degree \( y_{\sigma} \) containing \( y_{\sigma} \) rows of \( \phi \) corresponding to \( B_{\sigma} \) such that \( | \phi_{\sigma} | \equiv 0 \pmod{p} \). It can be shown that it is possible to make this selection of \( y_{\sigma} \) columns for each block \( B_{\sigma} \) in such a manner that every column appears for one and only one block. Hence we may assume without restriction that

\[ \phi = \begin{pmatrix} \phi_{1} & * \\ \phi_{2} & \ddots & * \\ & \ddots & \ddots & * \\ & & \phi_{y_{\sigma}} & \end{pmatrix}, \quad | \phi_{\sigma} | \equiv 0 \pmod{p}. \]

In what follows we shall denote by \( K_{\sigma, 1}, K_{\sigma, 2}, \ldots, K_{\sigma, y_{\sigma}} \) the \( p \)-regular classes of \( \Theta \) associated with \( B_{\sigma} \) by the preceding construction. We set

\[ Y_{\sigma} = (\varphi_{\alpha}(V_{\kappa, \alpha})/n_{\sigma, \alpha}). \]

We then have

\[ | Y_{\sigma} | = | \phi_{\sigma} || C_{\sigma} | / \prod_{a=1}^{y_{\sigma}} n_{a, \sigma}, \]

where \( C_{\sigma} \) is the Cartan matrix of \( B_{\sigma} \). Since \( | Y_{\sigma} | \) is \( p \)-integer and \( | C_{\sigma} | \) is a power of \( p \), it follows from (22) and (23) that \( | C_{\sigma} | \geq \prod_{a=1}^{y_{\sigma}} p^{b_{a, \sigma}}, \sigma \). On the other hand, we have

\[ | C | = \prod_{\sigma} | C_{\sigma} | = \prod_{\sigma} (\prod_{a=1}^{y_{\sigma}} p^{b_{a, \sigma}}, \sigma). \]
Hence \( |C_\sigma| = \prod \mathfrak{p}^{\nu_\mathfrak{p}^{\nu_\mathfrak{p}}}. \) This implies \( |Y_\sigma| \equiv 0 \pmod{p} \). If we set \( \Phi_\sigma Y_\sigma = Q_\sigma \), then \( |Q_\sigma| \equiv 0 \pmod{p} \) and

\[
\Phi_\sigma C_\sigma \Phi_\sigma = Q_\sigma T_\sigma,
\]

where \( T_\sigma = (n_{\sigma, a} \delta_{ab}) \). If we work in the ring \( \mathcal{O} \) of \( p \)-integers of \( \Omega \), we obtain by (24) the following

**Theorem 8.** Let \( K_{\tau, 1}, K_{\tau, 2}, \ldots, K_{\tau, y_\tau} \) be the \( p \)-regular classes of \( \mathfrak{S} \) associated with the block \( B_\tau \). Then the elementary divisors of \( C_\sigma \) are the powers of \( p \) with the exponents \( n_{\sigma, a} \) (\( \alpha = 1, 2, \ldots, y_\tau \)).

We see easily that our theorem is identical with [1, Theorem 2].

Now we set

\[
M = \begin{pmatrix}
Y_1 & 0 \\
Y_2 & \\
\vdots & \\
0 & Y_\tau
\end{pmatrix}.
\]

Then

\[
S = \bar{\Phi}' M = \left( \frac{1}{n_{\sigma, a}} \sum_{i} \varphi_i (V_i^{-1}) \varphi (V_{\sigma, a}) \right)
= \left( \frac{1}{n_{\sigma, a}} \sum_{i} \chi_i (V_i^{-1}) \chi (V_{\sigma, a}) \right) = (s(\tau, \beta; \sigma, \alpha)),
\]

where each row is characterized by a pair of indices \( \tau, \beta \) and each column is characterized by a pair of indices \( \sigma, \alpha \). Since \( |Y_\sigma| \equiv 0 \pmod{p} \), we have

\[
|S| \equiv 0 \pmod{p}.
\]

By the simple computation we see that

\[
K_{\tau, a} E_\sigma = \sum_{i, \beta} \left( \frac{1}{n_{\sigma, a}} \sum_{i} \chi_i (V_{\sigma, a}) \chi_i (V_i^{-1}) \right) K_{\tau, \beta}
= \sum_{i, \beta} s(\tau, \beta; \sigma, \alpha) K_{\tau, \beta}.
\]

Let \( \mathcal{D} \) be the defect group of \( B_\sigma \). Since \( E_\sigma \ast \mathfrak{D} \), if \( \delta_{\tau, \beta} \notin \mathcal{D} \), then

\[
s(\tau, \beta; \sigma, \alpha) = 0 \pmod{p}.
\]

We see also that \( s(\tau, \beta; \sigma, \alpha) = 0 \pmod{p} \) if \( \delta_{\tau, \beta} \notin \delta_{\tau, \alpha} \). It follows from (25) that

\[
(K_{\tau, 1} E_1, K_{\tau, 2} E_1, \ldots, K_{\tau, y_\tau} E_\tau) = (K_{\tau, 1}, K_{\tau, 2}, \ldots, K_{\tau, y_\tau}) S.
\]

(25) implies that \( \{K_{\tau, a} E_\sigma \ast \mathfrak{D}\} \) are linearly independent. If
are taken in a suitable order corresponding to (4), we have by the above argument

\[ P^{-1}SP \equiv \begin{pmatrix} W_a & 0 \\ W_{a_1} & \ddots \\ \vdots & \ddots & W_{a_k} \end{pmatrix} \quad (\text{mod } p), \]

where \( P \) denotes a suitable permutation matrix. \( K_{r, \beta} \) and \( K_{r, \alpha} \) range over only the \( p \)-regular classes of defect \( \alpha \) in \( W_{a_i} = (s^*(r, \beta; \sigma, \alpha)) \), where \( s^*(r, \beta; \sigma, \alpha) = s(r, \beta; \sigma, \alpha) \pmod{p} \). (25) yields

\[ |W_{a_i}| \equiv 0. \]

Moreover we may assume by (5) that

\[ W_{a_i} = \begin{pmatrix} A_1 & 0 \\ A_2 & \ddots \\ \vdots & \ddots & A_i \\ 0 & \cdots & 0 \end{pmatrix}, \]

where \( K_{r, \beta} \) and \( K_{r, \alpha} \) range over only the \( p \)-regular classes with the defect group \( \mathcal{D}_{r, \alpha} \) in \( d_r = (s^*(r, \beta; \sigma, \alpha)) \). Hence

\[ |A_i| \equiv 0. \]

Consequently we have the

**Lemma 11.** There exists at least one class \( K_{r, \beta} \) with \( \mathcal{D}_{r, \beta} = \mathcal{D}_{r, \alpha} \) such that \( s^*(r, \beta; \sigma, \alpha) \equiv 0 \) in (26).

**Theorem 9.** Let \( K_{r, \alpha} \) (\( \alpha = 1, 2, \ldots, y_o \)) be the \( p \)-regular classes of \( \mathcal{D} \) associated with a block \( B_o \) of defect \( d \) with the defect group \( \mathcal{D} \). Then \( \mathcal{D}_{r, \alpha} \subseteq \mathcal{D} \) (\( \alpha = 1, 2, \ldots, y_o \)) and there exists exactly one class \( K_{r, \alpha} \) with \( \mathcal{D}_{r, \alpha} = \mathcal{D} \).

**Proof.** Lemma 11 implies \( \mathcal{D}_{r, \alpha} \subseteq \mathcal{D} \). It follows from (27) that \( E^* \) is expressed as a linear combination of \( K_{r, \alpha} E^* \) (\( \alpha = 1, 2, \ldots, y_o \)). Hence there exists at least one class, say, \( K_{r, \alpha} \) with \( \mathcal{D}_{r, \alpha} = \mathcal{D} \). Suppose that \( \mathcal{D}_{r, \alpha} = \mathcal{D} \). Then \( \chi_i(V_{r, 1}) \equiv \chi_i(V_{r, 2}) \equiv 0 \pmod{p} \) for \( \chi_i \) in \( B_o \) whose degree \( z_i \) is divisible by \( p^{d+1} \). Let \( \chi_j \) and \( \chi_i \) be two characters in \( B_o \) such that \( z_j \equiv 0 \), \( z_i \equiv 0 \pmod{p^{d+1}} \). Since \( \omega_j(V_{r, 1}) \equiv \omega_i(V_{r, 1}) \pmod{p} \), we have \( \chi_j(V_{r, 1}) \equiv -\frac{z_j}{z_i} \chi_i(V_{r, 1}) \pmod{p} \). Similarly, \( \chi_j(V_{r, 2}) \equiv \frac{z_j}{z_i} \chi_i(V_{r, 2}) \pmod{p} \).
NOTES ON BLOCKS OF GROUP CHARACTERS

\[ \frac{z_i}{z_i} \chi_i(V, a) \pmod{p}. \] We set \( Z_\sigma = (\chi_i(V, a)) \), where row index \( i \) ranges over all \( \chi_i \) in \( B_\sigma \). It follows by the above argument that \( Z_\sigma \) has the rank \( r < y_\sigma \) when it is considered \( \pmod{p} \). But this gives a contradiction and hence the theorem is proved.

**Corollary 1.** Let \( C_\sigma \) be the Cartan matrix of a block \( B_\sigma \) of defect \( d \). \( C_\sigma \) has one elementary divisor \( p^r \) while all other elementary divisors of \( C_\sigma \) are powers of \( p \) with exponents smaller than \( d \).

**Corollary 2.** If there exist \( k \) \( p \)-regular classes \( K_\sigma \) in \( S \) with \( S_\sigma = \emptyset \), then \( S \) possesses at most \( k \) blocks with the defect group \( D \).

If \( B_\sigma \) is a block of defect 0, then \( B_\sigma \) consists of exactly one ordinary character \( \chi_i \) and one modular character \( \varphi_\sigma \). Moreover \( \chi_i(V) = \varphi_\sigma(V) \) for any \( p \)-regular element \( V \). Since \( \chi_i \) with \( z_i \equiv 0 \pmod{p^r} \) belongs to a block of defect 0, \( \chi_i \) forms a block \( B_\sigma \) of its own.

**Theorem 10.** Let \( K_{\sigma, a} \) \( (\alpha = 1, 2, \ldots, y_a) \) be the \( p \)-regular classes of \( S \) associated with a block \( B_\sigma \) with the defect group \( D \) and let \( r_{\sigma, \rho} \) be the number of classes \( K_{\sigma, a} \) with \( \bar{\Sigma}_\sigma = \bar{\Sigma}_\rho \) \( (\rho = a_i) \). Then \( r_{\sigma, \rho} \) depends only the subgroup \( \bar{\Sigma}_\rho \) and the block \( B_\sigma \).

**Proof.** Let \( K'_{\sigma, a} \) \( (\alpha = 1, 2, \ldots, y_a) \) be a second set of \( p \)-regular classes of \( S \) associated with \( B_\sigma \) and let \( r'_{\sigma, \rho} \) be the number of classes \( K'_{\sigma, a} \) with \( \bar{\Sigma}'_\sigma = \bar{\Sigma}_\rho \). We have for \( K_{\sigma, a} \) with \( \bar{\Sigma}_\sigma = \bar{\Sigma}_\rho \)

\[ K_{\sigma, a} E_\sigma = \sum_{\beta} t_{a\beta} K'_{\rho, \beta} E_\rho. \]

Here the sum extends over only those \( K'_{\rho, \beta} E_\rho \) with \( \bar{\Sigma}'_\rho, \beta \subseteq \bar{\Sigma}_\rho \), since \( K_{\sigma, a} E_\sigma \in \bar{\Sigma}(\bar{\Sigma}_\rho) \). Moreover there exists at least one \( K'_{\rho, \beta} \) with the defect group \( \bar{\Sigma}_\rho \) such that \( t_{a\beta} \neq 0 \). Suppose that \( r_{\sigma, \rho} > r'_{\sigma, \rho} \). Then we can conclude that the \( r_{\sigma, \rho} \) \( K_{\sigma, a} E_\sigma \) are linearly dependent (mod \( \bar{\Sigma}_\rho \)) and hence \( |A_\rho| = 0 \). This contradicts (29), so that \( r_{\sigma, \rho} \leq r'_{\sigma, \rho} \).

Similarly, we have \( r_{\sigma, \rho} \geq r'_{\sigma, \rho} \) and hence \( r_{\sigma, \rho} = r'_{\sigma, \rho} \).

**References**


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