Some Homotopy Groups of the Homogeneous Space $E_6/F_4$

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SOME HOMOTOPY GROUPS OF THE HOMOGENEOUS SPACE $E_6/F_4$

YOSHIHiro HIRATO

1. Introduction

Let $F_4$ and $E_6$ be the compact, connected, simply connected, simple, exceptional Lie groups of rank 4 and 6 respectively. We consider the homogeneous space $E_6/F_4$. Cohen and Selick constructed in [3] a map $\lambda : \Omega^2 S^{17} \to \Omega S^9$ which is $\text{ad}(\sigma_9)$ on the bottom cell where $\text{ad} : \pi_{16} (S^9) \to \pi_{15} (\Omega S^9)$ is an adjoint isomorphism and $\sigma_9$ a generator of $\pi_{16} (S^9)$. They also showed that there does not exist a spherical fibration $S^9 \to E \to S^{17}$ giving rise to the $\lambda$. Thus the homotopy fibre of $\lambda$ is expected to be homotopy equivalent to $\Omega^2 (E_6/F_4)$.

Conlon has determined $\pi_i (E_6/F_4)$ for $i \leq 23$ in [2]. In this paper we calculate $\pi_i (E_6/F_4 : p)$ for $i \leq 39$ where we denote by $\pi_i (Y : p)$ the $p$-primary component of $\pi_i (Y)$. The calculation will be done by making use of the fibration

$$X \xrightarrow{p} S^9 \xrightarrow{p} E_6/F_4$$

where $X$ is the homotopy fibre of the natural inclusion of $S^9$ in $E_6/F_4$. Our results are stated as follows.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$i \leq 8$</th>
<th>9</th>
<th>10</th>
<th>11</th>
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<th>15</th>
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<td>0</td>
<td>2</td>
<td>0</td>
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<td>$(2)^3$</td>
</tr>
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<td>24</td>
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<td>30</td>
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<tr>
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<td>8 + 2</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>16 + 2</td>
<td>2</td>
<td>$(2)^3$</td>
<td>2</td>
<td>8 + 2</td>
<td>8</td>
<td>2</td>
</tr>
</tbody>
</table>

| 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 |
| 32 + 8 + 2 | $(2)^2$ | $(2)^2$ | 4 + $(2)^3$ | 8 + 2 | $(2)^2$ | 2 | 0 |

Here an integer ‘$n$’ indicates a cyclic group $\mathbb{Z}_n$ of order $n$, the symbol ‘$\infty$’ an infinite cyclic group, the symbol ‘+’ the direct sum of groups and ‘$(n)^k$’ indicates the direct sum of $k$-copies of $\mathbb{Z}_n$. These results are stated in Theorem 4.4 in which we also give their generators.

The results on $\pi_i (E_6/F_4 : 2)$ are expected to determine the homotopy type of homotopy fibre of $\lambda$.

We use freely the notation in [14].

I would like to thank Professor Mamoru Mimura for his advice and criticism throughout the preparation of the manuscript, Professor Hirosi Toda.

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for suggesting me to use the homotopy fibre of the inclusion map $S^9 \rightarrow E_6/F_4$ of the bottom cell, and also the referee for pointing out the work by I. M. James.

2. Preliminaries

We denote by $\mathcal{O}P$ the octonionic projective plane. As is well known (see Conlon [2]), we have

$$E_6/F_4 \simeq \Sigma(\mathcal{O}P) \cup e^{26} \quad \text{and} \quad \Sigma(\mathcal{O}P) \simeq S^9 \cup h_9 e^{17}$$

where $h_9 = \sigma_9 + \alpha_2(9) + \alpha_1(9) \in \pi_{16}(S^9) = \mathbb{Z}_{16}\{\sigma_9\} \oplus \mathbb{Z}_3\{\alpha_2(9)\} \oplus \mathbb{Z}_5\{\alpha_1(9)\}$.

We denote by $i_\Omega : S^9 \rightarrow S^9 \cup h_9 e^{17}$ the inclusion and by $\omega \in \pi_{17}(\Sigma(\mathcal{O}P), S^9)$ the homotopy class of the characteristic map of 17-dimensional cell of $\Sigma(\mathcal{O}P)$. For the boundary homomorphism $\partial : \pi_{17}(\Sigma(\mathcal{O}P), S^9) \rightarrow \pi_{16}(S^9)$, we have

$$\partial(\omega) = h_9.$$ 

Then by Theorem 1.4 of [5], we have

$$\pi_{25}(\Sigma(\mathcal{O}P), S^9) = \mathbb{Z}\{[\omega, \iota_9]\} \oplus \mathbb{Z}_2\{\omega \bar{\nu}_{16}\} \oplus \mathbb{Z}_2\{\omega \bar{\varepsilon}_{16}\}$$

and

$$\pi_{26}(\Sigma(\mathcal{O}P), S^9) = \mathbb{Z}_2\{[\omega, \eta_9]\} \oplus \mathbb{Z}_2\{\omega \bar{\nu}^3_{16}\} \oplus \mathbb{Z}_2\{\omega \mu_{16}\} \oplus \mathbb{Z}_2\{\omega \eta_{16} \varepsilon_{17}\}.$$ 

Here we denote by $\hat{\alpha}$ an element of $\pi_n(CS^{16}, S^{16})$ such that $\partial(\hat{\alpha}) = \alpha$ for the boundary homomorphism $\partial : \pi_n(CS^{16}, S^{16}) \rightarrow \pi_{n-1}(S^{16})$.

We calculate $\partial : \pi_{25}(\Sigma(\mathcal{O}P), S^9) \rightarrow \pi_{24}(S^9)$ and $\partial : \pi_{26}(\Sigma(\mathcal{O}P), S^9) \rightarrow \pi_{25}(S^9)$ where

$$\pi_{24}(S^9) = \mathbb{Z}_{16}\{\rho^\prime\} \oplus \mathbb{Z}_2\{\sigma_9 \bar{\nu}_{16}\} \oplus \mathbb{Z}_2\{\sigma_9 \varepsilon_{16}\} \oplus \mathbb{Z}_2\{\bar{\varepsilon}_9\} \oplus \mathbb{Z}_3\{\alpha_4(9)\} \oplus \mathbb{Z}_5\{\alpha_2(9)\}$$

and

$$\pi_{25}(S^9) = \mathbb{Z}_2\{\sigma_9 \nu^3_{16}\} \oplus \mathbb{Z}_2\{\sigma_9 \mu_{16}\} \oplus \mathbb{Z}_2\{\sigma_9 \eta_{16} \varepsilon_{17}\} \oplus \mathbb{Z}_2\{\mu_9 \sigma_{18}\}.$$ 

By Theorem 8.18 in Chapter X of [18] and by the fact that the odd primary component of $\pi_{17}(S^9)$ is trivial, we have $[\alpha_2(9), \iota_9] = [\iota_9, \iota_9] \alpha_2(17) = 0$ and $[\alpha_1(9), \iota_9] = [\iota_9, \iota_9] \alpha_1(17) = 0$. Thus we have

$$\partial[\omega, \iota_9] = -[\partial \omega, \iota_9] \quad \text{by (2.1) of [6]}$$

$$= -[\sigma_9, \iota_9] - [\alpha_2(9), \iota_9] - [\alpha_1(9), \iota_9]$$

$$= -[\iota_9, \iota_9] \sigma_{17}$$

$$= (\sigma_9 \eta_{16} + \bar{\nu}_9 + \bar{\varepsilon}_9) \sigma_{17} \quad \text{by (7.1) of [14]}$$

$$= \sigma_9 \bar{\nu}_{16} + \sigma_9 \varepsilon_{16} \quad \text{by Lemma 6.4 and 10.7 of [14]}.$$
So we have
\[ \partial[\omega, \eta_9] = -[\partial \omega, \iota_9]\eta_{24} \]
\[ = \sigma_9 \nu_{16} \eta_{24} + \sigma_9 \varepsilon_{16} \eta_{24} \]
\[ = \sigma_9 \nu_{16}^3 + \sigma_9 \eta_{16} \varepsilon_{17} \quad \text{by Lemma 6.3 of [14].} \]

By the naturality of the boundary homomorphism, we have
\[ \partial(\omega \bar{\nu}_{16}) = h_9 \bar{\nu}_{16} = \sigma_9 \bar{\nu}_{16}, \quad \partial(\omega \bar{\varepsilon}_{16}) = \sigma_9 \varepsilon_{16}, \]
\[ \partial(\omega \nu_{16}^3) = \sigma_9 \nu_{16}^3, \quad \partial(\omega \mu_{16}) = \sigma_9 \mu_{16}, \quad \partial(\omega \eta_{16} \varepsilon_{17}) = \sigma_9 \eta_{16} \varepsilon_{17}. \]

By the argument above, we have
\[ \text{Ker} f \partial : \pi_25(\Sigma(\Omega P), S^9) \to \pi_24(S^9) \]
\[ = \mathbb{Z} \{[\omega, \iota_9] + \omega \bar{\nu}_{16} + \omega \bar{\varepsilon}_{16} \} \]
and
\[ \text{Coker} f \partial : \pi_26(\Sigma(\Omega P), S^9) \to \pi_25(S^9) \]
\[ = \mathbb{Z}_2 \{\mu_9 \sigma_{18}\}. \]
Therefore we have
\[ \pi_25(\Sigma(\Omega P)) = \mathbb{Z}\{\theta\} \oplus \mathbb{Z}_2 \{i_{\Omega^*}(\mu_9 \sigma_{18})\} \]
where \( \theta \) satisfies
\[ j_{\Omega^*}(\theta) = [\omega, \iota_9] + \omega \bar{\nu}_{16} + \omega \bar{\varepsilon}_{16} \]
for the homomorphism \( j_{\Omega^*} : \pi_25(\Sigma(\Omega P)) \to \pi_25(\Sigma(\Omega P), S^9) \).

Araki has determined integral cohomology of \( E_6/F_4 \) in [1] as follows:
\[ H^*(E_6/F_4; \mathbb{Z}) \cong \wedge(x_9, x_{17}). \]

Then by Theorem 3.3 of [6], we have the following.

**Proposition 2.1.**

\[ E_6/F_4 \simeq \Sigma(\Omega P) \cup_\beta e^{26}, \]
where \( \beta \equiv \theta \mod i_{\Omega^*}(\mu_9 \sigma_{18}) \).

Let \( X \) denote the homotopy fibre of the natural inclusion of the \( S^9 \) in \( E_6/F_4 \). Thus the Serre spectral sequence implies that the integral cohomology ring satisfies
\[ H^*(X; \mathbb{Z}) \cong \Gamma(y_{16}), \]
where \( \Gamma(y_{16}) \) denotes the divided polynomial algebra on a generator \( y_{16} \) of degree 16. Hence we have
\[ X \simeq S^{16}_g \cup e^{32} \cup e^{48} \cup \ldots, \]
where \( g \in \pi_31(S^{16}) \cong \mathbb{Z}\{[t_{16}, \iota_{16}]\} \oplus \mathbb{Z}_{32}\{\rho_{16}\} \oplus \mathbb{Z}_2\{\varepsilon_{16}\} \oplus \mathbb{Z}_3\{\alpha_4(16)\} \oplus \mathbb{Z}_5\{\alpha_2(16)\} \).

By the equality \( y_{16}^2 = 2y_{32} \), we have
\[ g = \pm[t_{16}, \iota_{16}] + a_1 \rho_{16} + a_2 \varepsilon_{16} + a_3 \alpha_4(16) + a_4 \alpha_2(16), \]
where \( a_i \) are integers. We consider the homotopy exact sequence associated with the fibration

\[
X \overset{i}{\to} S^9 \overset{p}{\to} E_6 / F_4.
\]

Then for the inclusion \( j : S^{16} \to X \), we have

\[
i_* j_*(\iota_{16}) = ah_9,
\]

where \( a \) is an integer prime to 2, 3 and 5.

We consider

\[
h_9 \circ g = h_9(\pm[\iota_{16}, \iota_{16}] + a_1 \rho_{16} + a_2 \bar{\varepsilon}_{16} + a_3 \alpha_4(16) + a_4 \alpha_2(16))
\]

\[
= \pm h_9[\iota_{16}, \iota_{16}] + a_1 \sigma_9 \rho_{16} + a_2 \sigma_9 \bar{\varepsilon}_{16} + a_3 \alpha_2(9) \alpha_4(16) + a_4 \alpha_1(9) \alpha_2(16).
\]

By (7.2), Corollary 7.12 and Theorem 8.18 in Chapter X of [18], we have

\[
h_9[\iota_{16}, \iota_{16}] = [h_9, h_9]
\]

\[
= [\sigma_9, \sigma_9] + [\alpha_2(9), \alpha_2(9)] + [\alpha_1(9), \alpha_1(9)]
\]

\[
= [\iota_9, \iota_9] \sigma_{17}^2 + [\iota_9, \iota_9] \alpha_2(17)^2 + [\iota_9, \iota_9] \alpha_1(17)^2.
\]

By (7.1), Lemma 6.4 and 10.7 of [14] and by the fact that odd primary component of \( \pi_{17}(S^9) \) is trivial, we have

\[
h_9[\iota_{16}, \iota_{16}] = [h_9, h_9]
\]

\[
= [\iota_9, \iota_9] \sigma_{17}^2
\]

\[
= (\sigma_9 \eta_{16} + \bar{\nu}_9 + \bar{\varepsilon}_9) \sigma_{17}^2
\]

\[
= 0.
\]

By Theorem 7.6 of [15], the homomorphism \( E_6 : \pi_{25}(S^3 : 3) \to \pi_{31}(S^9 : 3) \) is trivial. So we have \( \alpha_2(9) \alpha_4(16) = E_6(\alpha_2(3) \alpha_4(10)) = 0 \). By the fact \( \pi_{31}(S^9 : 5) = 0 \), we have \( \alpha_1(9) \alpha_2(16) = 0 \). Hence we have

\[
h_9 \circ g = a_1 \sigma_9 \rho_{16} + a_2 \sigma_9 \bar{\varepsilon}_{16}.
\]

On the other hand, since \( j_*(g) = 0 \in \pi_{31}(X) \), we have

\[
h_9 \circ g = i_* j_*(g) = 0.
\]

Then by the fact that \( \sigma_9 \rho_{16} \) and \( \sigma_9 \bar{\varepsilon}_{16} \) are generators of order 16 and 2 respectively, we have

\[
g \equiv \pm[\iota_{16}, \iota_{16}] \mod 16 \rho_{16}, \alpha_4(16), \alpha_2(16).
\]
3. Homotopy Groups of the $X$

From now on, we restrict our attention to the 2-primary component and so omit for simplicity the notation ‘2’ indicating 2-primary component of homotopy group.

By the argument in the previous section, we have

$$E_6/F_4 \cong \frac{1}{2} S^9 \cup_{\sigma_9} e^{17} \cup e^{26}$$

and

$$X \cong \frac{1}{2} S^{16} \cup_g e^{32} \cup e^{48} \cup \ldots,$$

where

$$g \equiv [\iota_{16}, \iota_{16}] \mod 16\rho_{16}.$$  

For dimensional reasons, we have

$$\pi_i(X) \cong \pi_i(S^{16} \cup_g e^{32})$$

for $i \leq 46$. We consider the homotopy exact sequence

$$\cdots \to \pi_i(S^{16}) \xrightarrow{j_*} \pi_i(S^{16} \cup_g e^{32}) \to \pi_i(S^{16} \cup_g e^{32}, S^{16}) \xrightarrow{\partial} \pi_{i-1}(S^{16}) \xrightarrow{j_*} \cdots$$

associated with the pair $(S^{16} \cup_g e^{32}, S^{16})$. The collapsing map $\pi : (S^{16} \cup_g e^{32}, S^{16}) \to (S^{32}, *)$ induces a homomorphism

$$\pi_* : \pi_i(S^{16} \cup_g e^{32}, S^{16}) \to \pi_i(S^{32})$$

which is an isomorphism for $i \leq 45$ by the Blakers-Massey theorem. Therefore the following sequence is exact for $i \leq 45$:

$$\cdots \to \pi_i(S^{16}) \xrightarrow{j_*} \pi_i(S^{16} \cup_g e^{32}) \xrightarrow{\pi_*} \pi_i(S^{32}) \xrightarrow{\Delta'} \pi_{i-1}(S^{16}) \xrightarrow{j_*} \cdots,$$

where

$$\Delta' = \partial \circ \pi_*^{-1} : \pi_i(S^{32}) \xrightarrow{\pi_*} \pi_i(S^{16} \cup_g e^{32}, S^{16}) \xrightarrow{\partial} \pi_{i-1}(S^{16}).$$

For any element $\alpha \in \pi_n(S^{31})$, we have

$$(3.1) \quad \Delta'(\Sigma \alpha) \equiv [\iota_{16}, \iota_{16}]\alpha \mod (16\rho_{16})\alpha.$$  

By use of this formula, we calculate $\Delta' : \pi_{n+1}(S^{32}) \to \pi_n(S^{16})$ for $n \leq 39$.

We recall here some necessary results on $\pi_{n+i}(S^n)$ for $i \leq 30$ determined by Toda [14], Mimura-Toda [10], Mimura [8], Mimura-Mori-Oda [9], and Oda [12].

Table 1.
\[
\begin{array}{cccccccc}
& i & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\
\pi_{i+1}(S_{32}^2) & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
generator & & & & & & & & & & \\
\pi_i(S_{16}^2) & & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z} & \mathbb{Z}_2 \\
generator & & & & & & & \iota_{16} & & \eta_{16} \\
\pi_i(S_9^2) & \mathbb{Z} & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_8 & 0 & 0 & \mathbb{Z}_2 & \mathbb{Z}_16 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\
generator & \iota_9 & \eta_9 & \eta_9^2 & \nu_9 & 0 & 0 & \nu_9^2 & \sigma_9 & \sigma_9 \eta_{16}, \nu_9, \varepsilon_9 \\
\hline
& 18 & 19 & 20 & 21 & 22 & 23 \\
& 0 & 0 & 0 & 0 & 0 & 0 \\
& \mathbb{Z}_2 & & \mathbb{Z}_8 & 0 & 0 & \mathbb{Z}_2 & \mathbb{Z}_16 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\
& \eta_{16}^2 & \nu_{16} & & \nu_{16}^2 & & \sigma_{16} & & & \\
& \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_8 \oplus \mathbb{Z}_2 & \mathbb{Z}_8 \oplus \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & \mathbb{Z}_16 \oplus \mathbb{Z}_4 \\
& \sigma_9 \eta_{16}^2, \nu_9^2, \mu_9, \eta_9 \varepsilon_{10} & \sigma_9 \nu_{16}, \eta_9 \mu_{10} & \zeta_9, \nu_9 \nu_{17} & \sigma_9 \nu_{16}^2 & \sigma_9 \gamma, \kappa_9 \\
\hline
& 24 & 25 \\
& 0 & 0 \\
& \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\
& \nu_{16}, \varepsilon_{16} & \nu_{16}^3, \mu_{16}, \eta_{16} \varepsilon_{17} \\
& \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\
& \rho', \sigma_9 \nu_{16}, \sigma_9 \varepsilon_{16}, \varepsilon_9 & \sigma_9 \nu_{16}^3, \sigma_9 \mu_{16}, \sigma_9 \eta_{16} \varepsilon_{17}, \mu_9 \sigma_{18} \\
\hline
& 26 & 27 & 28 & 29 \\
& 0 & 0 & 0 & 0 \\
& \mathbb{Z}_2 & \mathbb{Z}_8 & 0 & 0 \\
& \eta_{16} \mu_{17} & \zeta_{16} & & \\
& \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_8 \oplus \mathbb{Z}_2 & \mathbb{Z}_8 \oplus \mathbb{Z}_2 & \mathbb{Z}_8 \\
& \sigma_9 \eta_{16} \mu_{17}, \nu_9 \kappa_{12}, \mu_9, \eta_9 \mu_{10} \sigma_{19} & \sigma_9 \zeta_{16}, \eta_9 \mu_{10} & \zeta_9, \sigma_9 & \kappa_9 \\
\end{array}
\]
### SOME HOMOTOPY GROUPS OF THE HOMOGENEOUS SPACE $E_6/F_4$

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<td>$\sigma^2_{16}, \kappa_{16}$</td>
<td>$[\iota_{16}, \iota_{16}, \rho_{16}, \bar{\epsilon}_{16}$</td>
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<td>$\mathbb{Z}_{16} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
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<tr>
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<td>$\sigma_9\rho_{16}, \epsilon_9\kappa_{17}, \nu_9\bar{\epsilon}<em>{12}, \sigma_9\bar{\epsilon}</em>{16}$</td>
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<tr>
<td>$\eta^<em><em>16\eta</em>{32}, (\Sigma\eta^</em>)\eta_{32}, \bar{\epsilon}<em>{16}, \sigma</em>{16}\eta_{23}\mu_{24}, \nu_{16}\kappa_{19}, \bar{\mu}_{16}$</td>
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<tr>
<td>$\delta_9, \bar{\mu}<em>9\sigma</em>{26}, \sigma^*<em>9, \sigma_9\mu</em>{16}, \sigma^2_0\eta_{23}\mu_{24}, \sigma_9\nu_{16}\kappa_{19}, \sigma_9\bar{\epsilon}_{16}$</td>
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<table>
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<tr>
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<tr>
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</tr>
<tr>
<td>$\sigma_9\xi_{16}, \sigma_9\nu^*<em>16, \sigma_9\eta</em>{16}\mu_{17}, \mu_{3,9}, \eta_9\mu_{10}\sigma_{27}$</td>
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</tbody>
</table>
As for the boundary formula (3.1) we have

**Lemma 3.1.** (1) $\Delta'(t_{32}) \equiv [\nu_{16}, \iota_{16}] \mod 16 \rho_{16}$,
(2) $\Delta'(\eta_{32}) \equiv \Sigma \eta^* \mod \omega_{16}, \sigma_{16} \mu_{23}$,
(3) $\Delta'(\eta_{32}^2) \equiv (\Sigma \eta^*) \eta_{32} \mod \omega_{16} \eta_{32}, \sigma_{16} \mu_{23} \eta_{32}$,
(4) $\Delta'(\nu_{32}) = \pm (\Sigma^3 \lambda - 2\nu^*_1)$. 

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<table>
<thead>
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<td>$\kappa_{16}$</td>
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<tr>
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</tr>
<tr>
<td>$\sigma_9 \zeta_{16}, \sigma_9 \omega_{16} \nu_{32}, \sigma_9 \sigma_{28}, \nu^2_0 \kappa_{15}, \eta_0 \mu_{3,10}$</td>
<td>$\sigma_9 \kappa_{16}, \zeta_{3,9}, \nu_9 \sigma_{17}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>37</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>$\nu^2_{32}$</td>
</tr>
<tr>
<td>$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
</tr>
<tr>
<td>$\eta_{16} \kappa_{17}, \sigma^3_{16}, (\Sigma^3 \lambda) \nu_{34}, \nu^*<em>{16} \nu</em>{34}$</td>
</tr>
<tr>
<td>$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
</tr>
<tr>
<td>$\sigma^4_9, \sigma_9 \eta_{16} \kappa_{17}, \sigma_9 \nu^*<em>{16} \nu</em>{34}, \nu_9 \kappa_{17}, \varepsilon_9 \kappa_{17}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>38</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}_{16}$</td>
</tr>
<tr>
<td>$\sigma_{32}$</td>
</tr>
<tr>
<td>$\mathbb{Z}<em>{16} \oplus \mathbb{Z}</em>{16} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
</tr>
<tr>
<td>$\sigma^<em>_{16}, \Sigma \sigma^</em> \omega_{16} \nu^2_{32}, \varepsilon_{16} \kappa_{24}, \nu_{16} \sigma_{19}$</td>
</tr>
<tr>
<td>$\mathbb{Z}_{16} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
</tr>
<tr>
<td>$\sigma_9 \sigma^*<em>{16}, \sigma_9 \omega</em>{16} \nu^2_{32}, \sigma_9 \varepsilon_{16} \kappa_{24}, \sigma_9 \nu_{16} \sigma_{19}, \eta_9 \varepsilon_{10} \kappa_{18}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>39</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
</tr>
<tr>
<td>$\nu^*<em>{32}, \varepsilon</em>{32}$</td>
</tr>
<tr>
<td>$\mathbb{Z}_{16} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
</tr>
<tr>
<td>$\tilde{\rho}<em>{16}, \nu</em>{16} \kappa_{19}, \phi_{16}, \psi_{16}, \Sigma \tilde{\varepsilon}^* \sigma^* \Sigma \tilde{\rho}^* \varepsilon_{16}, \tilde{\nu}_{16}^*$</td>
</tr>
<tr>
<td>$\mathbb{Z}_{16} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
</tr>
<tr>
<td>$\sigma_9 \tilde{\rho}<em>{16}, \sigma_9 \nu</em>{16} \kappa_{19}, \sigma_9 \phi_{16}, \sigma_9 \psi_{16}, \phi_9 \sigma_{32}$</td>
</tr>
</tbody>
</table>

http://escholarship.lib.okayama-u.ac.jp/mjou/vol45/iss1/6
Lemma 3.2. We have the following table of $\pi_i(X)$ for $i \leq 39$. 

<table>
<thead>
<tr>
<th>$i$</th>
<th>$i \leq 15$</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_i(X)$</td>
<td>0</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_8$</td>
<td>0</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>generator</td>
<td>$j_*(\iota_{16})$</td>
<td>$j_*(\eta_{16})$</td>
<td>$j_*(\eta_{16}^2)$</td>
<td>$j_*(\nu_{16})$</td>
<td>$j_*(\nu_{16}^2)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$23$</td>
<td>$24$</td>
<td>$25$</td>
<td>$26$</td>
<td>$27$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mathbb{Z}_{16}$</td>
<td>$\mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_8$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$j_*(\sigma_{16})$</td>
<td>$j_<em>(\nu_{16})$, $j_</em>(\epsilon_{16})$</td>
<td>$j_<em>(\nu_{16}^3)$, $j_</em>(\mu_{16})$, $j_*(\eta_{16}\epsilon_{17})$</td>
<td>$j_*(\eta_{16}\mu_{17})$</td>
<td>$j_*(\xi_{16})$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$28$</td>
<td>$29$</td>
<td>$30$</td>
<td>$31$</td>
<td>$32$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0$</td>
<td>0</td>
<td>$\mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
<td>$\mathbb{Z}_{32} \oplus \mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$j_<em>(\sigma_{16}^2)$, $j_</em>(\kappa_{16})$</td>
<td>$j_<em>(\rho_{16})$, $j_</em>(\epsilon_{16})$</td>
<td>$j_<em>(\eta_{16}^2)$, $j_</em>(\omega_{16})$, $j_*(\sigma_{16}\mu_{23})$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Since \( \nu \) is the 2-local fibration
\[
\begin{align*}
Z_2 & \oplus Z_2 & \oplus Z_2 & \oplus Z_2 & \oplus Z_2 \\
& j_{*}(\eta_{16}^{*}\eta_{32}), j_{*}(\epsilon_{16}^{*}), j_{*}(\sigma_{16}\eta_{23}\mu_{24}), j_{*}(\nu_{16}\kappa_{19}), j_{*}(\mu_{16})
\end{align*}
\]

Then for the inclusion
\[
X \xrightarrow{\iota_{16}} S^{9} \xrightarrow{p} E_{6}/F_{4}.
\]

By use of this formula, we calculate \( i_{*} \cdot j_{*}(\iota_{16}) = \sigma_{9} \).

**Lemma 4.1.** (1) \( \sigma_{9}\eta_{16}^{*} \equiv \phi_{9} \mod \sigma_{9}^{2}\mu_{23}, 4\nu_{9}\bar{\kappa}_{12} \),
(2) \( \sigma_{9}\eta_{16}^{*}\eta_{32} \equiv \delta_{9} \mod \bar{\mu}_{9}\sigma_{26}, \sigma_{9}^{2}\eta_{23}\mu_{24} \),
(3) \( \sigma_{9}(\bar{\epsilon}_{16}^{*} + \bar{\nu}_{16}^{*}) = \phi_{9}\sigma_{32} \).

**Proof.** (1) is obtained in Part I, Proposition 3.4 (7) of [12].
(2) By (1), we have
\[
\sigma_{9}\eta_{16}^{*}\eta_{32} \equiv \phi_{9}\eta_{32} \mod \sigma_{9}^{2}\eta_{23}\mu_{24}.
\]

By Part I, Proposition 3.5 (9) of [12], we have
\[
\phi_{9}\eta_{32} \equiv \delta_{9} \mod \bar{\mu}_{9}\sigma_{26}, \nu_{9}\eta_{12}\bar{\kappa}_{13}.
\]

Since \( \nu_{9}\eta_{12} = 0 \) ((5.9) of [14]), we have
\[
\sigma_{9}\eta_{16}^{*}\eta_{32} \equiv \delta_{9} \mod \bar{\mu}_{9}\sigma_{26}, \sigma_{9}^{2}\eta_{23}\mu_{24}.
\]
SOME HOMOTOPY GROUPS OF THE HOMOGENEOUS SPACE $E_6/F_4$

(3) By the definition of $\varepsilon^*_16$ (see (3.4) of [9]) and (1) we have

$$\sigma_9(\varepsilon^*_16 + \bar{\nu}^*_16) = \sigma_9\eta^*_16\sigma_{32} \equiv \phi_9\sigma_{32} \text{ mod } \sigma_9^2\mu_{23}\sigma_{32}, 4\nu_9\bar{\kappa}_{12}\sigma_{32}.$$  

By (2.3) of [9], we have

$$\sigma_9^2\mu_{23}\sigma_{32} = 0.$$  

By Part III, Proposition 2.2 (5) of [12], we have

$$4\nu_9\bar{\kappa}_{12}\sigma_{32} = 0.$$  

Hence we have $\sigma_9(\varepsilon^*_16 + \bar{\nu}^*_16) = \phi_9\sigma_{32}$.

For $i_* : \pi_n(X) \rightarrow \pi_n(S^9)$, we have the following.

**Lemma 4.2.** (1) The homomorphisms $i_* : \pi_n(X) \rightarrow \pi_n(S^9)$ are epimorphisms for $n = 16, 21, 22, 39$. For the other values of $n$ ($17 \leq n \leq 38$), we have the following table of the kernels of $i_*$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>23</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coker $i_*$</td>
<td>$Z_2 \oplus Z_2$</td>
<td>$Z_2 \oplus Z_2 \oplus Z_2$</td>
<td>$Z_2$</td>
<td>$Z_8 \oplus Z_2$</td>
<td>$Z_4$</td>
<td>$Z_{16} \oplus Z_2$</td>
</tr>
<tr>
<td>generator</td>
<td>$\nu_9, \varepsilon_9$</td>
<td>$\nu_9^2, \mu_9, \eta_9 \varepsilon_{10}$</td>
<td>$\eta_9 \mu_{10}$</td>
<td>$\zeta_9, \bar{\nu}<em>9 \mu</em>{17}$</td>
<td>$\kappa_9$</td>
<td>$\rho', \varepsilon_9$</td>
</tr>
<tr>
<td>25</td>
<td>26</td>
<td>27</td>
<td>28</td>
<td>29</td>
<td>30</td>
<td>31</td>
</tr>
<tr>
<td>$Z_2$</td>
<td>$Z_2 \oplus Z_2 \oplus Z_2$</td>
<td>$Z_2$</td>
<td>$Z_8 \oplus Z_2$</td>
<td>$Z_8$</td>
<td>$Z_2$</td>
<td>$Z_2 \oplus Z_2$</td>
</tr>
<tr>
<td></td>
<td>$\mu_9 \sigma_{18}$</td>
<td>$\nu_9 \bar{\kappa}<em>{12}, \mu_9, \eta_9 \mu</em>{10} \sigma_{19}$</td>
<td>$\eta_9 \mu_{10}$</td>
<td>$\zeta_9, \bar{\sigma}_9$</td>
<td>$\bar{\kappa}_9$</td>
<td>$\eta_9 \bar{\kappa}_{10}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\nu_9 \varepsilon_{17}, \nu_9 \bar{\sigma}_{12}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>33</td>
<td>34</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Z_{16} \oplus Z_8 \oplus Z_2$</td>
<td>$Z_2 \oplus Z_2$</td>
<td>$Z_2 \oplus Z_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\bar{\rho}<em>9, \nu_9 \bar{\kappa}</em>{12}, \bar{\kappa}<em>9 \nu</em>{29} - \nu_9 \bar{\kappa}_{12}$</td>
<td>$\bar{\mu}<em>9 \sigma</em>{26}, \bar{\sigma}_9'$</td>
<td>$\mu_{3,9}, \eta_9 \mu_{10} \sigma_{27}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>36</td>
<td>37</td>
<td>38</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$</td>
<td>$Z_8 \oplus Z_2$</td>
<td>$Z_2 \oplus Z_2$</td>
<td>$Z_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\bar{\kappa}<em>9 \nu</em>{29}, \bar{\sigma}<em>9 \sigma</em>{28}, \nu_9^2 \bar{\kappa}<em>{15}, \eta_9 \mu</em>{3,10}$</td>
<td>$\zeta_{3,9}, \bar{\nu}<em>9 \sigma</em>{17}$</td>
<td>$\bar{\nu}<em>9 \bar{\kappa}</em>{17}, \varepsilon_9 \bar{\kappa}_{17}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\eta_9 \varepsilon_{10} \bar{\kappa}_{18}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(2) For $n = 16, 31, 34$, we have the following table of the kernels of $i_*$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>16</th>
<th>31</th>
<th>34</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ker $i_*$</td>
<td>$Z$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
</tr>
<tr>
<td>generator</td>
<td>$16j_*(\nu_{16})$</td>
<td>$16j_*(\rho_{16})$</td>
<td>$4j_<em>(\nu_{16}^</em>)$</td>
</tr>
</tbody>
</table>

For other values of $n$ ($n \leq 38$), the homomorphisms $i_*$ are monomorphisms.

**Proof.** (1) By (4.1), Lemmas 3.2 and 4.1, we obtain the results easily.

(2) By (4.1), Lemmas 3.2 and 4.1, we can determine Ker{$i_* : \pi_n(X) \rightarrow \pi_n(S^9)$} easily except for the case $n = 31$. 

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We consider \( i_* : \pi_{31}(X) \to \pi_{31}(S^9) \) where
\[
\pi_{31}(S^9) = \{ \sigma_9 \rho_{16}, \varepsilon_9 \kappa_{17}, \nu_9 \sigma_{12}, \sigma_9 \varepsilon_{16} \} \cong \mathbb{Z}_{16} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.
\]

For the case \( g = [\iota_{16}, \iota_{16}] + 16\rho_{16} \), we have \( \pi_{31}(X) = \{ j_*(\rho_{16}), j_*(\varepsilon_{16}) \} \cong \mathbb{Z}_{32} \oplus \mathbb{Z}_2 \) and \( j_*([\iota_{16}, \iota_{16}]) = 16j_*(\rho_{16}) \). By (4.1) we have \( i_*j_*(\rho_{16}) = \sigma_9 \rho_{16} \) and \( i_*j_*\varepsilon_{16} = \sigma_9 \varepsilon_{16} \). By (4.1) and the argument in Section 2, we have \( i_*j_*([\iota_{16}, \iota_{16}]) = \sigma_9 [\iota_{16}, \iota_{16}] = 0 \). Hence we have \( \text{Ker}\{ i_* : \pi_{31}(X) \to \pi_{31}(S^9) \} = \{ 16j_*(\rho_{16}) \} \cong \mathbb{Z}_2 \) and \( j_*([\iota_{16}, \iota_{16}]) = 16j_*(\rho_{16}) \).

For the case \( g = [\iota_{16}, \iota_{16}] \), we have \( \pi_{31}(X) = \{ j_*(\rho_{16}), j_*(\varepsilon_{16}) \} \cong \mathbb{Z}_{32} \oplus \mathbb{Z}_2 \). Then by the above argument, we have \( \text{Ker}\{ i_* : \pi_{31}(X) \to \pi_{31}(S^9) \} = \{ 16j_*(\rho_{16}) \} \cong \mathbb{Z}_2 \).

The following lemma will be used later.

**Lemma 4.3.** Let \((E, p, B)\) be a fibration, \( F \) a fiber \( p^{-1}(*) \) and \( \Delta \) the boundary homomorphism in the homotopy exact sequence of the fibration. Then for any element \( \alpha \) of \( \pi_{i+1}(B) \), we have
\[
\alpha \in \{ p, i, \Delta(\alpha) \}.
\]

**Proof.** Let \( E^{i+1}_+ \) (resp. \( E^{i+1}_- \)) be the upper-(resp. lower-)hemisphere of \( S^{i+1} \). Since \( p_* : \pi_{i+1}(E, F) \to \pi_{i+1}(B, *) \) is an isomorphism, there exists \( a : (E^{i+1}_+, S^i) \to (E, F) \) such that \( p \circ a \) and \( a|_{S^i} \) are representatives of \( \alpha \) and \( \Delta(\alpha) \) respectively. Then we define a map \( \tilde{a} : S^{i+1} \to E \cup CF \) by \( \tilde{a}|_{E^{i+1}_+} = a \) and \( \tilde{a}|_{E^{i+1}_-}(x, t) = (a|_{S^i}(x), 1 - 2t) \in CF \). We define a map \( \tilde{p} : E \cup CF \to B \) by \( \tilde{p}|_E = p \) and \( \tilde{p}|_{CF} = \ast \). Then by the definition of Toda bracket, \( \tilde{p} \circ \tilde{a} \) represents an element of \( \{ p, i, \Delta(\alpha) \} \). Since \( \tilde{p} \circ \tilde{a} \simeq p \circ a \), we have \( \alpha \in \{ p, i, \Delta(\alpha) \} \).

Let us state our main result.

**Theorem 4.4.** We have the following table of \( \pi_i(E_6/F_4) \) for \( i \leq 39 \).

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \pi_i(E_6/F_4) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \leq 8 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \text{generator} )</td>
<td>( p_*(\iota_9) )</td>
</tr>
<tr>
<td>( 16 )</td>
<td>( \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( [16j_<em>(\iota_{16})], p_</em>(\nu_9), p_*(\varepsilon_9) )</td>
</tr>
<tr>
<td>( 20 )</td>
<td>( \mathbb{Z}_8 \oplus \mathbb{Z}_2 )</td>
</tr>
<tr>
<td>( 21 )</td>
<td>( p_<em>(\zeta_9), p_</em>(\nu_9 \nu_{17}) )</td>
</tr>
</tbody>
</table>

12
Here we denote by $[\alpha]$ an element of $\pi_i(E_6/F_4)$ such that $\Delta([\alpha]) = \alpha \in \pi_{i-1}(X)$. The following relations hold:

$$2[16j_*(\iota_{16})]\rho_{17} = -p_*(\bar{\rho}_9),$$
$$2[4j_*(\nu_{16}^3)] \equiv p_*(a\bar{\kappa}_9\nu_{29}^3 + (a + 1)\nu_9^2\bar{\kappa}_{15}) \mod p_*(\bar{\sigma}_9\sigma_{28}),$$

where $a = 0$ or 1.

**Proof.** By Lemma 4.2, we can determine $\pi_i(E_6/F_4)$ for $i \leq 39$ easily except for the case $i = 32, 35$.

Consider the case $i = 32$; by Lemma 4.2, we have an exact sequence

$$0 \to \mathbb{Z}_{16} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2 \xrightarrow{p_*} \pi_{32}(E_6/F_4) \xrightarrow{\Delta} \mathbb{Z}_2 \to 0,$$
Therefore we have
\[
p_*(\rho') \in p \circ \{\sigma_9, 16\mu_{16}, \sigma_{16}\}
\]
by the definition of \(\rho'\) ([14])
\[
= -\{p, \sigma_9, 16\mu_{16}\} \circ \sigma_{17}
\]
\[
= -\{p, i, 16j_*(\iota_{16})\} \circ \sigma_{17}
\]
by (4.1)
\[
\ni -\{p, i, 16j_*(\iota_{16})\} \circ \sigma_{17}
\]
\[
\ni -\{p, i, \Delta[16j_*(\iota_{16})]\} \circ \sigma_{17}
\]
\[
\ni -[16j_*(\iota_{16})]\sigma_{17}
\]
by the definition of \(\rho'\) ([14])

By Lemma 10.7 of [14] and (4.1), we have
\[
p_*(\rho) = [-16j_*(\iota_{16})]\sigma_{17}
\]
Then we have
\[
p_*(\rho) \in \{p_*(\rho'), 16\mu_{24}, \sigma_{24}\}
\]
by (3.2) of [9]
\[
= -[16j_*(\iota_{16})]\sigma_{17}, 16\mu_{24}, \sigma_{24}\}
\]
\[
\ni -[16j_*(\iota_{16})] \circ \sigma_{17}, 16\mu_{24}, \sigma_{24}\}
\]
\[
\ni -[16j_*(\iota_{16})] \sigma_{17}, 16\mu_{24}, \sigma_{24}\}
\]
\[
\ni -[16j_*(\iota_{16})] \sigma_{17}, 16\mu_{24}, \sigma_{24}\}
\]

The indeterminacy of \(\{p_*(\rho'), 16\mu_{24}, \sigma_{24}\}\) is
\[
p_*(\rho') \circ \pi_{32}(S^{24}) + \pi_{25}(E_6/F_4) \circ \sigma_{25} = \{p_*(\rho\varepsilon_{24}), p_*(\rho'\varepsilon_{24}), p_*(\rho_9\rho_{23})\}.
\]

By Part III, Proposition 2.2 (1) of [12] and (4.1), we have
\[
p_*(\rho'\varepsilon_{24}) = 0
\]
and
\[
p_*(\rho\varepsilon_{24}) = p_*(\rho_9\rho_{23}) = 0.
\]

By Lemma 2.1 (4) of [13], we have
\[
\rho_9\sigma_{18}^2 \equiv \rho'\eta_{24}\sigma_{25} + \sigma_9\mu_{23} \text{ mod } \sigma_9\nu_{16}\mu_{25}, \sigma_9\nu_{16}\nu_{17}\sigma_{25}.
\]
By (7.20) of [14], we have \(\sigma_9\nu_{16}\mu_{25} = 0\). By Lemma 10.7 of [14], we have \(\sigma_9\nu_{16}\nu_{17}\sigma_{25} = 0\). By Lemma 6.4 of [14] and Part III, Proposition 2.2 (1) of [12], we have \(\rho'\eta_{14}\sigma_{25} = \rho'\varepsilon_{24} + \rho'\varepsilon_{24} = \sigma_9\mu_{23}\). Hence we have
\[
\rho_9\sigma_{18}^2 = 2\sigma_9\mu_{23} = 0.
\]
So we have
\[
p_*(\rho') \circ \pi_{32}(S^{24}) + \pi_{25}(E_6/F_4) \circ \sigma_{25} = 0
\]
and
\[
p_*(\rho_9) = -2[16j_*(\iota_{16})]\rho_{17}.
\]

Therefore we have
\[
\pi_{32}(E_6/F_4) = \{[16j_*(\iota_{16})]\rho_{17}, p_*(\nu_9\kappa_{12}), p_*(\kappa_9\nu_{29} - \nu_9\kappa_{12})\} \cong \mathbb{Z}_{32} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2.
\]
Consider the case \( i = 35 \); by Lemma 4.2, we have an exact sequence
\[
0 \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \stackrel{p_*}{\rightarrow} \pi_{35}(E_6/F_4) \rightarrow \mathbb{Z}_2 \rightarrow 0,
\]
where \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) is generated by \( \kappa_9 \nu_{29}^2, \sigma_9 \nu_{28}, \nu_{16}^3 \kappa_{15}, \eta_9 \mu_{3,10} \) and \( \mathbb{Z}_2 \) is generated by \( 4j_*(\nu_{16}^3) \). By the exactness there exists an element \([4j_*(\nu_{16}^3)] \in \pi_{35}(E_6/F_4)\) such that \( \Delta([4j_*(\nu_{16}^3)]) = 4j_*(\nu_{16}^3)\). We consider \( \{\sigma_9, 4\nu_{16}^3, 2\iota_{34}\} \). The indeterminacy of the Toda bracket is
\[
\sigma_9 \circ \pi_{35}(S^{16}) + 2\pi_{35}(S^9) = \{\sigma_9 \tilde{\zeta}_{16}, \sigma_9 \omega_{16} \nu_{32}, \sigma_9 \tilde{\sigma}_{16}\}.
\]
So the Toda bracket \( \{\sigma_9, 4\nu_{16}^3, 2\iota_{34}\} \) is represented by
\[
a\kappa_9 \nu_{29}^2 + b \sigma_9 \nu_{28} + c \nu_{16}^3 \kappa_{15} + d \eta_9 \mu_{3,10} \in \{\sigma_9, 4\nu_{16}^3, 2\iota_{34}\}
\]
for some integers \( a, b, c, d \). By Part I, Proposition 3.5 (6) of [12], we have \( \bar{\sigma}_{10} \sigma_{29} = 0 \). Then the facts that \( \langle \sigma, 4\nu_{16}^3, 2\iota \rangle = \nu^2 \kappa \) by Theorem 1 of [4] and that \( \pi_{26}^S(S^0) = \{\nu^2 \kappa, \eta_9 \mu_{3, \ast}\} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) imply that
\[
\{\sigma_9, 4\nu_{16}^3, 2\iota_{34}\} \ni a \kappa_9 \nu_{29}^2 + (a_1 + 1) \nu_{16}^3 \kappa_{15} + a_2 \sigma_9 \nu_{28},
\]
where \( a_i = 0 \) or 1. We have
\[
p_*(\sigma_9, 4\nu_{16}^3, 2\iota_{34}) = -\{p, \sigma_9, 4\nu_{16}^3 \circ 2\iota_{35}
\]
\[
= -\{p, i_*(\iota_{16}), 4\nu_{16}^3 \circ 2\iota_{35}
\]
\[
\supset -\{p, i_*(\nu_{16}^3) \circ 2\iota_{35}
\]
\[
\supset [4j_*(\nu_{16}^3)] \circ 2\iota_{35}
\]
\[
= 2[4j_*(\nu_{16}^3)]
\]
Since \( p_*(\sigma_9) = 0 \) by (4.1), we have
\[
p_*(\sigma_9 \circ \pi_{35}(S^{16}) + 2\pi_{35}(S^9)) = p_*(\{\sigma_9 \tilde{\zeta}_{16}, \sigma_9 \omega_{16} \nu_{32}, \sigma_9 \tilde{\sigma}_{16}\}) = 0.
\]
Therefore we have
\[
p_*(a_1 \kappa_9 \nu_{29}^2 + (a_1 + 1) \nu_{16}^3 \kappa_{15} + a_2 \sigma_9 \nu_{28}) = 2[4j_*(\nu_{16}^3)],
\]
where \( a_1 = 0 \) or 1. So we have
\[
\pi_{35}(E_6/F_4) = \{[4j_*(\nu_{16}^3)], p_*([a + 1] \kappa_9 \nu_{29}^2 + a \nu_{16}^3 \kappa_{15}), p_*(\sigma_9 \nu_{28}), p_*(\eta_9 \mu_{3,10})\}
\]
\[
\cong \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.
\]

\[\square\]

REFERENCES


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