On the spaces with normal projective connexions and some imbedding problem of Riemannian spaces

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CONNEXIONS AND SOME IMBEDDING PROBLEM
OF RIEMANNIAN SPACES

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Introduction.

In a previous paper¹, the author obtained a theorem as follows: *If the group of holonomy of a space with a normal projective connection fixes a hyperquadric, the space is one corresponding to the class of Riemannian spaces projective to each other including an Einstein space*, in the domain of such points that they are not the images of the points on the hyperquadric into the space. Such a space has also been studied by S. Sasaki and K. Yano², but the properties in the neighborhood of the image of the hyperquadric invariant under the group of holonomy of the space into it have never been investigated by any one. One of the most interesting problems which arises in connection with this space is the determination of the conditions under which a given Riemannian space $V^n$ can be imbedded into a Riemannian space $V^{n+1}$ as a hypersurface $F_n$ such that the group of holonomy of the space with a normal projective connexion corresponding to $V^{n+1}$ fixes a hyperquadric $Q_n$ and the image of $Q_n$ into $V^{n+1}$ is $F_n$.

On the other hand, *any Riemannian space $V^n$ can be imbedded into a suitable Einstein space $A_{n+1}$ whose scalar curvature is a given constant* (from now on, we call this *Campbell’s theorem*)³. As regards the signification of the theorem by means of the groups of holonomy, the author has obtained some results in connection with the space with a normal conformal connexion⁴.

¹) T. Ötsuki, On projectively connected spaces whose groups of holonomy fix a hyperquadric, Jour. of the Math. Soc. of Japan, Vol. 1, No. 4, 1950, pp. 251—263.


³) J. E. Campbell, A course of differential geometry, 1926.

In the present paper, we shall investigate the same problem by means of the groups of holonomy of the spaces with normal projective connexions, considering the above-mentioned imbedding problem of \( V_n \) into \( V_{n+1} \).

§1. Fundamental equations.

Let there be given a space with a normal projective connexion \( X \), corresponding to a given Riemannian space \( V_n \) with positive definite line element

\[
(1,1) \quad ds^2 = g_{ij}(x)dx^idx^j \quad (i, j = 1, 2, \ldots, n) \tag{1,2}
\]

in each of its coordinate neighborhoods. If we take suitable semi-natural frames \( R(A, A_i) \), the projective connexion of the space \( X \) is given, as is well known, by means of Christoffel symbols \( \Gamma^i_{ik} \) made by \( g_{ij} \) by the following equations:

\[
(1,3) \quad \omega^i_j = \Gamma^i_{ik}dx^k, \quad \omega^0_i = \Gamma^0_{ij}dx^j = \frac{1}{n-1}K_{ij}dx^j
\]

and we put

\[
(1,4) \quad \begin{cases} 
K^{j}_{ik} = \frac{\partial \Gamma^{j}_{ik}}{\partial x^k} - \frac{\partial \Gamma^{j}_{ik}}{\partial x^j} + \Gamma^{m}_{jk} \Gamma^{i}_{m} - \Gamma^{m}_{ij} \Gamma^{i}_{mk}, \\
K_{ij} = K^{j}_{ik}, \quad K = g^{ij}K_{ij}.
\end{cases}
\]

If the group of holonomy \( H \) of the given space fixes a hyperquadric \( Q_{n-1} \) (we may not always assume that \( Q_{n-1} \) is non-degenerate), \( Q_{n-1} \) can be represented in the tangent projective space of each point \( A \) with respect to the natural frame by

\[
(1,5) \quad Q_{n-1} : g_{\lambda \mu}x^\lambda x^\mu = 0 \quad (\lambda, \mu = 0, 1, 2, \ldots, n),
\]

where \( g_{\lambda \mu} \) satisfies the relation\(^5\)


\(^6\) See Note 1), pp. 251–255.
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\[ \begin{align*}
\frac{\partial}{\partial x^k} G_{\alpha \gamma} - 2G_{\alpha \gamma} &= 0, \\
\frac{\partial}{\partial x^k} G_{\tau_\alpha} - \Pi_{\beta\gamma} G_{\rho} - G_{\gamma k} &= 0, \\
\frac{\partial}{\partial x^k} G_{ij} - \Pi_{\beta k} G_{\rho j} - \Pi_{\rho k} G_{ij} &= 0
\end{align*} \] (1,5)

and

\[ \begin{align*}
\Pi_{jk} &= \Gamma_{jk} + \delta_{j}^{i} \tau_{k} + \delta_{k}^{i} \tau_{j}, \\
\Pi_{jk} &= -\frac{1}{n-1} K_{jk} + \tau_{jk} - \tau_{j} \tau_{k}, \\
\tau &= -\frac{1}{2(n+1)} \log g, \quad \tau_{j} = -\frac{\partial \tau}{\partial x^{j}}, \\
\tau_{jk} &= -\frac{\partial \tau_{j}}{\partial x^{k}} - \Pi_{jk} \tau_{k}.
\end{align*} \] (1,6)

We have obtained (1,5) at the points where \( G_{\gamma} = 0 \), that is, at the points which do not belong to the surface of image \( F_{\alpha-1} \) of \( Q_{\alpha-1} \) into \( X_{\alpha} \). But, by virtue of the continuity of \( G_{\lambda} \), we may consider that (1,5) is satisfied at each point of \( X_{\alpha} \). Conversely, if (1,5) is integrable, the group of holonomy of \( X_{\alpha} \) fixes a hyperquadric \( Q_{\alpha-1} \).

Now, in order to represent (1,5) by the quantities of \( V_{\alpha} \), let us put

\[ G_{\alpha 0} = 2\phi, \quad G_{0 \alpha} = \phi \] (1,7)

and putting the relation and (1,6) into (1,5), we get

\[ \begin{align*}
\frac{\partial}{\partial x^{i}} G_{\phi} &= \phi, \\
\frac{\partial}{\partial x^{i}} G_{\phi i} &= \phi, \\
\frac{\partial}{\partial x^{i}} G_{ij} &= 2\phi_{j} G_{ij} + \phi_{j} G_{ik} + \phi_{i} G_{sj} \\
&\quad + \phi_{i} \left( -\frac{1}{n-1} K_{ik} + \tau_{i j} - \phi_{j} \phi_{j} \right) \\
&\quad + \phi_{j} \left( -\frac{1}{n-1} K_{ik} + \tau_{i j} - \phi_{i} \phi_{k} \right)
\end{align*} \] (1,5')

where a semicolon "\( ; \)" denotes the covariant differentiation of \( V_{\alpha} \).

Now, let us consider a change of coordinate system: \( (x) \rightarrow (\bar{x}) \), and \( R(A, A_{0}) \) and \( \bar{R}(\bar{A}, \bar{A}_{0}) \) be the natural frames in the coordinate
systems respectively. Putting
\[ \tilde{A}_\alpha = a_{\mu}^\alpha A_\mu, \quad a_\rho = 0, \quad a_0 = \rho, \]
\[ A_\mu = b_\mu^{\beta} \tilde{A}_\beta \quad (A_0 = \tilde{A}), \]
and
\[ dA_\lambda = \omega_\lambda^{\mu} A_\mu, \quad d\tilde{A}_\sigma = \tilde{\omega}_\sigma^{\beta} A_\beta, \]
where \( \omega_\lambda^{\mu} \) are different from those of (1,3), we get
\[ d\tilde{A} = d\rho A + \rho dx^i A_i, \]
\[ = d \log \rho \tilde{A} + \rho \frac{\partial x^i}{\partial \tilde{x}^j} d\tilde{x} A_i, \]
and hence
\[ a_j' = \rho \frac{\partial x^i}{\partial \tilde{x}^j}. \]
Accordingly, we get
\[ b_0' = \rho^{-1}, \quad b_i' = \rho^{-1} \frac{\partial \tilde{x}^j}{\partial x^i}, \quad b_0' = 0, \]
\[ b_i' = -\rho^{-2} \frac{\partial \tilde{x}^j}{\partial x^i} d_i', \]
and
\[ \tilde{\omega}_0' = d \log \rho - \rho^{-1} d_i' d\tilde{x}^j. \]
We get likewise the relation
\[ \tilde{\omega}_i' = d\omega_i b_\mu + \omega_i^{\lambda} \omega_\lambda^{\mu} b_\mu \]
\[ = nd \log \rho + d \left( \frac{\partial x^i}{\partial \tilde{x}^j} \right) \frac{\partial \tilde{x}^i}{\partial \tilde{x}^j} + \omega_i' - \rho d_i' dx^j, \]
and hence
\[ \tilde{\omega}_i' = nd \log \rho + \rho^{-1} d_i' d\tilde{x}^j + d \left( \frac{\partial x^i}{\partial \tilde{x}^j} \right) \frac{\partial \tilde{x}^i}{\partial \tilde{x}^j} \]
since \( \omega_i' = 0 \). Accordingly, from the relations above and that \( \tilde{R}(\tilde{A}, A) \)
is the natural frame, we obtain
\[ 0 = \tilde{\omega}_i' - n\tilde{\omega}_0' = (n + 1)\rho^{-1} d_i' d\tilde{x} + d \left( \frac{\partial x^i}{\partial \tilde{x}^j} \right) \frac{\partial \tilde{x}^i}{\partial \tilde{x}^j}, \]
from which we get the relation
\[ d_i' = -\frac{\rho}{n + 1} \frac{\partial}{\partial \tilde{x}^j} \log \left| \frac{\partial x^i}{\partial \tilde{x}^j} \right|. \]
Furthermore, putting $\omega_0^0 = 0$, we get

$$d \log \rho = \rho^{-1} d \omega^0 d \tilde{x}^1 = -\frac{1}{n+1} d \log \left| \frac{\partial x^i}{\partial \tilde{x}^j} \right|,$$

that is

$$\rho = \frac{\partial x^i}{\partial \tilde{x}^j}^{-\frac{1}{n+1}} \tag{1.8}$$

Now, in the coordinate system $(\tilde{x})$, we have

$$\bar{G}_{\alpha\beta} = a_{\alpha} a_{\beta} G_{\lambda\mu},$$

especially

$$\bar{G}_{00} = \rho^2 G_{00} = \frac{\partial x^i}{\partial \tilde{x}^j}^{-\frac{2}{n+1}} G_{\alpha\beta}.$$

Accordingly, $| g_{ij} |^{\frac{1}{n+1}} G_{00}$ is a scalar. Let us denote this by $y(x)$. Then, we get from (1.5') the relations

$$\frac{1}{2} G_{00} = \varphi = g^{\frac{1}{n+1}} y(x), \tag{1.9}$$

$$G_{0i} = \bar{p}_i = g^{\frac{1}{n+1}} (q_i + 2y_i); \tag{1.10}$$

and

$$g^{\frac{1}{n+1}} \bar{p}_{1i} = q_{1i,j} + 2y_{1i,j} + 2q_j r_i + 2r_j (q_i + 2y_i)$$

$$= (q_i + 2y_i) r_j + (q_j + 2y_j) r_i + 2y r_{(i,j)} - \frac{2y}{n-1} K_{ij} + g^{\frac{1}{n+1}} G_{ij},$$

where $g = | g_{ij} |$ and

$$q_i = \frac{\partial y}{\partial x^i}. \tag{1.11}$$

Now, if we define a tensor by

$$T_{ij} = g^{\frac{1}{n+1}} G_{ij} - q_i r_j - q_j r_i - 2y r_{ij}, \tag{1.12}$$

which is a symmetric covariant tensor of $\nabla_n$, then the last relation becomes

$$q_{1i,j} = -\frac{2y}{n-1} K_{ij} + T_{ij}. \tag{1.13}$$

Putting (1.10), (1.12) into the last relation of (1.5'), we get
\[ g^{ij} G_{ij} \equiv T_{ij;k} + q_{i;k} \tau_j + q_{j;k} \tau_i + \tau_i \tau_{jk;k} 
+ 2y(\tau_{i;k} + \tau_{j;k}) + 2q_k \tau_i \tau_j + 2g^{ij} \tau_k G_{ij} \]
\[ = 2g^{ij} \tau_k G_{ij} + \tau_i \{ T_{ik} + q_k \tau_i + \tau_k(q_i + 2y \tau_i) \} \]
\[ + \tau_j \{ T_{ik} + q_k \tau_i + \tau_k(q_i + 2y \tau_i) \} \]
\[ + (q_i + 2y \tau_i) \left(-\frac{1}{n-1} K_{kj} + \tau_{i;1} - \tau_{v;j} \right) \]
\[ + (q_j + 2y \tau_j) \left(-\frac{1}{n-1} K_{ik} + \tau_{i;1} - \tau_{i;k} \right) \]

that is
\[ T_{ij;k} = -\frac{1}{n-1} (q_i K_{kj} + q_j K_{ik}) \]
\[ - \tau_i (q_{j;k} + \frac{2y}{n-1} K_{kj} - T_{kj}) - \tau_j (q_{i;k} + \frac{2y}{n-1} K_{ik} - T_{ik}) \].

Hence, using the relation (1,13), the last one becomes
\[ T_{ij;k} = -\frac{1}{n-1} (q_i T_{kj} + q_j T_{ik}) \]

Thus, we see that the fundamental equations (1,5) characterising the space can be represented by means of the quantities of the Riemannian space \( V_n \) as follows:

\[ (1,11) \quad \frac{\partial y}{\partial x^i} = q_i, \]
\[ (1,13) \quad q_i \tau_j = -\frac{2y}{n-1} K_{ij} + T_{ij}, \quad (\alpha) \]
\[ (1,14) \quad T_{ij;k} = -\frac{1}{n-1} (q_i T_{kj} + q_j T_{ik}) \]

and \( G_{\alpha\gamma} \) are determined by \( y, q_i, T_{ij} \) so that

\[ (1,9) \quad G_{\alpha\gamma} = 2g^{-\frac{1}{n+1}} y, \]
\[ (1,10) \quad G_{\alpha\gamma} = G_{\alpha\gamma} = g^{-\frac{1}{n+1}} (q_i + 2y \tau_i), \quad (\beta) \]
\[ (1,12) \quad G_{ij} = g^{-\frac{1}{n+1}} (T_{ij} + q_i \tau_j + q_j \tau_i + 2y \tau_i \tau_j), \]

where
\[ \tau = -\frac{1}{2(n+1) \log g}, \quad \tau_i = \frac{\partial \tau}{\partial x^i}. \]
Consequently, we get the following

**Theorem 1.** In order that the group of holonomy of the space with a normal projective connexion corresponding to a given Riemannian space $V_n$ fixes a hyperquadric, that the system of equations (α) is integrable is necessary and sufficient.

§2. Relations between the spaces in which (α) is integrable and the Einstein spaces.

In this paragraph, we shall give a proof of the first theorem described in Induction.

For a given Riemannian space $V_n$, let the system of equations (α) be integrable. On the region of points where $y(x) \neq 0$, let us consider the following tensor

$$\bar{g}_{ij} = \frac{1}{2y} T_{ij} - \frac{1}{4y^3} q_i q_j.$$ (2.1)

Then, by means of (α), we get

$$\bar{g}_{ijk;\ell} = \frac{1}{2y} T_{ijk;\ell} - \frac{1}{2y^3} T_{ijk} q_\ell - \frac{1}{4y^3} (q_{i;\ell} q_j + q_{j;\ell} q_i)$$

$$+ \frac{1}{2y^3} q_i q_j q_\ell$$

$$= - \frac{1}{2(n-1)y} (q_i K_{ij} + q_I K_{jk}) - \frac{1}{2y^3} T_{ij} q_\ell$$

$$- \frac{1}{4y^3} \left( - \frac{2}{n-1} K_{ij} + T_{ij} \right) q_\ell - \frac{1}{4y^3} \left( - \frac{2}{n-1} K_{jk} + T_{jk} \right) q_i$$

$$+ \frac{1}{2y^3} q_i q_j q_\ell$$

$$= - \frac{1}{2y^3} T_{ij} q_\ell - \frac{1}{4y^3} T_{ik} q_j - \frac{1}{4y^3} T_{kj} q_i + \frac{1}{2y^3} q_i q_j q_\ell,$$

that is

$$\bar{g}_{ijk;\ell} = - \frac{1}{y} (q_\ell \bar{g}_{ij} + \frac{1}{2} q_\ell \bar{g}_{ij} + \frac{1}{2} q_i \bar{g}_{\ell j}).$$ (2.2)

Accordingly, if $|\bar{g}_{ij}| \neq 0$, the Riemannian space $\bar{V}_n$ with line element (not always positive definite)

$$ds^2 = \bar{g}_{ij} dx^i dx^j$$

is projective to $V_n$, in other words, Christoffel symbols $\bar{\Gamma}^i_{jk}$ made by $\bar{g}_{ij}$ satisfy the relation
\[ \tilde{r}_{jk} = r_{jk} + \delta_j^i \frac{\partial}{\partial x^i} \log y^{-\frac{1}{2}} + \delta_k^i \frac{\partial}{\partial x^i} \log y^{-\frac{1}{2}}. \]

For we get easily from (2,3)
\[
\tilde{g}_{ij,k} = \frac{\partial \tilde{g}_{ij}}{\partial x^k} - \tilde{g}_{kj,i} \tilde{r}_{ik} - \tilde{g}_{ik,j} \tilde{r}_{kj}.
\]

\[
= \tilde{g}_{kj,i} (\tilde{r}_{ik} - \tilde{r}_{ki}) + \tilde{g}_{ik,j} (\tilde{r}_{kj} - \tilde{r}_{jk})
\]

\[
= -\frac{1}{y} \left( q_i \tilde{g}_{ij} + \frac{1}{2} q_i \bar{g}_{ij} + \frac{1}{2} q_j \bar{g}_{ik} \right).
\]

which becomes (2,3)

Now, regarding the curvature tensor \( \bar{K}_{jk} \) of \( V_n \), we get easily
\[
\bar{K}_{jk} = K_{jk} + \left( \frac{1}{2y} \log y^{-\frac{1}{2}} \right) q_j \log y^{-\frac{1}{2}}.
\]

Accordingly, by contraction we get
\[
K_{jk} = K_{jk} - (n - 1) \left( \frac{1}{2y} \log y^{-\frac{1}{2}} \right) q_j \log y^{-\frac{1}{2}},
\]

and hence, by means of (a), we get
\[
K_{jk} = K_{jk} + \frac{n - 1}{2y} q_j \log y^{-\frac{1}{2}} - \frac{1}{4y^2} q_j q_k
\]

\[
= (n - 1) \left( \frac{1}{2y} T_{jk} - \frac{1}{4y^2} q_j q_k \right) = (n - 1) \bar{g}_{jk}.
\]

From the last relation we see that \( V_n \) is an Einstein space with non-vanishing curvature \( (n > 2) \) or a surface with constant curvature.

Conversely, if \( V_n \) is an Einstein space, then, since we have
\[
K_{ij} = \frac{K}{n} g_{ij},
\]

(a) becomes readily
\[
\frac{\partial y}{\partial x^i} = q^i,
\]

\[
q_{i,j} = 2y \left( \tilde{g}_{ij} - \frac{K}{n(n - 1)} g_{ij} + \frac{1}{2y} q_i q_j \right),
\]

\[
\tilde{g}_{ij,k} = -\frac{1}{y} \left( q_i \tilde{g}_{ij} + \frac{1}{2} q_i \bar{g}_{kj} + \frac{1}{2} q_j \bar{g}_{ik} \right).
\]
which are satisfied by
\[ y = \text{const.} \neq 0, \quad q_i = 0, \quad \bar{g}_{ij} = \frac{K}{n(n-1)} g_{ij}. \]

On the other hand, by means of the relations \((\beta)\) and \((2,1)\), we have
\[
\begin{vmatrix}
2y & q_j + 2y \tau_j \\
q_i + 2y \tau_i & T_{ij} + q_i \tau_j + q_j \tau_i + 2y \tau_i \tau_j
\end{vmatrix}
= g^{-1}
\begin{vmatrix}
2y & 0 \\
q_i + 2y \tau_i & T_{ij} - \frac{1}{2y} q_i q_j
\end{vmatrix}
= (2y)^{n+1} g^{-1} | \bar{g}_{ij} |.
\]

From the relation above, we see that the condition \(| \bar{g}_{ij} | \neq 0\) is equivalent to that the invariant hyperquadric \(Q_{n-1}\) is non-degenerate.

Accordingly, we obtain the following theorem.

**Theorem 2.** If the group of holonomy of the space with a normal projective connexion corresponding to a given Riemannian space \(V_n\) fixes a non-degenerate hyperquadric \(Q_{n-1}\), the space is projective to an Einstein space with non-vanishing curvature in the region of points which do not belong to the image of \(Q_{n-1}\) into \(V_n\). The converse is also true.

§3. The image of \(Q_{n-1}\).

Let \(F_{n-1}\) be the image surface of \(Q_{n-1}\) into \(V_n\), then it is given by the equation \(y = 0\). If \(y\) is not constant, we take a coordinate system \((x^1, x^2, \ldots, x^n)\) such that
\[
x^a = y, \quad g_{aa} = g^{bn} = 0
\]
(a, b, c \ldots = 1, 2, \ldots, n - 1)

and denote the Riemannian spaces given by the hypersurfaces \(F_{n-1} (y)\) on which \(y = \text{const.}\) by \(V_{n-1} (y)\) whose fundamental tensors are \(g_{ab}(x, y)\).

Furthermore, we denote Christoffel symbols of \(V_{n-1} (y)\) determined by \(g_{ab}(x; y)\) by \(\{^{a}_{bc}\}\) and the covariant differentiation with respect to \(\{^{a}_{bc}\}\) by a comma.

Now, if we put
\[
\sqrt{g_{aa}} = \psi(x, y),
\]
we can easily obtain the relation.
(3.3) \[ \frac{\partial}{\partial y} g_{ab} = -2\psi h_{ab} \]

where \( h_{ab} \) is the second fundamental tensor of \( F_{n-1}(y) \), and

\[
\begin{align*}
\Gamma^a_{bc} &= \{e^a, e_b, e_c\}, \\
\rho^a_{ab} &= \frac{1}{\psi} h_{ab}, \\
\Gamma^a_{bn} &= -\psi h^a_b, \\
\rho^a_{an} &= \frac{1}{\psi} \psi^c_{,a}, \\
\rho^a_{na} &= -\psi g^{ab} \psi^c_{,a}, \\
\rho^a_{nn} &= \frac{1}{\psi} \frac{\partial \psi}{\partial y}.
\end{align*}
\]

Furthermore, making use of Gauss-Codazzi equations\(^7\)

\[
\begin{cases}
K^a_{abcd} = R^a_{abcd} - h_{ab} h_{cd} + h_{ad} h_{bc}, \\
\psi K^a_{abc} = h_{ab,c} - h_{ac,b},
\end{cases}
\]

and the relation

\[
K^a_{bn} = \frac{1}{\psi} \frac{\partial h_{ab}}{\partial y} + h^a_{b} h_{bc} - \frac{1}{\psi} \psi^a_{,ab},
\]

we can obtain the relation

\[
\begin{cases}
K_{ab} = \frac{1}{\psi} \frac{\partial}{\partial y} h_{ab} - h h_{ab} + 2h^a h_{bc} + R_{ab} - \frac{1}{\psi} \psi^a_{,ab}, \\
K_{na} = \psi (h_{na} - h^a_{,b}), \\
K_{nn} = \psi \frac{\partial h}{\partial y} - \psi^2 h^a h^b - \psi g^{ab} \psi^c_{,a}, \\
h = h^a.
\end{cases}
\]

Now, since we have the relation

\[ q_a = 0, \quad q_n = 1 \]

in this coordinate system, we obtain by means of (3.4) the relation

\[ q_{a1} = -\Gamma^a_{ab} = -\frac{1}{\psi} h_{ab}. \]

Hence, by virtue of (3.6), (1,13) becomes

\[
-\frac{1}{\psi} h_{ab} = -\frac{2}{n-1} \left( \frac{1}{\psi} \frac{\partial}{\partial y} h_{ab} - h h_{ab} + 2h^a h_{bc} + R_{ab} - \frac{1}{\psi} \psi^a_{,ab} \right)
+ T_{ab},
\]

or

\[ 7 \) J. A. Schouten and D. J. Struik, Einführung in die neueren Methoden der Differentialgeometrie, 1935.
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\[(3.7) \quad \frac{\partial}{\partial y} h_{ab} = \frac{n-1}{2y} (h_{ab} + \psi T_{ab}) + \psi(h h_{ab} - 2h_{eb} h_{ae} - R_{ab}) + \psi_{,ab} \]

From the relation above, we get easily

\[(3.7') \quad \frac{\partial}{\partial y} h_{a} = \frac{n-1}{2y} (h_{a} + \psi T_{a}) + \psi(h h_{a} - R_{a}) + g^{bc} \psi_{,ac} \]

and by contraction

\[(3.8) \quad \frac{\partial}{\partial y} h = \frac{n-1}{2y} (h + \psi T) + \psi(h^2 - R) + g^{ab} \psi_{,ab} \]

We get likewise

\[q_{\alpha;\alpha} = - \Gamma_{\alpha\alpha} = - \frac{1}{\psi} \psi_{,\alpha} \]

\[= - \frac{2y}{n-1} (h_{,\alpha} - h_{\alpha,\alpha}) + T_{\alpha\alpha} \]

If we put

\[(3.9) \quad L_{\alpha} = \frac{1}{\psi} T_{\alpha\alpha}, \]

which is a covariant tensor of $V_{n-1}(y)$, then the relation above becomes

\[(3.10) \quad L_{\alpha} + \frac{1}{\psi^2} \psi_{,\alpha} - \frac{2y}{n-1} (h_{,\alpha} - h_{\alpha,\alpha}) = 0. \]

Lastly, putting (3.8) into the relation

\[q_{\alpha;\alpha} = - \Gamma_{\alpha\alpha} = - \frac{1}{\psi} \frac{\partial \psi}{\partial y} \]

\[= - \frac{2y}{n-1} (\psi \frac{\partial h}{\partial y} - \psi^3 h_{b} h_{b} - \psi g^{ab} \psi_{,ab}) + T_{\alpha\alpha}, \]

we get

\[\frac{1}{\psi} \frac{\partial \psi}{\partial y} = \psi(h + \psi T) + \frac{2y}{n-1} (h^3 - h_{e} h_{b} - R) - T_{\alpha\alpha}. \]

Introducing a scalar of $V_{n-1}(y)$ such that

\[(3.11) \quad S = \frac{1}{\psi^3} - T_{\alpha\alpha}, \]

the relation above becomes

\[(3.12) \quad \frac{\partial \psi}{\partial y} = \psi^3 h + \psi^3 (T - S) + \frac{2y}{n-1} (h^3 - h_{e} h_{b} - R). \]
Now, let us represent (1,14) by means of the quantities of $V_{n-1}(y)$.

We get by (3,4) the relation

$$T_{an1n} = \frac{\partial T_{an}}{\partial y} - \Gamma_{an}^i T_{in} - \Gamma_{bn}^i T_{ni}$$

$$= \psi \frac{\partial L_a}{\partial y} + \psi h_a^i T_{eb} + \psi h_b^i T_{ac} - \psi_{,a} L_b - \psi_{,b} L_a$$

$$= - \frac{1}{n-1} (q_a T_{bn} + q_b T_{an}) = 0,$$

or

$$(3.13) \quad \frac{\partial T_{an}}{\partial y} = - \psi (h_a^i T_{eb} + h_b^i T_{ac}) + \psi_{,a} L_b + \psi_{,b} L_a,$$

from which we get by (3,3) the relation

$$\frac{\partial T}{\partial y} = 2 \psi_a L^a$$

Furthermore, we get by (3,4), (3,6) the relation

$$T_{an1n} = \frac{\partial T_{an}}{\partial y} - \Gamma_{an}^i T_{in} - \Gamma_{bn}^i T_{ni}$$

$$= \psi \frac{\partial L_a}{\partial y} + \psi h_a^i L_b - \psi_{,a} S + \psi g_{ae} \psi_{,a} T_{ab}$$

$$= - \frac{1}{n-1} K_{aa} = - \frac{\psi}{n-1} (h_a - h_a^b),$$

that is

$$(3.14) \quad \frac{\partial L_a}{\partial y} = - \psi h_a^b L_b - \psi_{,a} T_{a} + \psi_{,a} S - \frac{1}{n-1} (h_a - h_a^b).$$

We get likewise by (3,8) the relation

$$T_{an1n} = \frac{\partial T_{an}}{\partial y} - 2 \Gamma_{an}^i T_{ni} = \psi \frac{\partial S}{\partial y} + 2 \psi \psi_{,a} L^a$$

$$= - \frac{2 \psi}{n-1} \left\{ \frac{n-1}{2y} (h + \psi T) + \psi (h_a - h_a^b T_a) - R \right\},$$

that is

$$(3.15) \quad \frac{\partial S}{\partial y} = - 2 \psi_{,a} L^a - \frac{1}{y \psi} (h + \psi T) - \frac{2}{n-1} (h_a^b - h_a^b T_a - R).$$

We get readily the following relations.
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\[ T_{ab;e} = T_{a;e} - h_{ac}L_b - h_{cb}L_a \]
\[ = - \frac{1}{n-1} (q_a K_{eb} + q_b K_{ae}) = 0, \]

\[ T_{an;1} = \psi L_{a,b} - \psi h_{ab}S + \psi h_{bc}T_{ac} \]
\[ = - \frac{1}{n-1} \left( \frac{1}{\psi} \frac{\partial h_{ab}}{\partial y} - h_{ab} + \frac{1}{\psi} \frac{\partial}{\partial y} (h_{ab} + \psi T_{ab}) \right), \]

whose last side becomes by means of (3,7)
\[ = - \frac{1}{2y\psi} (h_{ab} + \psi T_{ab}), \]
hence we have

(3,16) \[ T_{ab;e} - L_a h_{be} - L_b h_{ae} = 0, \]

(3,17) \[ L_a,n + T_{a} h_{be} - h_{ab}S + \frac{1}{2y\psi} (h_{ab} + \psi T_{ab}) = 0, \]

Lastly, we get by (3,4), (3,6) the relation

\[ T_{n;i1} = \psi S_{a} + 2\psi h_{ab}L_a \]
\[ = - \frac{2}{n-1} K_{aa} = - \frac{2\psi}{n-1} (h_{a} - h_{a,b}), \]

that is

(3,18) \[ S_{a} + 2h_{ab}L_a + \frac{2}{(n-1)\psi} (h_{a} - h_{a,b}) = 0. \]

Hence, if we replace \( n \) with \( n + 1 \), we obtain the following

Theorem 3. In order that we can imbed a given Riemannian space \( V_n \) with line element

\[ ds^2 = g_{\lambda}(x)dx^\lambda dx^\mu \]

into a Riemannian space \( V_{n+1} \) as a hypersurface so that the group of holonomy of the space \( X_{n+1} \) with a normal projective connexion corresponding to \( V_{n+1} \) fixes a hyperquadric \( Q_n \) and the image of \( Q_n \) into \( V_{n+1} \) is the hypersurface, a necessary and sufficient condition is that the following system of equations with respect to \( g_{\lambda \mu}, h_{\lambda \mu}, \psi, L_a, T_{ab} \), \( S \)

8) From now on we assume that the indices take the following values:

\[ a, b, c, d, \ldots, \lambda, \mu, \nu, \rho, \ldots = 1, 2, \ldots, n. \]
\( \frac{\partial g_{ab}}{\partial y} = -2\psi h_{ab} \),

\( \frac{\partial h_{ab}}{\partial y} = \frac{n}{2y} (h_{ab} + \psi T_{ab}) + \psi(h h_{ab} - 2h_a^b h_{b\lambda} - R_{ab}) + \psi_{,ab} , \)

\( \frac{\partial \psi}{\partial y} = \psi^3 h + \psi^3 (T - S) + \frac{2y\psi^3}{n} (h^2 - h_a^b h_{b\lambda} - R) , \)

\( \frac{\partial T_{ab}}{\partial y} = -\psi(h a^b T_{\lambda b} + h_b T_{ab}) + \psi_{,a} L_a + \psi_{,b} L_b , \)

\( \frac{\partial L_a}{\partial y} = -\psi h_{a}^b L_{\lambda} - T_{a}^b \psi_{,\lambda} + \psi_{,a} S - \frac{1}{n} (h_{a}, - h_{a}^b ,) , \)

\( \frac{\partial S}{\partial y} = -2\psi_{,\lambda} L_{\lambda} - \frac{1}{y\psi} (h + \psi T) - \frac{2}{n} (h^2 - h_a^b h_{b\lambda} - R) \)

is integrable under the conditions

\( \frac{L_a}{\partial y} \psi^3 - \frac{2y}{n} (h_{a} - h_{a}^b ,) = 0, \)

\( L_{a,b} + T_{a}^b h_{\lambda} - h_{ab} S + \frac{1}{2y\psi^3} (h_{ab} + \psi T_{ab}) = 0, \)

\( T_{ab,c} - L_a h_{bc} - L_b h_{ae} = 0, \)

\( S_{a} + 2h_{a}^b L_{\lambda} + \frac{2}{n\psi} (h_{a} - h_{a}^b ,) = 0 \)

and under the initial condition

\[ [g_{ab}(x, y)]_{y=0} = g_{ab}(x). \]

Then, in the coordinate neighborhood \( x^1, \ldots, x^n, y \), the line element of \( V_{n+1} \) is given by

\[ ds^2 = g_{ab}(x, y) dx^a dx^b + (\psi(x, y) dy)^2. \]

**§4. Invariant hyperquadric and Campbell's theorem.**

In this paragraph, we shall investigate the problem to imbed a given \( V_n \) in an Einstein space \( A_{n+1} \) so that the relation between \( V_n \) and \( A_{n+1} \) is the one stated in Theorem 3.

Now, if a Riemannian space \( V_{n+1} \) with line element

\[ ds^2 = g_{ab}(x, y) dx^a dx^b + (\psi(x, y) dy)^2 \]

is an Einstein space with scalar curvature \((n + 1)k\), the following relations hold good:
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\[
\frac{\partial g_{ab}}{\partial y} = -2\psi h_{ab},
\]

(4.1) \( \frac{\partial h_{ab}}{\partial y} = k\psi g_{ab} + \psi(hh_{ab} - 2h^a\mathbf{h}\mathbf{b} - R_{ab}) + \psi_{,ab}, \)

(4.2) \( V_a \equiv h_{,a} - h^b_{,a} = 0, \)

(4.3) \( z \equiv (n - 1)k + h^2 - h^a h^b - R = 0. \)

The converse is also true. The proof is easy by means of (3.6).

Accordingly, in order that \( V_{n+1} \) in Theorem 3 is an Einstein space, besides (I), (II), the above relations (4.1), (4.2), (4.3) are necessary and sufficient. Therefore we shall replace these relations by other ones such that we can easily treat our problem.

From (L) and (4.1) we obtain

(4.4) \[ T_{ab} = -\frac{1}{\psi^2} h_{ab} + \frac{2yk}{n} g_{ab}, \]

\[ T = -\frac{h}{\psi^2} + 2yk \]

and from (II, ) and (4.2) we obtain the relation

(4.5) \[ L_a = -\frac{1}{\psi^2} \psi_{,a} = \left( \frac{1}{\psi^2} \right)_a. \]

Conversely, if \( L_a = \left( \frac{1}{\psi^2} \right)_{,a} \) and (II, ) hold, then we get

\[ yV_a = 0. \]

from which we get \( V_a = 0 \) when \( V_a \) is continuous at \( y = 0. \)

Then, putting (4.4) into (L) and using (4.3), we get

\[ \frac{\partial \psi}{\partial y} = \left( 2yk - S + \frac{2yQ}{n} \right)\psi^3 = -\left( S - \frac{2yk}{n} \right)\psi^3, \]

where

(4.6) \[ Q_{ab} = hh_{ab} - h^a h^b - R_{ab}, \quad Q = g^{\mu\nu} Q_{\mu\nu}. \]

By (I, ), (L), (L), (4.4), (4.5), we obtain the relation

\[ \frac{\partial T_{ab}}{\partial y} = -\frac{1}{\psi^2} \frac{\partial h_{ab}}{\partial y} + \frac{h_{ab}}{\psi^2} \frac{\partial \psi}{\partial y} + \frac{2k}{n} g_{ab} - \frac{4yk\psi}{n} h_{ab} = -kg_{ab} - (Q_{ab} - h^a h^b) - \frac{1}{\psi^2} \psi_{,ab}. \]
that is

\[
\left(1 - \frac{2}{n}\right)k g_{ab} + \frac{1}{\psi^2} \psi_{ab} - \frac{2}{\psi^2} \psi_{a} \psi_{b} - R_{ab} + h_{ab} \left\{ R + \psi S - \frac{2}{n} \psi \left(k + \frac{Q}{n}\right) \right\} = 0.
\]

Furthermore, by (I) and (4.2) we get

\[
\frac{\partial}{\partial y^i} L_\alpha = \left\{ S - 2y \left(k + \frac{Q}{n}\right) \right\} \psi_{\alpha} + \psi \left(S_{,\alpha} - 2y k_{,\alpha} \right) \]

that is

\[
S_{,\alpha} - \frac{2}{n} Q_{,\alpha} - \frac{2}{\psi^2} h_{\alpha} \psi_{\alpha} = 0.
\]

By means of (I) and (4.3), we get

\[
\frac{\partial S}{\partial y} = \frac{2}{\psi^2} g^{\alpha \mu} \psi_{,\alpha} \psi_{,\mu} - \frac{2k}{n}
\]

and from (II) we get

\[
- \frac{1}{\psi^2} \psi_{,ab} + \frac{2}{\psi^2} \psi_{,a} \psi_{,b} + \left( - \frac{1}{\psi^2} h_{\alpha} \psi_{\alpha} + \frac{2y k_{,\alpha}}{n} \delta_{\alpha} \right) h_{ab} - h_{ab} S + \frac{k}{n \psi^2} g_{ab} = 0.
\]

Lastly, from (II) and (III) we obtain the relations

\[
h_{ab,c} = \frac{1}{\psi^c} (\psi_{,c} h_{ab} + \psi_{,a} h_{cb} + \psi_{,b} h_{ac}),
\]

\[
S_{,\alpha} - \frac{2}{\psi^2} h_{\alpha} \psi_{\alpha} = 0.
\]

Hence we obtain the following

**Theorem 4.** A necessary and sufficient condition in order that we can imbed a given Riemannian space \( V_n \) with line element
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\[ ds^2 = g_{ab}(x)dx^a dx^b \]

into an Einstein space \( A_{n+1} \) so that the space \( A_{n+1} \) has the property of \( V_{n+1} \) in Theorem 3 is that the following system of equations

\[
(\text{III}_1) \quad - \frac{\partial g_{ab}}{\partial y} = -2\psi h_{ab} ,
\]

\[
(\text{III}_2) \quad - \frac{\partial h_{ab}}{\partial y} = \psi(kg_{ab} + hh_{ab} - 2h^a_h h_{b\lambda} - R_{ab}) + \psi_{,ab} ,
\]

\[
(\text{III}_3) \quad \frac{\partial \psi}{\partial y} = -\left(S - \frac{2yk}{n}\right)\psi^\lambda ,
\]

\[
(\text{III}_4) \quad \frac{\partial S}{\partial y} = 2\frac{\psi^\lambda}{\psi} g^{\lambda\mu} \psi_{,\lambda} \psi_{,\mu} - \frac{2}{n} k
\]

is integrable under the conditions

\[
(\text{IV}_1) \quad \xi_{ab} = \frac{1}{\psi^r} \psi_{,ab} - \frac{2}{\psi^r} \psi_{,a} \psi_{,b} - \frac{k}{n} g_{ab} + h^a_h h_{b\lambda} + \psi h_{ab} \left(S - \frac{2yk}{n}\right) = 0 ,
\]

\[
(\text{IV}_2) \quad \eta_{ab} = \left(1 - \frac{1}{n}\right)kg_{ab} + hh_{ab} - h^a_h h_{b\lambda} - R_{ab} = 0 ,
\]

\[
(\text{IV}_3) \quad \zeta_{abc} = h_{abc} - \frac{1}{\psi^r} (\psi_{,a} h_{bc} + \psi_{,b} h_{ca} + \psi_{,c} h_{ab}) = 0 ,
\]

\[
(\text{IV}_4) \quad \sigma_a = S_a - \frac{2}{\psi^r} h^a_h \psi_{,\lambda} = 0
\]

and under the initial condition

\[ [g_{ab}(x, y)]_{y=0} = g_{ab}(x) . \]

The proof is evident from the computation above and the relation

\[ z = g^{\lambda\mu} \eta_{\lambda\mu} , \quad V_a = \zeta_{\lambda\lambda} - \zeta_{a\lambda} . \]

§5. Some properties of \( \xi_{ab}, \eta_{ab}, \zeta_{abc} \) and \( \sigma_a \).

In this paragraph, we shall investigate properties of the tensors \( \xi_{ab}, \eta_{ab}, \zeta_{abc} \) and the vector \( \sigma_a \) made by any solutions \( g_{ab}, h_{ab}, \psi, S \) of the system of equations (III).

Let us denote the Riemannian spaces with line elements

\[ ds^2 = g_{\lambda\mu}(x, y)dx^\lambda dx^\mu \]
by $V_a(y)$. Then, we get by (III)
\[
\frac{\partial R_{bc}^a}{\partial y} = g^{\alpha \lambda}(\psi h_{bc})_{,\lambda} - (\psi h_{bc})_{,e} - (\psi h_{bc})_{,b},
\]
which becomes by means of (IV)
\[
\frac{\partial R_{bc}^a}{\partial y} = \psi(\zeta_{bc}^\alpha - \zeta_{bc}^\alpha_{,e} - \zeta_{bc}^\alpha_{,b}) - 2(\psi_{,b} h_{bc} + \psi_{,c} h_{bc}).
\]
We get likewise by means of (IV) the relation
\[
S_{a,b} = \sigma_{a,b} - \frac{4}{\psi^{\alpha}} \psi_{,b} h_{a}^\lambda \psi_{,\lambda} + \frac{2}{\psi^{\alpha}} (h_{a}^\lambda \psi_{,\lambda} + h_{b}^\lambda \psi_{,\lambda})
\]
which becomes by means of (IV)
\[
S_{a,b} = \sigma_{a,b} + \frac{2}{\psi^{\alpha}} \zeta_{a,b}^\lambda \psi_{,\lambda}
\]
from which we get easily
\[
\frac{\partial h_{ab}}{\partial y} = \psi(k - \psi_{,b} h_{a}^\lambda \psi_{,\lambda} + \eta_{ab}),
\]
from which we get easily
\[
\frac{\partial h}{\partial y} = \psi(k - \psi_{,b} h_{a}^\lambda \psi_{,\lambda} + \eta_{ab}) + g^{\alpha \mu} \psi_{,\mu}.
\]
Then, we obtain from (IV)
\[
\frac{\partial \xi_{ab}}{\partial y} = -\psi(S - \frac{2yk}{n}) \left\{ \psi_{,ab} - \frac{2}{\psi} \psi_{,a} \psi_{,b} + \psi(S - \frac{2yk}{n}) h_{ab} \right\}
\]
and hence by means of (IV), (5,1), (5,2)
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\[ -\psi^2 \left( S - \frac{2yK}{n} \right) (\xi_{ab} - \gamma_{ab}) \]

\[ - \psi \psi_{a\beta} \left( \sigma_a + \frac{2}{\psi^x} \kappa_a^{\lambda} \lambda^{;\lambda} \right) - \psi \psi_{,\beta} \left( \sigma_\beta + \frac{2}{\psi^x} \kappa_\beta^{;\beta} \lambda^{;\lambda} \right) \]

\[ - \psi^2 \left[ \sigma_{a,b} + \sigma_{b,a} + \frac{2}{\psi^x} (\xi_{a,\lambda} + \xi_{b,\lambda}) \psi_{,\lambda} \right. \]

\[ + \psi \left( h_{a,b}^{\lambda} + h_{b,a}^{\lambda} \right) \psi_{,\lambda} \mu \]

\[ - \psi_{b,a} \left( \xi_{ab}^{;\lambda} - \xi_{a,b}^{;\lambda} - \psi_{a,b}^{;\lambda} \right) - \psi_{b,a} \xi_{ab}^{;\lambda} \]

\[ + \frac{2}{\psi^x} (h_{a,b}^{;\lambda} + h_{b,a}^{;\lambda}) \]

that is

\[ (5,3) \]

\[ \frac{\partial \xi_{ab}}{\partial y} = -\psi^2 \left( S - \frac{2yK}{n} \right) (\xi_{ab} - \gamma_{ab}) - \psi (\psi_{a} \sigma_{b} + \psi_{b} \sigma_{a}) \]

\[ - \frac{\psi^2}{2} (\sigma_{a,b} + \sigma_{b,a}) - \psi_{b,a} \xi_{ab}^{;\lambda} + \psi \left( h_{a,b}^{;\lambda} + h_{b,a}^{;\lambda} \right). \]

Now, we have generally the relation

\[ \frac{\partial R_{ab}}{\partial y} = \frac{\partial}{\partial y} \left( \frac{\partial R_{ab}}{\partial x^\lambda} - \frac{\partial R_{ab}}{\partial x^\lambda} + \frac{\partial R_{ab}}{\partial x^\lambda} - \frac{\partial R_{ab}}{\partial x^\lambda} \right) \]

\[ = \left( \frac{\partial}{\partial y} \right)_{,\lambda} R_{ab}^{;\lambda} - \left( \frac{\partial}{\partial y} \right)_{,b} R_{ab}^{;\lambda} \]

\[ = \left( \frac{\partial}{\partial y} \right)_{,\lambda} R_{ab}^{;\lambda} - \frac{1}{2} \left( \frac{\partial}{\partial y} \right)_{,h} R_{ab}^{;\lambda} - \frac{1}{2} \left( \frac{\partial}{\partial y} \right)_{,h} R_{ab}^{;\lambda}. \]

Putting (5,1) into the relation, it becomes

\[ \frac{\partial R_{ab}}{\partial y} = \{ \psi (\xi^a_{ab} - \xi^a_{b,a} - \xi^a_{a,b}) - 2(\psi_{a} h_{b}^{;\lambda} + \psi_{b} h_{a}^{;\lambda}) \} \]

\[ + \left\{ \frac{\psi}{2} \xi^a_{a,b} + \psi_{a} h + \psi_{b} h_{a}^{;\lambda} \right\} + \left\{ \frac{\psi}{2} \xi^a_{b,a} + \psi_{b} h + \psi_{a} h_{b}^{;\lambda} \right\} \]

\[ = \psi (\xi^a_{ab} - \xi^a_{b,a} - \xi^a_{a,b}) + \psi_{a} (\xi^a_{ab} - \xi^a_{b,a} - \xi^a_{a,b}) \]

\[ - \psi_{a} h_{b}^{;\lambda} - \psi_{b} h_{a}^{;\lambda} + 2 \psi_{a} \psi_{b} h \]

\[ - 2 \psi_{a} (\xi^a_{b,a} + \frac{1}{\psi^x} \psi_{a} h + \frac{2}{\psi^x} \psi_{b} h_{a}^{;\lambda}) \]

\[ - 2 \psi_{a} (\xi^a_{a,b} + \frac{1}{\psi^x} \psi_{a} h + \frac{2}{\psi^x} \psi_{b} h_{a}^{;\lambda}) \]

\[ + \frac{\psi}{2} (\xi^a_{a,b} + \xi^a_{b,a}) + \frac{1}{2} (\psi_{a} \xi^a_{b,a} + \psi_{b} \xi^a_{b,a}) \]

\[ + \frac{\psi}{2} (\xi^a_{a,b} + \xi^a_{b,a}) + \frac{1}{2} (\psi_{a} \xi^a_{b,a} + \psi_{b} \xi^a_{b,a}) \]
that is

\[
\frac{\partial R_{ab}}{\partial y} = \psi(\zeta^a_{\lambda} - \zeta^b_{\lambda} - \zeta^a_{\lambda} - \zeta^b_{\lambda}) + \frac{\psi}{2} \left( \zeta^a_{\lambda} + \zeta^b_{\lambda} \right)
\]

(5.4)

Then, by means of (IIIa), (IIIc), (5.4) we get the relation

\[
\frac{\partial \gamma_{ab}}{\partial y} = -2 \left( 1 - \frac{1}{n} \right) \psi k h_{ab} + \psi \left( h^a_{\lambda} h^b_{\lambda} + \frac{1}{\psi} g^{\lambda\mu} \psi_{,\lambda} \right) h_{ab}
\]

\[
+ \psi \left( \frac{k}{n} g_{ab} - h^a_{\lambda} h^b_{\lambda} + \frac{1}{\psi} \psi_{,ab} \right) h_{ab}
\]

\[
- \psi \left( \frac{k}{n} g_{ab} + \frac{1}{\psi} \psi_{,ab} \right) h_{ab} - \psi \left( \frac{k}{n} g_{ab} + \frac{1}{\psi} \psi_{,ab} \right) h^a_{\lambda}
\]

\[
- \frac{\partial R_{ab}}{\partial y}
\]

\[
= \psi h_{ab} \left( \frac{1}{\psi} g^{\lambda\mu} \psi_{,\lambda} \mu - 2 \psi^2 g^{\lambda\mu} \psi_{,\lambda} \psi_{,\mu} - k + h^a_{\lambda} h^b_{\lambda} + \psi^2 \right)
\]

\[
- \psi \left( \frac{1}{\psi} \psi_{,ab} - 2 \psi^2 \psi_{,a} \psi_{,b} - \frac{k}{n} g_{ab} + h^a_{\lambda} h^b_{\lambda} - \psi_{,ab} \right)
\]

\[
- \psi (h^a_{\lambda} \gamma_{ab} + h^b_{\lambda} \gamma_{ab}) - \psi (\zeta_{ab\lambda} - \zeta_{a\lambda} - \zeta_{b\lambda})
\]

\[
- \frac{\psi}{2} (\zeta^a_{\lambda} + \zeta^b_{\lambda})
\]

\[
- \psi_{,a} \left( \frac{3}{2} \zeta^a_{\lambda} - 2 \zeta^b_{\lambda} \right) - \psi_{,b} \left( \frac{3}{2} \zeta^b_{\lambda} - 2 \zeta^a_{\lambda} \right),
\]

that is

\[
\frac{\partial \gamma_{ab}}{\partial y} = \psi \left( h^a_{\lambda} + \frac{1}{\psi} \psi_{,a} \right) - \psi h (\xi_{ab} - \gamma_{ab})
\]

(5.5)

\[
- \psi (h^a_{\lambda} \gamma_{ab} + h^b_{\lambda} \gamma_{ab}) - \psi (\zeta_{ab\lambda} - \zeta^a_{\lambda} - \zeta_{b\lambda} - \zeta_{a\lambda})
\]
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\[ - \frac{\psi}{2} (\zeta^{\lambda}_{a,b} + \zeta^{\lambda}_{b,a}) - \psi^{\lambda}_{\lambda} \zeta^{\lambda}_{a,b} \]

\[ - \psi^{\lambda}_{a} \left( \frac{3}{2} \zeta^{\lambda}_{b} - 2 \zeta^{\lambda}_{b} \right) - \psi^{\lambda}_{b} \left( \frac{3}{2} \zeta^{\lambda}_{a} - 2 \zeta^{\lambda}_{a} \right). \]

Now, using (IV), we have

\[ \psi_{abc} = (\psi^{\lambda}_{ab})_{,c} + \frac{2}{\psi} (\psi^{\lambda}_{a} \psi_{bc} + \psi^{\lambda}_{b} \psi_{ac}) - \frac{2}{\psi^{2}} \psi^{\lambda}_{a} \psi^{\lambda}_{b} \psi_{c} \]

\[ + \frac{k}{n} g_{ab} \psi_{c} - h_{a}^{\lambda} h_{b}^{\lambda} \psi_{c} - 2\psi (S - \frac{2yk}{n}) h_{ab} \psi_{c} \]

\[ - (\psi^{\lambda}_{a} \psi_{b}^{\lambda} + \psi_{a}^{\lambda} h_{b}^{\lambda} + \psi_{b}^{\lambda} h_{a}^{\lambda} + g_{ab} \psi_{c} h_{ad}) h_{ba} \]

\[ - (\psi^{\lambda}_{b} \psi_{a}^{\lambda} + \psi_{b}^{\lambda} h_{a}^{\lambda} + \psi_{a}^{\lambda} h_{b}^{\lambda} + \psi_{a}^{\lambda} h_{bd}) h_{ba} \]

\[ - \psi^{\lambda}_{ab} \left( \sigma_{c} + \frac{2}{\psi^{2}} h_{\lambda}^{\psi} \psi_{\lambda} \right) \]

\[ - \psi (S - \frac{2yk}{n}) (\psi^{\lambda}_{a} \psi_{b} + \psi^{\lambda}_{b} \psi_{a} + \psi_{a} h_{ac} + \psi_{b} h_{ac}) \]

Hence we have by the relation above

\[ \frac{\partial \zeta_{abc}}{\partial y} = \left( \frac{\partial h_{ab}}{\partial y} \right)_{,c} - h_{ab} \frac{\partial \zeta^{\lambda}_{ac}}{\partial y} - h_{ac} \frac{\partial \zeta^{\lambda}_{bc}}{\partial y} \]

\[ - \frac{\partial}{\partial y} \left\{ \frac{1}{\psi} \left( \psi_{a}^{\lambda} h_{bc} + \psi_{b}^{\lambda} h_{ac} + \psi_{c}^{\lambda} h_{ab} \right) \right\} \]

\[ = \psi_{b} \left( \frac{k}{n} g_{ab} - h_{a}^{\lambda} h_{b}^{\lambda} + \psi_{b} \right) \]

\[ - (\psi^{\lambda}_{a} \psi_{b}^{\lambda} + \psi_{a}^{\lambda} h_{b}^{\lambda} + \psi_{b}^{\lambda} h_{a}^{\lambda} + g_{ab} \psi_{c} h_{ad}) h_{ba} \]

\[ - (\psi^{\lambda}_{b} \psi_{a}^{\lambda} + \psi_{b}^{\lambda} h_{a}^{\lambda} + \psi_{a}^{\lambda} h_{b}^{\lambda} + \psi_{a}^{\lambda} h_{bd}) h_{ba} \]

\[ + \psi^{\lambda}_{ab} \left( \sigma_{c} + \frac{2}{\psi^{2}} h_{\lambda}^{\psi} \psi_{\lambda} \right) \]

\[ - h_{ac} \left\{ \psi \left( \zeta^{\lambda}_{b} - \zeta_{b}^{\lambda} - \zeta_{c}^{\lambda} \right) - 2 \left( \psi_{a}^{\lambda} h_{c}^{\lambda} + \psi_{b}^{\lambda} h_{c}^{\lambda} \right) \right\} \]

\[ - h_{ac} \left\{ \psi \left( \zeta_{b}^{\lambda} - \zeta_{b}^{\lambda} - \zeta_{c}^{\lambda} \right) - 2 \left( \psi_{a}^{\lambda} h_{c}^{\lambda} + \psi_{b}^{\lambda} h_{c}^{\lambda} \right) \right\} \]

\[ + \sum_{a,b,c,d} \left[ h_{ab} \left\{ \psi \left( S - \frac{2yk}{n} \right) + \psi^{\lambda}_{c} \sigma_{c} + 2 h_{c}^{\psi} \psi_{\lambda} \right\} \right. \]

\[ \left. - \psi_{c} \left( \frac{k}{n} g_{ab} - h_{a}^{\lambda} h_{b}^{\lambda} + \psi_{ab} + \frac{1}{\psi} \psi_{ab} \right) \right\} \]

\[ = - \psi \left( h_{ac} \left( \zeta_{b}^{\lambda} + \zeta_{b}^{\lambda} - \zeta_{c}^{\lambda} \right) - \psi h_{bc} \left( \zeta_{a}^{\lambda} + \zeta_{a}^{\lambda} - \zeta_{c}^{\lambda} \right) \right) \]

\[ - \psi \left( \frac{2}{n} yk \right) \zeta_{abc} + (\psi \xi_{ab} + \psi \eta_{ab}) \xi_{c} \]

\[ + \psi^{\lambda}_{ab} \sigma_{c} + h_{bc} \sigma_{c} \]

\[ - \psi \left( \frac{1}{\psi} \psi_{c} \psi_{ab} - \frac{2}{\psi^{2}} \psi_{c} \psi_{a} \psi_{b} - \frac{k}{n} g_{ab} \right) \]

\[ + h_{c} h_{bc} + \psi h_{ac} \left( S - \frac{2}{n} yk \right) + \eta_{ab} \)}
Lastly, regarding \( \sigma_a \) we get by (IV.) relation

\[
\frac{\partial \sigma_a}{\partial y} = \left( \frac{\partial S}{\partial y} \right)_{\sigma_a} = \frac{1}{\psi} \frac{\partial h_{ab}}{\partial y} + 2h_{\lambda} \left( \frac{\partial}{\partial y} \frac{1}{\psi} \right)_{\lambda} \\
= 4 \frac{\psi}{\psi} \frac{g^\lambda}{\psi} \frac{\partial h_{ab}}{\partial y} + 2h_{\lambda} \left( \frac{\partial}{\partial y} \frac{1}{\psi} \right)_{\lambda} \\
- 2 \frac{\psi}{\psi} \frac{h_{ab}}{\partial y} \left( \frac{k}{n} \delta^\lambda + h^\lambda h^\mu + \nu^\lambda + \frac{1}{\nu} \frac{g^\mu}{\psi} \frac{\partial}{\partial y} \frac{1}{\psi} \right) \\
+ 2h_{\lambda} \left( \frac{S - 2}{n} yk \right) h_{ab} + \psi \left( \sigma + \frac{2}{n} h^\lambda h_{\lambda} \right) \\
= 2\psi h_{\lambda} \sigma_{\lambda} \\
+ 2 \frac{\psi}{\psi} \frac{g^\lambda}{\psi} \frac{\partial h_{ab}}{\partial y} \left( \frac{1}{\psi} \frac{\partial h_{ab}}{\partial y} - \frac{k}{n} g_{\lambda a} \right) \\
+ h_{\lambda} h_{\mu a} \psi \left( \frac{S - 2}{n} yk \right) h_{\lambda a} - \nu_{ab} \right),
\]

that is

\[
(5,7) \quad \frac{\partial \sigma_a}{\partial y} = 2\psi h_{\lambda} \sigma_{\lambda} + 2 \frac{\psi}{\psi} \frac{g^\lambda}{\psi} \frac{\partial \xi_{ab}}{\partial y} \left( \psi_{ab} - \nu_{ab} \right)
\]

Thus we see that \( \xi_{ab}, \nu_{ab}, \zeta_{abc}, \sigma_a \) made by any solutions of the system of equations (III) satisfy a system of equations (5,3), (5,5), (5,6), (5,7) linear with respect to these quantities and their derivatives. Therefore, if we have at \( y = 0 \)

\[
\xi_{ab} = 0, \quad \nu_{ab} = 0, \quad \zeta_{abc} = 0, \quad \sigma_a = 0,
\]

the relation holds good in a proper neighborhood of \( y = 0 \). Hence we obtain a more exactly theorem as follows:
Theorem 5. In order that we can imbed a given Riemannian space \( V_n \) with line element
\[
d s^2 = g_{\alpha\beta}(x)dx^\alpha dx^\beta
\]
into an Einstein space in the sense as stated in Theorem 4, a necessary and sufficient condition is that the following equations with respect to \( h_{ab} (= h_{bd}) \), \( \psi \), \( S \) is integrable for the space \( V_n \):

\[
\begin{align*}
\frac{1}{\psi} \psi_{,a} - \frac{2}{\psi^2} \psi_{,a} \psi_{,c} - \frac{k}{n} g_{ab} + h^a_b h_{ab} + \psi h_{ab} S &= 0, \\
h_{ab,\cdot} c - \frac{1}{\psi} (\psi_{,a} h_{bc} + \psi_{,b} h_{ac} + \psi_{,c} h_{ab}) &= 0, \\
S_{,a} - \frac{2}{\psi^2} h^a_b \psi_{,\cdot} &= 0
\end{align*}
\]

under the condition

\[
(1 - \frac{1}{n}) k g_{ab} + h h_{ab} - h^a_b h_{ab} - R_{ab} = 0.
\]

§ 6. Integrability conditions of (5,8), (5,9) \((n > 2)\).

In order to investigate the integrability of (5,8), (5,7), let us replace them by the following equivalent system of equations

\[
\begin{align*}
(6,1) & \quad \psi_{,a} = \psi \rho_a, \\
(6,2) & \quad \rho_{a,\cdot} b = \rho_{a,\cdot} b + \frac{k}{n} g_{ab} - h^a_b h_{ab} - \psi S h_{ab}, \\
(6,3) & \quad h_{ab,\cdot} c = \rho_{c} h_{ab} + \rho_a h_{bc} + \rho_b h_{ac}, \\
(6,4) & \quad S_{,a} = \frac{2}{\psi^2} h^a_b \rho_{\cdot}
\end{align*}
\]

and

\[
(IV_2) \quad \psi_{ab} \equiv (1 - \frac{1}{n}) k g_{ab} + h h_{ab} - h^a_b h_{ab} - R_{ab} = 0.
\]

Now, we get from (6,1) the relation
\[
\psi_{,a(b)} = \psi \rho_{(a,\cdot) b} + \psi_{,b} \rho_{a} = \psi \{\rho_{(a,\cdot) b} + \rho_{a} \rho_{b}\} = 0
\]

and from (6,2) the relation
\[
\rho_{a,\cdot(b) c} = \rho_{a(b) \cdot c} - h^a_{\cdot, c} h_{b\cdot \lambda} - \psi S h_{a(b) \cdot} - \psi h_{a(b) S_{,c}}
\]
that is
\[
\left\{ R^\lambda_{a c} - h^\lambda_{ac} + h^\lambda_{ad} - \frac{k}{n} (g_{ae} \delta^c_d - g_{de} \delta^c_e) \right\} \rho_\lambda = 0.
\]

We get from (6.3) the relation
\[
h_{ab,cd} = -\frac{1}{2} R^\lambda_{a c} h_{\lambda b} - \frac{1}{2} R^\lambda_{b c} h_{\lambda a} = h_{ab,cd} = \rho_a h_{bd} + \rho_b h_{ad} + (\rho_a \rho_b + \frac{k}{n} g_{ad} - h_{ac} h^d_{cd} - \psi S h_{ad} h_{cd}) h_{cb} + (\rho_a \rho_b + \frac{k}{n} g_{bd} - h_{bc} h^d_{cd} - \psi S h_{bd} h_{cd}) h_{ca} = h_{a b} \left( h^\lambda_{ac} h^\lambda_{bd} + \frac{k}{n} \delta^c_d g_{a c} \right) + h_{a b} \left( h^\lambda_{ac} h^\lambda_{da} + \frac{k}{n} \delta^c_d g_{a c} \right).
\]

that is
\[
\left\{ R^\lambda_{a c} - h^\lambda_{ac} + h^\lambda_{ad} - \frac{k}{n} (g_{ae} \delta^c_d - g_{de} \delta^c_e) \right\} \rho_\lambda = 0.
\]

We get lastly from (6.4) the relation
\[
S_{(c d)} = -\frac{2}{\sqrt{\rho}} \rho_{c d} h_{\lambda c} h_{\lambda d} + \frac{2}{\sqrt{\rho}} \rho_{(c d)} h_{\lambda c} = -\frac{2}{\sqrt{\rho}} \rho_{c d} h_{\lambda c} h_{\lambda d} + \frac{2}{\sqrt{\rho}} \left( \rho_{(c d)} h_{\lambda c} + \frac{k}{n} g_{c d} - h^\lambda_{ac} h_{\lambda d} - \psi S h_{ac} h_{\lambda d} \right) h_{a c} = 0.
\]

Hence, if we put
\[
F_{abcd} = R_{abcd} - h_{ac} h_{bd} + h_{ad} h_{bc} - \frac{k}{n} (g_{ac} g_{bd} - g_{ad} g_{bc}),
\]
the results above are represented by
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\[(6,6) \quad F_\lambda^{\alpha \beta \gamma \delta} \rho_\lambda = 0,\]
\[(6,7) \quad F_\lambda^{\alpha \beta \gamma \delta} h_\beta^\gamma + F_{\alpha \beta \lambda \delta} h_\alpha^\lambda = 0,\]

and

\[(6,8) \quad F_{\alpha \beta \gamma \delta} = - F_{\beta \alpha \delta \gamma} = - F_{\alpha \beta \delta \gamma} = F_{\gamma \delta \alpha \beta}.
\]

Accordingly, we obtain the following theorem.

**Theorem 6.** A condition of integrability of the system of equations \((6,1) - (6,4), (IV_3)\) is that the system of algebraic relations with respect to \(\rho_\alpha, h_{\alpha \beta}, \psi, S\) derived successively from \((IV_3), (6,6), (6,7)\) by differentiation and by substitution of \((6,1) - (6,4)\) is compatible.

Now, if \(V_n\) is an Einstein space, then by definition we have the relation

\[R_{\alpha \beta} = \frac{R}{n} g_{\alpha \beta}.
\]

Then if we put

\[h_{\alpha \beta} = \psi g_{\alpha \beta},
\]

\((IV_3)\) becomes

\[\gamma_{\alpha \beta} = \left\{\left(1 - \frac{1}{n}\right)k + (n - 1)\psi^2 - \frac{R}{n}\right\} g_{\alpha \beta} = 0,
\]

hence we have the relation

\[\psi = \left(\frac{R}{n(n-1)} - \frac{k}{n}\right)^{\frac{1}{2}}.
\]

Furthermore, \((6,7)\) is satisfied by means of \((6,8)\). If we put \(\psi = \text{const.}, S = \text{const.}\), then we get from \((6,2)\) the relation

\[\frac{k}{n} - \psi^2 - \psi S = 0.
\]

We can easily determine \(\psi, S\) so that the last relation holds good. Thus we obtain the following corollary.

**Corollary.** Any Einstein space \(A_n\) can be imbedded into an Einstein space \(A_{n+1}\) in the sense as stated in Theorem 4 and so that \(A_n\) is totally geodesic or umbilical in \(A_{n+1}\).

As easily seen, the spaces \(V_n\) which can be imbedded in an Ein-
stein space $A_{n+1}$ and are totally geodesic or umbilical in it are Einstein spaces.

§ 7. Integrability conditions of (5.3), (5.9) $(n = 2)$.

In the case $n = 2$, let us denote the Gaussian total curvature by

$$K = \frac{R_{1212}}{g}.$$  

Then, since we have the relations

$$hh_{ab} - h_{a}h_{b} = \frac{1}{g} g_{ab} \left| h_{\lambda \mu} \right|$$

$$R_{ab} = Kg_{ab},$$

(IV.2) becomes

$$\gamma_{ab} = \frac{k}{2} g_{ab} + h_{ab} - h_{a}h_{b} - R_{ab}$$

$$ = \left\{ \frac{1}{g} h_{\lambda \mu} \right\} - \left( K - \frac{k}{2} \right) g_{ab} = 0.$$  

Hence we have a equivalent condition

$$K - \frac{k}{2} - \frac{1}{g} \left| h_{\lambda \mu} \right| = 0.$$  

On the other hand, as regards $F_{abcd}$ we have

$$F_{1212} \equiv R_{1212} - \left| h_{\lambda \mu} \right| - \frac{k}{2} g = \left( K - \frac{k}{2} \right) g - \left| h_{\lambda \mu} \right|.$$  

Accordingly, we see that (6.6), (6.7) are identically satisfied if (7.1) holds good.

By differentiation, we get from (7.1) the relation

$$K_{,a} + \frac{1}{g^2} \left| h_{\lambda \mu} \right| \frac{\partial g}{\partial x^{a}} - \frac{1}{g} \left( h_{11} \frac{\partial h_{22}}{\partial x^{a}} + h_{22} \frac{\partial h_{11}}{\partial x^{a}} - 2h_{22} \frac{\partial h_{12}}{\partial x^{a}} \right)$$

$$ = K_{,a} - \frac{2}{g} \left( h_{11} h_{12, a} + h_{12, a} h_{22} - 2h_{22} h_{12, a} \right) = 0.$$  

Putting (6.3) into the last relation, we get

$$K_{,a} - \frac{2}{g} \left\{ \left| h_{\lambda \mu} \right| \rho_{a} + \rho_{1}(h_{1a}h_{2a} - h_{12}h_{2a}) + \rho_{2}(h_{11}h_{2a} - h_{12}h_{1a}) \right\} = 0,$$

that is
Furthermore, putting (7,1) in the relation, we get

\[(7,2) \quad K_{,a} - 4\left(K - \frac{k}{2}\right)\rho_a = 0.\]

i) **The case \(K = \text{constant}.**

If we put \(k = 2K\), we have the sole condition (7,1), since (7,2) becomes a trivial one. Then the system of equations (6,1) - (6,4), (IV.) is clearly integrable.

ii) **The case \(K \neq \text{constant}.**

Furthermore, differentiating (7,2), we get the relation

\[K_{,a} = 4K_{,b}\rho_a - 4\left(K - \frac{k}{2}\right)\rho_{a,b} = 0,\]

into which we put (6,2), we get the relation

\[K_{,a} = 4K_{,b}\rho_a - 4\left(K - \frac{k}{2}\right)\left\{\rho_a\rho_b + \frac{k}{2}g_{ab} + \frac{h_{,a}}{g}g_{ab} - (h + \psi S)h_{ab}\right\} = 0.\]

Putting (7,1), (7,2) into the last relation, we obtain the relation

\[(7,3) \quad K_{,ab} - \frac{5}{4\left(K - \frac{k}{2}\right)}K_{,a}K_{,b} - 4\left(K - \frac{k}{2}\right)\left\{Kg_{ab} - (h + \psi S)h_{ab}\right\} = 0.\]

On the other hand, we get by (6,3)

\[h_{,a} = \rho_a h + 2h_{,a}^b \rho_a,\]

that is

\[\left(\frac{h}{\psi}\right)_{,a} = \frac{2}{\psi} h_{,a} \rho_a.\]

Comparing this with (6,4), we have the relation

\[(7,4) \quad S = \frac{h}{\sqrt{\rho}} + 2C \quad (C = \text{constant}).\]

If we put (7,4) into (7,3), we obtain

\[(7,5) \quad K_{,ab} - \frac{5}{4\left(K - \frac{k}{2}\right)}K_{,a}K_{,b} - 4\left(K - \frac{k}{2}\right)Kg_{ab} + 8\left(K - \frac{k}{2}\right)(h + \psi C)h_{ab} = 0.\]
Let us define a tensor of $V$, depending on $k$ such that

$$L_{ab}(k) = \frac{K_{ab}}{4(K - \frac{k}{2})} - \frac{5K_aK_b}{16(K - \frac{k}{2})^2} - Kg_{ab},$$

then (7,5) is represented by

$$(7,5') \quad L_{a'b'}(k) + 2(h + \psi C)h_{ab} = 0.$$  

Now, we divide the case into the two following cases.

ii) The case $L_{a'b'}(k) = 0$.

Then, we have $h_{ab} = 0$ or $h + \psi C = 0$. In the first case, we get from (7,1) the relation $K = \text{const.}$ which is contradictory to our assumption. In the second case, we get easily

$$h_{a'} + C\psi_{a'} = 2h_{a'}\rho_{a'} + (h + C\psi)\rho_{a'} = 2h_{a'}\rho_{a'} = 0.$$  

Hence, solving these relations with respect to $h_{ab}$, we get

$$h_{11} = -\psi Cg_{\rho^a\rho^b}g_{\rho^a\rho^b},$$

$$h_{12} = \psi Cg_{\rho^1\rho^2}g_{\rho^1\rho^2},$$

$$h_{22} = -\psi Cg_{\rho^1\rho^1}g_{\rho^1\rho^1},$$

from which we get the relation

$$|h_{\lambda\mu}| = 0.$$  

Accordingly we get also $K = \text{const.}$ which is contradictory to our assumption.

ii) The case $L_{a'b'}(k) \neq 0$.

Then, we have

$$|L_{a'b'}| = 4(h + \psi C)^2|h_{\lambda\mu}|,$$

into which putting (7,1), we get

$$|L_{ab}| = 4(h + \psi C)^2g(K - \frac{k}{2}),$$

that is

$$h + \psi C = \pm \frac{1}{2} \sqrt{\frac{|L_{a'b'}(k)|}{g(K - \frac{k}{2})}}.$$
If we put

\[ F(k) = \frac{1}{2} \sqrt{\frac{L_{ab}(k)}{4(K - \frac{k}{2})}} , \]

(7,5') becomes

\[ L_{ab} + 2Fh_{ab} = 0. \]

Then, if \( F(k) = 0 \), it leads to a contradiction as (ii). Hence, by virtue of the above calculation, we get the relation

\[ \rho_a = \frac{K_a}{4(K - \frac{k}{2})}, \quad h_{ab} = \pm \frac{L_{ab}(k)}{2F(k)} , \]

(7,6)

\[ \psi^C = \pm F(k) \pm \frac{L_{ab}(k)}{2F(k)}, \]

\[ S = \pm \frac{F(k)}{\psi} + C \quad (k, C = \text{constant}). \]

Accordingly, in order that our system is integrable, it is necessary and sufficient that the relation derived from (7,5) by differentiation is satisfied for the space.

By means of (6,3), we get from (7,5''')

\[ L_{ab,\epsilon} + 2F_{\epsilon} h_{ab} \pm 2Fh_{ab,\epsilon} \]

\[ = L_{ab,\epsilon} + 2F_{\epsilon} h_{ab} \pm 2F(\rho_a h_{ab} + \rho_b h_{ab} + \rho_h h_{ad}) = 0, \]

into which putting (7,6), we obtain the relation

\[ L_{ab,\epsilon} - \frac{F_{\epsilon}}{\psi} - L_{ab} - \frac{1}{4(K - \frac{k}{2})}(K_{,a}L_{ab} + K_{,a}L_{bc} + K_{,b}L_{ac}) = 0. \]

Accordingly we obtain the following theorem.

**Theorem 7.** In order that we can imbed an two-dimensional Riemannian space \( V \) into an Einstein space \( A \) as a surface so that it is the image of the quadric which the group of holonomy of the space with a normal projective connexion corresponding to \( A \) fixes, the following condition is necessary and sufficient:

\( V \) is a surface with constant curvature, or for the tensor \( L_{ab}(k) \) and the scalar \( F(k) \) depending on a constant \( k \) the following relation holds good.
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\[ L_{a\nu, e} - L_{a\nu}\left\{ \log F\left( K - \frac{k}{2} \right)^{\frac{1}{2}} \right\} - \int e - \frac{1}{4\left( K - \frac{k}{2} \right)} (K_{a\nu}^e L_{e\nu} + K_{e\nu}^a L_{a\nu}) = 0. \]

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