On periodic P.I. rings and locally finite rings

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An element $x$ of a ring $R$ is called periodic if there exist distinct positive integers $m, n$ for which $x^m = x^n$. Especially, $x$ is called potent if $x^m = x$ for some positive integer $m > 1$. A ring $R$ is called periodic if all elements of $R$ are periodic. It is easily seen that a periodic ring $R$ has the property that every element of $R$ is expressible as a sum of a potent element and a nilpotent element. However it is not known whether a ring $R$ with this property is periodic or not. On the other hand, by a result of the author and H. Tominaga [6], if $R$ is a P. I. ring in which every element is the sum of two idempotents, then $R$ is periodic. In this paper, we shall prove that a P. I. ring $R$ in which every element is expressible as a sum of two periodic elements, is periodic.

We shall next consider the local finiteness of a periodic P. I. ring. A ring $R$ is said to be locally finite if any finitely generated subring of $R$ is a finite ring. Let $R$ be a periodic P. I. ring, and $S$ a finitely generated subring of $R$. We shall show that the additive group of $S$ is finitely generated and that some power of $S$ is a finite ring. Consequently a P. I. ring $R$ is locally finite if and only if $R$ is periodic and the additive group of $R$ is a torsion group. Using this, we shall give a characterization of a locally finite ring.

We begin with the following lemma.

**Lemma 1.** Let $R$ be a ring. Then $R$ is periodic if and only if all prime factor rings of $R$ are periodic.

**Proof.** Suppose that all prime factor rings of $R$ are periodic. For each $x \in R$, let $S(x) = \{x^n - x^{n+1}f(x) | n > 0 \text{ is an integer, } f(t) \in \mathbb{Z}[t] \}$, which is multiplicatively closed. By virtue of [3, Proposition 2], $R$ is periodic if and only if $0 \in S(x)$ for all $x \in R$. Assume, to the contrary, that there exists $a \in R$ such that $0 \notin S(a)$. Then, by Zorn’s lemma, we can find an ideal $I$ of $R$ which is maximal with respect to the property that $S(a) \cap I = \phi$. It is easy to check that $I$ is a prime ideal of $R$. Hence $R/I$ is periodic by hypothesis. But this contradicts the fact that $S(a) \cap I = \phi$.

A ring $R$ is said to be of bounded index (of nilpotence) if there is a positive integer $n$ such that $a^n = 0$ for any nilpotent element $a$ in $R$. The least
such integer is called the index of $R$. We shall show that a periodic ring of bounded index is a P. I. ring. Let $G$ denote the symmetric group of degree $n$. The identity
\[ s_n = \sum_{\sigma \in G} \text{sgn}(\sigma) X_{1\sigma} X_{2\sigma} \cdots X_{n\sigma} \]
is called the standard identity of degree $n$.

**Proposition 1.** Let $n$ be a positive integer and let $R$ be a periodic ring of index $n$. Then $R$ satisfies the polynomial identity $(s_2)^n$.

**Proof.** Let $J$ denote the Jacobson radical of $R$, and $x$ an element of $J$. Then there exist positive integers $p, q$ such that $x^{p+q} = x^p$. By [4, Theorem 1.2.3] all elements of $J$ are right-quasi-regular. Hence there exists $y \in R$ such that $(-x^p) + y + (-x^q)y = 0$. Then $x^p = x^p + x^q(-x^p + y - x^q) = (x^p - x^{p+q}) + (x^q - x^{p+q})y = 0$. This implies that $J$ is a nil ideal. Let $P$ be a primitive ideal of $R$. By [7, Theorem 2.3] $R/P = M_t(D)$ for some division ring $D$ and some positive integer $t \leq n$. Since $D$ is a periodic division ring, $D$ is commutative by [4, Lemma 3.1.3]. Hence $R/P$ satisfies the standard identity $s_{2n}$ of degree $2n$ by [8, Theorem 1.4.1]. Since $R/J$ can be embedded in the direct product of all primitive factor rings of $R$, $R/J$ also satisfies the identity $s_{2n}$, in other words, $s_{2n}(a_1, a_2, \ldots, a_{2n}) \in J$ for all elements $a_1, a_2, \ldots, a_{2n}$ in $R$. Since $J$ is a nil ideal of index at most $n$, we have that $s_{2n}(a_1, a_2, \ldots, a_{2n}) = 0$ for all $a_1, a_2, \ldots, a_{2n} \in R$. This completes the proof.

If $R$ is a periodic ring, each element $x$ in $R$ can be expressed in the form $y + w$, where $y^n = y$ for some $n = n(y) > 1$ and $w$ is nilpotent (e.g., see [2, Lemma 1]). However it is not known whether this property characterizes a periodic ring. On the other hand, by [6, Theorem 2], if $R$ is a P. I. ring in which every element is the sum of two idempotents then, for any $x \in R$, $x^3 - x$ is nilpotent. Hence $R$ is periodic by [3, Proposition 2]. We shall now prove the following

**Theorem 1.** Let $R$ be a P. I. ring. If every element of $R$ is expressed as a sum of two periodic elements, then $R$ is periodic.

**Proof.** By virtue of Lemma 1, we may assume that $R$ is a prime ring. Then, by [5, Theorem 1.4.2] the center $C$ of $R$ is nonzero. We claim that $C$ is periodic. Let $c$ be a nonzero element of $C$. Then, by hypothesis, there
exist \( x, y \in R \) such that \( c = x + y, x^n = x^n \) for some \( m > n > 0 \), and \( y^p = y^q \) for some \( p > q > 0 \). Then \((c - y)^m = (c - y)^n\), and so \( (c^n - c^n) = zy \) for some \( z \in C[y] \subseteq R \). If \( c^n - c^n \) is nilpotent, then \( c^n = c^n \), because \( C \) is an integral domain. Assume now that \( c^n - c^n \) is not nilpotent. Then \( e = \frac{y^p - y^q}{y^q} \) is a nonzero idempotent and \( y^q e = ey^q = y^q \). Therefore we have that \((c^n - c^n)^q (ae - a) = 0 \) for all \( a \in R \). Let us put \( L = \{ae - a \mid a \in R \} \). Then \( L \) is a left ideal of \( R \), and as seen above, \((c^n - c^n)^q L = 0 \). Since \((c^n - c^n)^q \neq 0 \) and since \( R \) is a prime ring, we obtain \( L = 0 \), that is, \( e \) is a right identity of \( R \). We can similarly prove that \( e \) is a left identity of \( R \). Hence \( e \) is the identity of \( R \). We shall now prove that the characteristic of \( R \) is nonzero. Assume, to the contrary, that the characteristic of \( R \) is zero. Then we may assume that \( R \) contains the ring \( Z \) of integers as a subring. By hypothesis, there exist two periodic elements \( v, w \in R \) such that \( 3 = v + w \). Obviously the subring \( S = Z[v, w] \) of \( R \) generated by \( v \) and \( w \) over \( Z \) is a commutative ring which is integral over \( Z \). By [1, Theorem 5.10] there exists a prime ideal \( P \) of \( S \) such that \( P \cap Z = 0 \). Consider now the factor ring \( S/\overline{P} \). Then \( S \) is an integral domain which is integral over \( Z \). So, without loss of generality, we may assume that \( S \) is a subring of the field \( C \) of complex numbers. In general, if \( a \) is a periodic element of \( C \), then the absolute value \( |a| \) of \( a \) is either 0 or 1. Hence we have \( 3 = |v + w| \leq |v| + |w| \leq 2 \), which is a contradiction. Therefore the characteristic of \( R \) is nonzero. Let \( F \) denote the prime field of \( C \). Since \( x \) and \( y \) are integral over \( F \), \( c = x + y \) is integral over \( F \). Hence \( c \) generates a finite subring of \( C \), and so \( c \) is periodic. Therefore we proved that \( C \) is a periodic field. By [8, Corollary 1.6.28], \( R \) is a simple P.I. ring. Hence, by Kaplansky’s theorem [8, Theorem 1.5.16], \( R \) can be identified with the matrix ring \( M_k(D) \) over a division ring \( D \) which is finite dimensional over \( C \). Then \( D \) is also periodic, and hence \( D \) is commutative. Thus we get \( C = D \). Therefore \( R = M_\delta(C) \) is periodic.

We shall next consider the finitely generated subrings of a periodic P.I. ring. Clearly a periodic P.I. ring need not be locally finite. For example, the subring

\[
\begin{pmatrix}
0 & Z \\
0 & 0
\end{pmatrix}
\]

of \( M_\delta(Z) \)

is a finitely generated periodic commutative ring, but this is not a finite ring. We shall prove the following:

Theorem 2. Let \( R \) be a periodic P.I. ring and let \( S \) be a finitely gener-
ated subring of \( R \). Then the additive group \( S^* \) of \( S \) is a finitely generated abelian group. Moreover there exists a positive integer \( n \) such that \( S^n \) is a finite ring. In particular, if \( S \) has an identity, then \( S \) is finite.

**Proof.** Let \( t(S) \) denote the torsion submodule of the \( \mathbb{Z} \)-module \( S \). Then \( t(S) \) is an ideal of \( S \) and \( S/t(S) \) is torsion-free. Let \( x \) be an element of \( S/t(S) \). Then \( x^{m+n} = x^m \) for some positive integers \( m, n \). Then we can easily see that \( x^m \) is an idempotent. Since \( (2x^{m+n})^{p+q} = (2x^m)^p \) for some positive integers \( p \) and \( q \), we obtain a positive integer \( h \) such that \( hx^{m+n} = 0 \). Since \( S/t(S) \) is torsion-free, we conclude that \( x^{m+n} = 0 \). Thus \( S/t(S) \) is a nil ring. Since \( S/t(S) \) is also a finitely generated P.I. ring, there exists a positive integer \( n \) such that \( (S/t(S))^n = 0 \) by [8, Proposition 1.6.34]. Hence we have \( S^n \subset t(S) \). Let \( c_1, c_2, \ldots, c_n \) generate the subring \( S \). Then \( A = \{ c_1, c_2, \ldots, c_n \} | 1 \leq i \leq m \} \) is a finite set, and hence there exists a positive integer \( k \) such that \( kA = 0 \). Hence we have \( kS^n = 0 \). Let \( B \) denote the set \( \{ c_1, c_2, \ldots, c_n \} | 1 \leq i \leq m, 1 \leq p \leq n \} \). Then we can easily see that

\[
kS = \sum_{b \in B} \mathbb{Z}kb.
\]

Hence \( kS \) is a finitely generated \( \mathbb{Z} \)-module. Let \( S' \) denote the ring \( S/kS \) and let us write \( k = \prod_{i=1}^t p_i^{k_i} \) where the \( p_i \) are distinct primes and \( k_i > 0 \) for all \( i \). Then, for each \( i \), \( S_i = \{ a \in S' | p_i^{k_i}a = 0 \} \) is a subring of \( S' \) and \( S' \) is the direct sum of \( S_1, S_2, \ldots, S_t \). We shall show that \( S' \) is finite. To show it, it suffices to prove that \( S_i \) is finite for each \( i = 1, 2, \ldots, t \). Hence, without loss of generality, we may assume that \( k = p^h \) for some prime \( p \) and some positive integer \( h \). Let us set \( I = pS' \). Then \( I^h = 0 \) and \( p^{h-1}I = 0 \). Then the ring \( S'/I \) is a finitely generated periodic algebra over \( \mathbb{Z}/p\mathbb{Z} \) satisfying a polynomial identity. Hence \( S/I \) is a finite dimensional algebra over \( \mathbb{Z}/p\mathbb{Z} \) by [4, Theorem 6.4.3]. Let \( S'/I = \{ a_0 + I, a_1 + I, \ldots, a_d + I \} \) where \( a_0 = 0 \), \( a_1, \ldots, a_d \) are elements of \( S' \). Then we can choose elements \( b_1, b_2, \ldots, b_r \) of \( I \) such that \( a_1, a_2, \ldots, a_d, b_1, b_2, \ldots, b_r \) generate \( S' \). For any \( i, j \) with \( 1 \leq i, j \leq d \), we have a unique integer \( t(i, j) \) with \( 1 \leq t(i, j) \leq d \) such that \( a_i a_j = a_{t(i, j)} \) modulo \( I \). Similarly we have a unique integer \( s(i, j) \) such that \( a_i + a_j = a_{s(i, j)} \) modulo \( I \). Let us now set \( x_{ij} = a_i a_j - a_{s(i, j)} \) and \( y_{ij} = a_i + a_j - a_{t(i, j)} \) for each \( 1 \leq i, j \leq d \). Let \( J \) denote the subring of \( S' \) generated by \( x_{ab}, y_{uv}, b_1, a_1 a_2, a_1 y_{uv}, a_2 y_{uv}, b_1 a_1, x_{ab} a_1, y_{uv} a_2, b_1 a_1 \) for \( 1 \leq a, \beta, \gamma \leq d, 1 \leq \mu, \nu \leq d, 1 \leq \lambda \leq f \). Then \( J \) is a finitely generated subring of \( I \). Since \( I^h = 0 \) and \( p^{h-1}I = 0 \), \( J \) must be finite. We can now easily see that each element \( x \) of \( S' \) can be uniquely expressed in the form \( a_i + z \), where \( 0 \leq i \leq d \) and \( z \in J \). This implies that \( I = J \). Therefore \( S' \) is a finite ring. Conse-
quently $S$ is a finitely generated $\mathbb{Z}$-module. Since the additive group of $S^n$ is a torsion group, $S^n$ is a finite ring. In particular, if $S$ has an identity, then $S^n = S$, and hence $S$ is finite.

As an immediate consequence of this theorem, we obtain the following:

**Corollary 1.** Let $R$ be a P. I. ring. Then $R$ is locally finite if and only if $R$ is periodic and the additive group of $R$ is a torsion group.

A ring $R$ is said to be of locally bounded index if every finitely generated subring of $R$ is of bounded index. Combining Corollary 1 with Proposition 1, we obtain the following characterization of a locally finite ring.

**Corollary 2.** A ring $R$ is locally finite if and only if $R$ is a periodic ring of locally bounded index and the additive group of $R$ is a torsion group.

The following example due to Golod and Shafarevitch shows that a finitely generated periodic ring with torsion additive group need not be finite.

**Example 1.** Let $p$ be a prime number. By [4, Theorem 8.1.3], there exists an infinite dimensional nil algebra $A$ over $\mathbb{Z}/p\mathbb{Z}$ generated by three elements. Clearly $A$ is generated by those three elements as a ring. Note that those elements generate infinite subsemigroup of the multiplicative semigroup of $R$.

As another corollary of Theorem 2, we obtain the following:

**Corollary 3.** Let $R$ be a P. I. ring. Then the following statements are equivalent:

1. $R$ is periodic.
2. For any finitely generated subring $S$ of $R$, there exists a positive integer $n$ such that $S^n$ is a finite subring.
3. For any finitely generated subring $S$ of $R$, there exists a finite ideal $I$ of $S$ such that $S/I$ is a nilpotent ring.
4. The ideal $t(R) = \{a \in R | na = 0 \text{ for some positive integer } n\}$ is locally finite and $R/t(R)$ is a nil ring.

**Proof.** The implication (1) $\Rightarrow$ (2) follows from Theorem 2 and (2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1). Let $x$ be an element of $R$, and $S$ denote the subring of $R$
generated by \( x \). Then there exists a finite ideal \( I \) of \( S \) such that \( S/I \) is nilpotent. This implies that some power of \( x \) generates a finite subring. Hence there exist distinct positive integers \( m, n \) such that \( x^m = x^n \).

(1) \( \iff \) (4). Assume that \( R \) is periodic. By Corollary 1 \( t(R) \) is locally finite. We also know that \( R/t(R) \) is a nil ring by the proof of Theorem 2.

Conversely, suppose that (4) holds, and let \( x \) be an element of \( R \). Then some power of \( x \) generates a finite subring of \( R \), and hence \( x \) is periodic.

A ring \( R \) is periodic if and only if each subsemigroup of \( R \) generated by a single element is finite. If \( R \) is a commutative periodic ring, then all finitely generated subsemigroups of \( R \) are finite. However Example 1 shows that this does not remain valid for noncommutative periodic rings. Thus we have the following

Conjecture. Let \( R \) be a periodic P. I. ring. Then all finitely generated subsemigroups of \( R \) are finite.

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(Received December 5, 1990)