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Abstract

In this paper we investigate the weighted ergodic properties of invertible Lamperti operators. Some results of Martín-Reyes, de la Torre and others in Málaga (Spain) are unified and generalized.

KEYWORDS: Weighted ergodic properties, invertible Lamperti operators, dominated ergodic theorem, almost everywhere convergence in the sense of Cesaro-alpha means, ergodic averages, ergodic Hilbert transform

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1. INTRODUCTION

Let \((X, \mathcal{F}, \mu)\) be a \(\sigma\)-finite measure space and let \(M(\mu)\) denote the space of all complex-valued measurable functions on \(X\). Two functions \(f\) and \(g\) in \(M(\mu)\) are not distinguished provided that \(f(x) = g(x)\) for almost all \(x \in X\). Hereafter all statements and relations will be assumed to hold modulo sets of measure zero. By a Lamperti operator \(T\) on \(M(\mu)\) we mean an operator of the form

\[
Tf(x) = h(x)\Phi f(x),
\]

where \(h \in M(\mu)\) is a fixed function and \(\Phi : M(\mu) \rightarrow M(\mu)\) is a linear and multiplicative operator. We recall that \(\Phi\) is a multiplicative operator if \(\Phi\) satisfies \(\Phi(fg) = (\Phi f)(\Phi g)\) for all \(f, g \in M(\mu)\).

In this paper we always assume \(T\) to be invertible on \(M(\mu)\). Hence it follows that \(0 < |h| < \infty\) a.e. on \(X\) and that \(\Phi\) is invertible on \(M(\mu)\). The following properties of \(T\) are known (cf. [11], [13]).

(I) If we put \(h_1 = h\), \(h_0 = 1\), \(h_{-1} = 1/\Phi^{-1}h\), \(h_n = h_1 \cdot \Phi h_{n-1}\) and \(h_{-n} = h_{-1} \cdot \Phi^{-1}h_{-n+1}\) \((n \geq 2)\), then for each \(j, k \in \mathbb{Z}\) we have

\[
T^j f = h_j \cdot \Phi^j f \quad \text{and} \quad h_{j+k} = h_j \cdot \Phi^j h_k.
\]

(II) By the Radon-Nikodym theorem, for each \(j \in \mathbb{Z}\) there exists a positive measurable function \(J_j\) in \(M(\mu)\) such that if \(0 \leq f \in M(\mu)\) then

\[
\int J_j \cdot \Phi^j f \, d\mu = \int f \, d\mu \quad \text{and} \quad J_{j+k} = J_j \cdot \Phi^j J_k \quad \text{for} \quad j, k \in \mathbb{Z}.
\]

Let \(\tau f = |h_1| \cdot \Phi f\) for \(f \in M(\mu)\). Then \(\tau\) is a positive invertible Lamperti operator, and for each \(j \in \mathbb{Z}\) we have

\[
\tau^j f = |h_j| \cdot \Phi^j f \quad \text{and} \quad |\tau^j f| = |T^j f| \quad \text{for} \quad f \in M(\mu),
\]

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so that $\tau^j$ becomes the linear modulus of $T^j$.

We recall that if $T : L^p(\mu) \rightarrow L^p(\mu)$, where $1 \leq p \leq \infty$, is a positive linear operator with positive inverse then $T$ has the form (1) for $f \in L^p(\mu)$ (cf. [11]), and thus the operator has a unique extension to an invertible Lamperti operator on $M(\mu)$.

Let $w$ be a nonnegative extended real-valued measurable function on $X$. Then, since the measure $wd\mu$ on $\mathcal{F}$ is absolutely continuous with respect to $\mu$, $f = g$ a.e. $\mu$ implies that $f = g$ a.e. $wd\mu$. But the converse does not hold. Therefore, as it is easily seen, a Lamperti operator $T$ on $M(\mu)$ is no longer an operator on $M(wd\mu)$ in general. And even though it is the case, the operator $T$ is not necessarily invertible on $M(wd\mu)$. In the case where $T$ is invertible on $M(wd\mu)$ and the measure $wd\mu$ is $\sigma$-finite, the study of weighted ergodic properties of $T$ on $M(wd\mu)$ reduces to that of $T$ on $M(\mu)$; and there are many papers investigating successfully invertible Lamperti operators $T$ on $M(\mu)$. See e.g. [1], [2], [3], [5], [15], [17] and [23], etc. However it should seem that the study is not enough for the non-invertible case, although some papers have treated of not necessarily invertible Lamperti operators (see e.g. [11], [12]), and hence the author thinks that it would be interesting to investigate the weighted ergodic properties of $T$ on $M(wd\mu)$, without assuming the invertibility of $T$ on $M(wd\mu)$. This is the starting point of the paper. Here we remark that, by an easy observation, an invertible Lamperti operator $T$ on $M(\mu)$ defined by (1) becomes an operator on $M(wd\mu)$ if and only if $\Phi_{\chi_A} \leq \chi_A$, where we let $A = \{ x : w(x) = 0 \}$ and $\chi_A$ denotes the characteristic function of $A$.

For an invertible Lamperti operator $T$ on $M(\mu)$ we introduce two ergodic maximal operators $M^+(T)$ and $M(T)$ on $M(\mu)$ by the relations

\begin{equation}
M^+(T)f = \sup_{n \geq 0} |T_0, nf|
\end{equation}

and

\begin{equation}
M(T)f = \sup_{m,n \geq 0} |T_m, nf|,
\end{equation}

where we let

\[ T_{m,n} = \frac{1}{m+n+1} \sum_{i=-m}^{n} T^i. \]

For simplicity $\tau$ will denote a positive invertible Lamperti operator on $M(\mu)$, unless the contrary is explained explicitly. In Section 2 we first characterize those $\tau$ for which the ergodic maximal operator $M^+(\tau)$ [or $M(\tau)$] is bounded in $L^p(wd\mu)$, $1 < p < \infty$. Among other things we will observe that $M^+(\tau)$ is bounded in $L^p(wd\mu)$ if and only if $\tau$ is an operator on $M(wd\mu)$ and satisfies

\begin{equation}
\sup_{n \geq 0} \| \tau_{0,n} \|_{L^p(wd\mu)} < \infty.
\end{equation}
This generalizes Martín-Reyes and de la Torre’s dominated ergodic theorem [17]; they considered the particular case where \( \tau \) comes from a positive linear operator in \( L^p(\mu), 1 < p < \infty \), with positive inverse and \( w = 1 \) on \( X \). We then apply the results obtained to prove the a.e. convergence of the ergodic averages \( (1/n) \sum_{i=0}^{n-1} T^i f \) and ergodic partial sums \( \sum_{k=1}^{n} (T^k f - T^{-k} f)/k \).

In Section 3 we consider an invertible Lamperti operator \( T \) on \( M(\mu) \) such that

\[
K_\infty := \sup_{n \in \mathbb{Z}} \|T^n\|_{L^\infty(\mu)} < \infty.
\]

Under the additional hypothesis that \( \Phi \) has no periodic part (i.e. for any \( n \geq 1 \) and \( E \in \mathcal{F} \) with \( \mu E > 0 \) there exists a non-null measurable subset \( A \) of \( E \) such that \( \Phi^n \chi_A \neq \chi_A \)), we prove that the ergodic maximal operator \( M^+(T) \) is of weak type \( (p, p), 1 \leq p < \infty \), with respect to the measure \( w d\mu \) if and only if the linear modulus \( \tau \) of \( T \) is an operator on \( M(wd\mu) \) and satisfies norm condition (7). We also consider the ergodic maximal Hilbert transform \( H^*(T) \) on \( M(\mu) \) defined by the relation

\[
H^*(T)f = \sup_{n \geq 1} \left| \sum_{k=1}^{n} \frac{T^k f - T^{-k} f}{k} \right|.
\]

It will be proved that \( H^*(T) \) is of weak type \( (p, p), 1 \leq p < \infty \), with respect to the measure \( w d\mu \) if and only if the linear modulus \( \tau \) of \( T \) is an invertible operator on \( M(wd\mu) \) and satisfies

\[
\sup_{n \geq 0} \|\tau_{-n, n}\|_{L^p(wd\mu)} < \infty.
\]

These generalize results of Atencia, Martín-Reyes and de la Torre (cf. [1], [2], [3]); they considered the case where \( w \) and \( T \) are such that \( 0 < w \in L^1(\mu) \) and \( T \) is of the form \( Tf(x) = (f \circ \phi)(x) = f(\phi x) \), where \( \phi \) is an ergodic invertible measure preserving transformation on a nonatomic probability measure space. Our proof is an adaptation of their arguments.

Lastly we unify the weighted inequalities obtained here and recent results of [4], [5], [15] to prove the a.e. convergence of the ergodic sequence \( \{T^n f\} \) and the ergodic partial sums \( \{\sum_{k=1}^{n} (T^k f - T^{-k} f)/k\} \) in the sense of Cesàro-\( \alpha \) means.

Throughout the paper \( C \) will denote a positive constant not necessarily the same at each occurrence.

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2. Weighted strong type inequalities and applications

In this section we first consider a \textit{positive} invertible Lamperti operator \( \tau \) on \( M(\mu) \). Let \( \tau f = h_1 \cdot \Phi f \). Then (2) holds with \( \tau \) instead of \( T \), and we have \( 0 < h_j < \infty \) on \( X \) for each \( j \in \mathbb{Z} \).
Theorem 1. Let $0 \leq w \leq \infty$ on $X$ and let $1 < p < \infty$. Then the following statements are equivalent for a positive invertible Lamperti operator $\tau$ on $M(\mu)$.

(a) $\tau$ is an operator on $M(wd\mu)$ and there exists a positive constant $C$ such that for any $f \in L^p(wd\mu)$

\begin{equation}
\int |M^+(\tau)f|^p w \, d\mu \leq C \int |f|^p w \, d\mu.
\end{equation}

(b) $\tau$ is an operator on $M(wd\mu)$ and there exists a positive constant $C$ such that for any $f \in L^p(wd\mu)$

\begin{equation}
\sup_{n \geq 0} \int |\tau_n f|^p w \, d\mu \leq C \int |f|^p w \, d\mu.
\end{equation}

(c) There exists a positive constant $C$ such that for a.e. $x \in X$ and all $k \geq 0$

\begin{equation}
\left( \sum_{i=0}^{k} h_{-i}(x)^{-p} J_{-i}(x) \Phi^{-i} w(x) \right) \cdot \left( \sum_{i=0}^{k} [h_i(x)]^{-p} J_i(x) \Phi^i w(x) \right)^{\frac{1}{p-1}} \leq C(k + 1)^p.
\end{equation}

Theorem 2. Let $0 \leq w \leq \infty$ on $X$ and let $1 < p < \infty$. Then the following statements are equivalent for a positive invertible Lamperti operator $\tau$ on $M(\mu)$.

(a) $\tau$ is an invertible operator on $M(wd\mu)$ and there exists a positive constant $C$ such that for any $f \in L^p(wd\mu)$

\begin{equation}
\int |M(\tau)f|^p w \, d\mu \leq C \int |f|^p w \, d\mu.
\end{equation}

(b) $\tau$ is an invertible operator on $M(wd\mu)$ and there exists a positive constant $C$ such that for any $f \in L^p(wd\mu)$

\begin{equation}
\sup_{n \geq 0} \int |\tau_{-n} f|^p w \, d\mu \leq C \int |f|^p w \, d\mu.
\end{equation}

(c) There exists a positive constant $C$ such that for a.e. $x \in X$ and all $k \geq 0$

\begin{equation}
\left( \sum_{i=0}^{k} h_i(x)^{-p} J_i(x) \Phi^i w(x) \right) \cdot \left( \sum_{i=0}^{k} [h_i(x)]^{-p} J_i(x) \Phi^i w(x) \right)^{\frac{1}{p-1}} \leq C(k + 1)^p.
\end{equation}

As in [16] and [17], to prove these theorems we need the following result about weights on the integers.

Lemma 1 (cf. [14], [18], [21]). Let $0 \leq w \leq \infty$ on $\mathbb{Z}$. For a function $f$ on $\mathbb{Z}$, define the functions $f^*$ and $f^{**}$ on $\mathbb{Z}$ by the relations
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\[ f^*(i) = \sup_{n \geq 0} \left| \frac{1}{n+1} \sum_{j=0}^{n} f(i+j) \right| \]

and

\[ f^{**}(i) = \sup_{n, m \geq 0} \left| \frac{1}{m+n+1} \sum_{j=-m}^{n} f(i+j) \right| \]

Then we have:

(I) When \( 1 < p < \infty \), there exists a positive constant \( C \) such that
\[
\sum_{i=-\infty}^{\infty} (f^*(i))^p w(i) \leq C \sum_{i=-\infty}^{\infty} |f(i)|^p w(i) \quad \text{for all } f \text{ if and only if there exists a positive constant } C \text{ such that for all } j \in \mathbb{Z} \text{ and } k \geq 0
\]

\[
\left( \sum_{i=0}^{k} w(j-i) \right) \cdot \left( \sum_{i=0}^{k} w(j+i)^{-\frac{1}{p-1}} \right)^{p-1} \leq C(k+1)^p.
\]

(II) When \( 1 < p < \infty \), there exists a positive constant \( C \) such that
\[
\sum_{i=-\infty}^{\infty} (f^{**}(i))^p w(i) \leq C \sum_{i=-\infty}^{\infty} |f(i)|^p w(i) \quad \text{for all } f \text{ if and only if there exists a positive constant } C \text{ such that for all } j \in \mathbb{Z} \text{ and } k \geq 0
\]

\[
\left( \sum_{i=0}^{k} w(j+i) \right) \cdot \left( \sum_{i=0}^{k} w(j+i)^{-\frac{1}{p-1}} \right)^{p-1} \leq C(k+1)^p.
\]

(III) There exists a positive constant \( C \) such that for all \( f \) and \( \lambda > 0 \)

\[
\sum_{\{i : f^*(i) > \lambda\}} w(i) \leq C \frac{1}{\lambda} \sum_{i=-\infty}^{\infty} |f(i)| w(i)
\]

if and only if there exists a positive constant \( C \) such that for all \( j \in \mathbb{Z} \)

\[
\sup_{n \geq 0} \frac{1}{n+1} \sum_{i=0}^{n} w(j-i) \leq C w(j).
\]

Proof of Theorem 1. (c) \( \Rightarrow \) (a). Let \( A = \{ x : w(x) = 0 \} \). We apply (13) with \( k = 1 \) to see that \( \Phi^{-1} \chi_A \geq \chi_A \). Hence \( \Phi \chi_A \leq \chi_A \), and thus \( \tau \) becomes an operator on \( M(wd\mu) \). Let \( 0 \leq f \in L^p(wd\mu) \). For an \( N \geq 1 \) we put

\[
f_N^* = \max_{0 \leq n \leq N} \tau_0, n f.
\]

Then for each \( L \geq 1 \) we have, by (3),

\[
\int (f_N^*)^p w \, d\mu = \frac{1}{L+1} \int \sum_{i=0}^{L} (\tau_i f_N^*)^p (h_i^{-p}, J_i \Phi^i w) \, d\mu,
\]

Proof of Theorem 2. (c) \( \Rightarrow \) (a). Let \( A = \{ x : w(x) = 0 \} \). We apply (13) with \( k = 1 \) to see that \( \Phi^{-1} \chi_A \geq \chi_A \). Hence \( \Phi \chi_A \leq \chi_A \), and thus \( \tau \) becomes an operator on \( M(wd\mu) \). Let \( 0 \leq f \in L^p(wd\mu) \). For an \( N \geq 1 \) we put

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\]

Then for each \( L \geq 1 \) we have, by (3),

\[
\int (f_N^*)^p w \, d\mu = \frac{1}{L+1} \int \sum_{i=0}^{L} (\tau_i f_N^*)^p (h_i^{-p}, J_i \Phi^i w) \, d\mu,
\]
where by (2), (3) and (c),
\[ \tau^j f^*_N = h_j \cdot \Phi^j f^*_N = h_j \cdot \max_{0 \leq n \leq N} \frac{1}{n+1} \sum_{i=0}^{n} \Phi^i h_i \cdot \Phi^{j+i} f \]
\[ = \max_{0 \leq n \leq N} \frac{1}{n+1} \sum_{i=0}^{n} \tau^{j+i} f \]
and
\[ \left( \sum_{i=0}^{k} h_{j-i}^{-p} J_{j-i} \Phi^{j-i} w \right) \cdot \left( \sum_{i=0}^{k} \left[ h_{j+i}^{-p} J_{j+i} \Phi^{j+i} w \right]^{\frac{1}{p-1}} \right)^{p-1} \leq C(k+1)^p \text{ a.e.} \]
on \( X \) for all \( j \in \mathbb{Z} \) and \( k \geq 0 \). Thus we apply Lemma A to obtain that
\[ \int (f^*_N)^p w \, d\mu \leq \frac{C}{L+1} \int \sum_{i=0}^{L+N} (\tau^i f)^p (h_i^{-p} J_i \Phi^i w) \, d\mu \]
\[ = \frac{C}{L+1} \sum_{i=0}^{L+N} \int \Phi^i (f^p w) \cdot J_i \, d\mu \]
\[ = \frac{C}{L+1} (L+N+1) \int f^p w \, d\mu \quad \text{(by (3)).} \]
By letting \( L \uparrow \infty \) and then \( N \uparrow \infty \), it follows that
\[ \int [M^+(\tau) f]^p w \, d\mu \leq C \int f^p w \, d\mu. \]

(a) \( \Rightarrow \) (b) is obvious.
(b) \( \Rightarrow \) (c). Let \( \tau^* \) denote the invertible Lamperti operator on \( M(\mu) \) defined by the relation
\[ \tau^* f = \frac{J_{-1}}{h_{-1}} \Phi^{-1} f \quad \text{for} \quad f \in M(\mu). \]
Using (2) and (3), we have
\[ \tau^{*i} f = \frac{J_{-i}}{h_{-i}} \cdot \Phi^{-i} f \quad \text{for} \quad i \in \mathbb{Z}, \]
and
\[ \int (\tau^i f) g \, d\mu = \int f(\tau^{*i} g) \, d\mu \quad \text{for} \quad 0 \leq f, g \in M(\mu). \]
Let \( 1/p + 1/p' = 1 \). If \( 0 \leq f \in L^p(\mu) \) and \( k \geq 0 \) then by (b)
\[ \int \left[ w^{\frac{1}{p}} \cdot \tau_{0,2k}(f w^{-\frac{1}{p}}) \right]^p \, d\mu = \int w \cdot \left[ \tau_{0,2k}(f w^{-\frac{1}{p}}) \right]^p \, d\mu \]
\[ \leq C \int (f^p w^{-1}) w \, d\mu \leq C \int f^p \, d\mu, \]

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so that the mapping \( f \mapsto w_{p}^{\frac{1}{p}} \cdot \tau_{0, 2k}(f \cdot w_{p}^{-\frac{1}{p}}) \) is a bounded linear operator from \( L^p(\mu) \) into \( L^p(\mu) \) with norm less than or equal to \( C_{p'}^{\frac{1}{p}} \); and from (21) it follows that its adjoint operator defined on \( L^{p'}(\mu) \) is identical with the mapping \( g \mapsto w_{p}^{-\frac{1}{p}} \cdot \tau_{0, 2k}^{*}(g \cdot w_{p}^{-\frac{1}{p}}) \) for \( g \in L^{p'}(\mu) \). Thus if \( 0 \leq f \in L^p(\mu) \) then we have
\[
\int \left( w_{\frac{1}{p}}^{\frac{1}{p-1}} \cdot \left[ \tau_{0, 2k}^{*}(f^{p-1} \cdot w_{p}^{\frac{1}{p}}) \right]^{\frac{1}{p-1}} \right)^{p} \, d\mu = \int \left( w_{p}^{-\frac{1}{p}} \cdot \tau_{0, 2k}(f^{p-1} \cdot w_{p}^{\frac{1}{p}}) \right)^{p'} \, d\mu \leq C \int f^{p} \, d\mu.
\]
Let us assume for the moment that \( p \geq 2 \). Since \( p - 1 \geq 1 \), the operator \( U : L^p(\mu) \to L^{p'}(\mu) \) defined by the relation
\[
Uf = w_{p}^{\frac{1}{p}} \cdot \tau_{0, 2k}(|f| \cdot w_{p}^{-\frac{1}{p}}) + w_{\frac{1}{p-1}}^{\frac{1}{p-1}} \cdot \left[ \tau_{0, 2k}^{*}(|f|^{p-1} \cdot w_{p}^{\frac{1}{p}}) \right]^{rac{1}{p-1}}
\]
satisfies \( U(f_1 + f_2) \leq Uf_1 + Uf_2 \) for \( f_1, f_2 \in L^p(\mu) \), and clearly we have
\[
\|U\| \leq 2C.
\]
Then choose a function \( g \in L^p(\mu) \) with \( g > 0 \) on \( X \), and define a function \( G \) on \( X \) by the relation
\[
G = \sum_{i=0}^{\infty} \frac{U^{i}g}{(3C)^{i}}.
\]
It follows that \( 0 < G \in L^p(\mu) \) and that
\[
UG \leq \sum_{i=0}^{\infty} \frac{U^{i+1}g}{(3C)^{i}} < 3CG < \infty \text{ a.e.}\]
on \( X \). Therefore we get
\[
\tau_{0, 2k}(G \cdot w_{p}^{-\frac{1}{p}}) \leq (3CG) \cdot w_{p}^{-\frac{1}{p}} \text{ a.e.}
\]
on \( X \), and
\[
\tau_{0, 2k}^{*}(G^{p-1} \cdot w_{p}^{\frac{1}{p}}) \leq (3CG)^{p-1} \cdot w_{p}^{\frac{1}{p}} \text{ a.e.}
\]
on \( X \). Consequently if we put
\[
w_1 = G^{p-1} \cdot w_{p}^{\frac{1}{p}} \quad \text{and} \quad w_2 = G \cdot w_{p}^{-\frac{1}{p}},
\]
then
\[
w = \left( G^{p-1} \cdot w_{p}^{\frac{1}{p}} \right) \cdot \left( G \cdot w_{p}^{-\frac{1}{p}} \right)^{1-p} = w_1 \cdot w_2^{1-p},
\]
and further by (23) and (22),
\[
\tau_{0, 2k}w_1 \leq (3C)^{p-1}w_1 \quad \text{and} \quad \tau_{0, 2k}w_2 \leq 3Cw_2 \text{ a.e.}
\]
on \( X \).

Next, let \( 1 < p < 2 \). Since \( p' > 2 \) and \( w^{-1/p} = \left( \frac{1}{w^{p-1}} \right)^{1/p'} \), we can apply
the above argument to $p'$ and observe that there exist two functions $w_1$ and $w_2$ such that

$$w_{p-1} = w_1 \cdot w_2^{1-p'}, \quad \tau_{0,2k} w_1 \leq (3C)^{p'-1} w_1 \quad \text{and} \quad \tau_{0,2k}^* w_2 \leq 3C w_2.$$  

Since $w = \left( w_1 \cdot w_2^{1-p'} \right)^{1-p} = w_2 \cdot w_1^{1-p}$, we conclude that, in any case, $w$ has the representation

$$w = w_1 \cdot w_2^{1-p} \quad \text{with} \quad \tau_{0,2k}^* w_1 \leq C w_1 \quad \text{and} \quad \tau_{0,2k}^* w_2 \leq C w_2,$$

where $C$ is a positive constant independent of $k \geq 0$.

If $0 \leq i \leq k$ then we have

$$\sum_{s=0}^{k} \tau^i w_2 \leq (2k+1) \tau^{-i} (\tau_{0,2k}^* w_2) \leq 2C (k+1) \tau^{-i} w_2,$$

whence

$$(25) \quad \sum_{i=0}^{k} \left( (\tau^i w_1) \cdot (\tau^{-i} w_2)^{1-p'} \right) \leq \left( \frac{1}{2C} \cdot \tau_{0,k}^* w_1 \right)^{1-p} \sum_{i=0}^{k} \tau^i w_1.$$

Similarly, since $\sum_{s=0}^{k} \tau^s w_1 \leq 2C (k+1) \tau^{-i} w_1$ for $0 \leq i \leq k$, we get

$$(26) \quad \sum_{i=0}^{k} \left[ (\tau^i w_1)^{1-p'} \cdot \tau^{-i} w_2 \right] \leq \left( \frac{1}{2C} \cdot \tau_{0,k}^* w_1 \right)^{1-p} \sum_{i=0}^{k} \tau^i w_2.$$

Now we use the relations

$$\quad (\tau^i w_1) \cdot (\tau^{-i} w_2)^{1-p'} = \frac{J_{-i}}{h_{-i}} \cdot \Phi^{-i} w_1 \cdot (h_{-i} \Phi^{-i} w_2)^{1-p}$$

$$\quad = h_{-i}^{p'} J_{-i}^{1-p'} \cdot \Phi^{-i} (w_1^{1-p'} w_2) = h_{-i}^{p'} J_{-i}^{1-p'} \cdot \Phi^{-i} w$$

and

$$(\tau^i w_1)^{1-p'} \cdot \tau^{-i} w_2 = h_i^{p'} J_i^{1-p'} \cdot \Phi^i (w_1^{1-p'} w_2)$$

By these together with (25) and (26) we have

$$\sum_{i=0}^{k} h_{-i}^{p'} J_{-i} \Phi^{-i} w \right)^{p-1} \left( \sum_{i=0}^{k} [h_i^{p'} J_i \Phi^i w]^{p-1} \right)^{1-p}$$

$$\leq \left( \frac{1}{2C} \right)^{p-1} \left( \sum_{i=0}^{k} \tau^i w_1 \right) \left( \frac{1}{2C} \right)^{p-1} \left( \sum_{i=0}^{k} \tau^i w_2 \right)^{p-1} \quad \text{a.e.}$$

on $X$, which completes the proof.

Proof of Theorem 2. This is similar to that of Theorem 1, and hence we omit the details.
Remark 1. (i) Let $A = \{x : w(x) = 0\}$ and $B = \{x : w(x) = \infty\}$. Then each of statements (a), (b) and (c) of Theorem 1 implies that $\Phi_{\chi A} \leq \chi A$ and $\Phi_{\chi B} \geq \chi B$. But in general we have $\Phi_{\chi A} \neq \chi A$ and $\Phi_{\chi B} \neq \chi B$. On the other hand, each of statements (a), (b) and (c) of Theorem 2 implies that $\Phi_{\chi A} = \chi A$ and $\Phi_{\chi B} = \chi B$. In this case we may assume without loss of generality that $X = \{x : 0 < w(x) < \infty\}$. Then it follows that $M(\omega d\mu) = M(\mu)$ and

$$\int \frac{\Phi^i w}{w} J_i \cdot (\Phi^i f) \, \omega d\mu = \int f \omega d\mu$$

for all $i \in \mathbb{Z}$ and $0 \leq f \in M(\mu)$. By using this together with Theorem of [16], we could give another proof of Theorem 2.

(ii) For a function $f$ on $\mathbb{Z}$ if we define the function $f^k$ on $\mathbb{Z}$ by

$$f^k(i) = \sup_{n \geq 0} \left| \frac{1}{n+1} \sum_{j=0}^{n} f(i-j) \right|,$$

then it follows clearly that

$$f^{**}(i) \leq f^*(i) + f^k(i) \leq 2 f^{**}(i) \quad (i \in \mathbb{Z}).$$

Using these inequalities together with Lemma A, we could prove that Theorem 1 implies Theorem 2.

Theorem 3. Let $0 \leq w \leq \infty$ on $X$ and let $1 < p < \infty$. If $\tau$ is the linear modulus of an invertible Lamperti operator $T$ on $M(\mu)$, then the following statements hold.

(a) If $\tau$ becomes an operator on $M(\omega d\mu)$ and satisfies $\sup_{n \geq 0} \|\tau_{0,n}\|_{L^p(\omega d\mu)} < \infty$, then for any $f \in L^p(\omega d\mu)$ the limit

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} T^i f$$

exists a.e. on the set $\{x : w(x) > 0\}$.

(b) If $\tau$ becomes an invertible operator on $M(\omega d\mu)$ and satisfies $\sup_{n \geq 0} \|\tau_{-n,n}\|_{L^p(\omega d\mu)} < \infty$, then for any $f \in L^p(\omega d\mu)$ the limit

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{(T^k f - T^{-k} f)/k}{k}$$

exists a.e. on the set $\{x : w(x) > 0\}$.

Proof. (a) By using Theorem 1 it follows from [7] that

$$\lim_{n \to \infty} \frac{1}{n} \tau^{|n| f} = \lim_{n \to \infty} \frac{1}{n} T^n f = 0 \quad \text{a.e.}$$

on the set $\{x : w(x) > 0\}$ for any $f \in L^p(\omega d\mu)$. Since the set $\{g + (f - T f) : Tg = g, f \in L^p(\omega d\mu)\}$ is a dense subspace of $L^p(\omega d\mu)$ by a mean ergodic theorem, we then apply Banach's convergence principle (see e.g. [8]) to infer that (a) holds.
(b) By Remark 1 (i), $T$ and $\tau$ can be considered to be invertible Lamperti operators on $M(\omega d\mu) = M(\mu)$. Thus (b) is a consequence of [19]. The proof is complete.

3. WEIGHTED WEAK TYPE INEQUALITIES AND APPLICATIONS

In this section we assume that an invertible Lamperti operator $T$ on $M(\mu)$ satisfies

\begin{equation}
K_\infty := \sup_{n \in \mathbb{Z}} \|T^n\|_{L^\infty(\mu)} < \infty.
\end{equation}

Hence from (2) we observe that

\begin{equation}
\frac{1}{K_\infty} \leq |h_n| \leq K_\infty \quad \text{a.e.}
\end{equation}

on $X$ for each $n \in \mathbb{Z}$. For $f \in M(\mu)$ we let

\[ M^+(\Phi)f = \sup_{n \geq 0} |\Phi_0, nf| \quad \text{and} \quad M(\Phi)f = \sup_{m, n \geq 0} |\Phi_m, nf|, \]

where

\[ \Phi_{m, n}f = \frac{1}{m + n + 1} \sum_{i=-m}^{n} \Phi^i f. \]

If $\tau$ denotes the linear modulus of $T$, then by (2), (4) and (28) we have

\begin{equation}
\frac{1}{K_\infty} \Phi_{m, n} \leq \tau_{m, n} \leq K_\infty \Phi_{m, n},
\end{equation}

so that

\begin{equation}
\frac{1}{K_\infty} M^+(\Phi) \leq M^+(\tau) \leq K_\infty M^+(\Phi) \quad \text{and}
\end{equation}

\begin{equation}
\frac{1}{K_\infty} M(\Phi) \leq M(\tau) \leq K_\infty M(\Phi).
\end{equation}

Using these relations we first prove the following weighted weak type inequalities.

**Theorem 4.** Let $0 \leq w \leq \infty$ on $X$ and let $1 \leq p < \infty$. If $T$ is an invertible Lamperti operator on $M(\mu)$ satisfying (27) and $\Phi$ has no periodic part, then the following statements are equivalent.

(a) $T$ becomes an operator on $M(\omega d\mu)$ and there exists a positive constant $C$ such that for any $f \in L^p(\omega d\mu)$ and $\lambda > 0$

\begin{equation}
\int_{\{x : M^+(T)f(x) > \lambda\}} w d\mu \leq C \frac{1}{\lambda^p} \int |f|^p w d\mu.
\end{equation}

(b) The linear modulus $\tau$ of $T$ becomes an operator on $M(\omega d\mu)$ and there exists a positive constant $C$ such that for any $f \in L^p(\omega d\mu)$

\[ \sup_{n \geq 0} \int |\tau_{0, n} f|^p w d\mu \leq C \int |f|^p w d\mu. \]
Theorem 5. Let \( 0 \leq w \leq \infty \) on \( X \). If \( T \) is an invertible Lamperti operator on \( M(\mu) \) satisfying (27) and \( \Phi \) has no periodic part, then the following statements are equivalent when \( 1 < p < \infty \), and statements (a) and (b) are equivalent when \( p = 1 \).

(a) \( T \) becomes an invertible operator on \( M(w\mu) \) and there exists a positive constant \( C \) such that for any \( f \in L^p(w\mu) \) and \( \lambda > 0 \)

\[
\int_{\{x : H^*(T)f(x) > \lambda\}} w \, d\mu \leq C \frac{1}{\lambda^p} \int |f|^p w \, d\mu.
\]

(b) The linear modulus \( \tau \) of \( T \) becomes an invertible operator on \( M(w\mu) \) and there exists a positive constant \( C \) such that for any \( f \in L^p(w\mu) \)

\[
\sup_{n \geq 0} \int |\tau_{-n} nf|^p w \, d\mu \leq C \int |f|^p w \, d\mu.
\]

(c) \( T \) becomes an invertible operator on \( M(w\mu) \) and there exists a positive constant \( C \) such that for any \( f \in L^p(w\mu) \)

\[
\int |H^*(T)f|^p w \, d\mu \leq C \int |f|^p w \, d\mu.
\]

Proof of Theorem 4. Let \( 1 < p < \infty \).

(b) \( \Rightarrow \) (a). Since \( |M^+(T)f| \leq M^+(\tau)|f| \) for \( f \in M(\mu) \), this implication is obvious from Theorem 1.

(a) \( \Rightarrow \) (b). By (29) it suffices to prove that

\[
\sup_{n \geq 0} \|\Phi_{0,n}\|_{L^p(w\mu)} < \infty.
\]

To do so, we apply Theorem 1. We see that it is enough to prove the existence of a positive constant \( C \) such that for a.e. \( x \in X \) and all \( k \geq 0 \)

\[
\left( \sum_{i=0}^{k} J_{-i}(x) \Phi^{-i} w(x) \right) \cdot \left( \sum_{i=0}^{k} [J_i(x)\Phi^i w(x)]^{\frac{1}{p} - 1} \right)^{p-1} \leq C(k + 1)^p.
\]

As in the proof of Lemma of [20], we may assume without loss of generality that there exists a one-to-one onto mapping \( S \) from \( X \) to \( X \) such that

(i) \( A \in \mathcal{F} \) if and only if \( SA \in \mathcal{F} \),
(ii) \( \mu(SA) > 0 \) if and only if \( \mu A > 0 \),
(iii) \( \Phi^i f = f \circ S^i \) for all \( i \in \mathbb{Z} \) and \( f \in M(\mu) \).

For simplicity, from now on, we will always assume that the one-to-one onto mapping \( S : X \rightarrow X \) satisfies the above conditions (i), (ii) and (iii).

For an integer \( k \) with \( k \geq 0 \) we define a nonnegative extended real-valued function \( d_k \) on \( X \) by the relation

\[
d_k(x) = \sum_{i=0}^{k} \left[ J_i(x) w(S^i x) \right]^{\frac{1}{p} - 1}.
\]
Write \( D_{-\infty} = \{ x : d_k(x) = 0 \}, \ D_\infty = \{ x : d_k(x) = \infty \}, \) and
\[
D_n = \{ x : 2^n \leq \frac{1}{2(k+1)} d_k(x) < 2^{n+1} \} \quad \text{for} \quad n \in \mathbb{Z}.
\]

Then we have
\[
X = D_{-\infty} \cup D_\infty \cup \left( \bigcup_{n \in \mathbb{Z}} D_n \right);
\]
and it is clear that (34) holds on \( D_{-\infty} \). On the other hand, (a) implies that
\[
\{ x : w(Sx) = 0 \} \subset \{ x : w(x) = 0 \},
\]
and therefore we get
\[
\sum_{i=0}^{k} J_{-i}(x)w(S^{-i}x) = 0 \quad \text{on} \quad D_\infty.
\]

It follows that (34) holds on \( D_\infty \). To prove (34) on each \( D_n, n \in \mathbb{Z} \), we apply the hypothesis that \( \Phi \) has no periodic part. By this hypothesis, \( D_n \) has the form
\[
D_n = \bigcup_{n=1}^{\infty} B_j,
\]
where the \( B_j \) satisfy
\[
B_j \cap S^\ell B_j = \emptyset \quad \text{for} \quad 1 \leq \ell \leq 2(k+1).
\]

Let us fix \( B_j \), and let \( A \) denote a measurable subset of \( B_j \) with \( 0 < \mu A < \infty \).

Then define a function \( f \) on \( X \) by the relation
\[
f(S^i x) = \begin{cases} 
    h_i(x)^{-1} \cdot [J_i(x)w(S^i x)]^{\frac{1}{p-1}} & \text{if} \quad x \in A \quad \text{and} \quad 0 \leq i \leq k \\
    0 & \text{otherwise.}
\end{cases}
\]

Since \( A \subset B_j \subset D_n \) and \( h_{i+j}(S^{-j} x) = h_j(S^{-j} x)h_i(x) \) by (2), it follows that for \( x \in A \) and \( 0 \leq j \leq k \),
\[
M^+(T)f(S^{-j} x) \geq \frac{1}{2(k+1)} \left| \sum_{i=0}^{k} h_{i+j}(S^{-j} x)f(S^{i+j}(S^{-j} x)) \right|
\]
\[
= \frac{1}{2(k+1)} \left| \sum_{i=0}^{k} h_j(S^{-j} x)h_i(x)f(S^i x) \right|
\]
\[
\geq \frac{1}{2(k+1)} \cdot \frac{1}{K_\infty} \sum_{i=0}^{k} [J_i(x)w(S^i x)]^{\frac{1}{p-1}} \quad \text{(by (28))}
\]
\[
= \frac{1}{K_\infty} \cdot \frac{1}{2(k+1)} d_k(x) \geq \left( \frac{1}{K_\infty} \right) \cdot 2^n \quad \text{(by (36))}.
\]

Hence if we set
\[
E(-1) := \bigcup_{i=0}^{k} S^{-i}A \quad \text{and} \quad E(1) := \bigcup_{i=0}^{k} S^iA,
\]

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then

\[ M^+(T)f \geq \left( \frac{1}{K_\infty} \right) 2^n \quad \text{on} \ E(-1). \]

Thus (a) implies that

\[ \int_{E(-1)} w \, d\mu \leq C \left( \frac{K_\infty}{2^n} \right)^p \int |f|^p w \, d\mu, \]

where by the definition of \( f \)

\[ \int |f|^p w \, d\mu = \int_{E(1)} |f|^p w \, d\mu = \sum_{i=0}^{k} \int_{S^i A} |f|^p w \, d\mu \]

\[ = \sum_{i=0}^{k} \int_{A} |f(S^ix)|^p w(S^ix)J_i(x) \, d\mu \quad \text{(by (3))} \]

\[ \leq K_\infty^p \sum_{i=0}^{k} \int_{A} [J_i(x)w(S^ix)]^{\frac{1}{p-1}} \, d\mu \quad \text{(by (28)),} \]

and by (3)

\[ \int_{E(-1)} w \, d\mu = \sum_{i=0}^{k} \int_{S^{-i} A} w \, d\mu = \sum_{i=0}^{k} \int_{A} w(S^{-i}x)J_{-i}(x) \, d\mu. \]

Consequently we get

\[ 2^{np} \int_{A} \sum_{i=0}^{k} J_{-i}(x)w(S^{-i}x) \, d\mu \leq C \cdot K_\infty^{2p} \int_{A} \sum_{i=0}^{k} [J_i(x)w(S^ix)]^{\frac{1}{p-1}} \, d\mu. \]

On the other hand, since \( A \subset B_j \subset D_n \), it follows that

\[ \frac{1}{\mu A} \int_{A} \frac{1}{k+1} \sum_{i=0}^{k} [J_i(x)w(S^ix)]^{\frac{1}{p-1}} \, d\mu \leq 2^{n+2}. \]

Combining this with (40) yields

\[ \left( \frac{1}{\mu A} \int_{A} \frac{1}{k+1} \sum_{i=0}^{k} J_{-i}(x)w(S^{-i}x) \, d\mu \right) \cdot \left( \frac{1}{\mu A} \int_{A} \frac{1}{k+1} \sum_{i=0}^{k} [J_i(x)w(S^ix)]^{\frac{1}{p-1}} \, d\mu \right)^{p-1} \leq C \cdot 2^{2p} K_\infty^{2p}. \]

Since this holds for every \( A \), arbitrary measurable subset of \( B_j \) with positive finite measure, we conclude that for a.e. \( x \in B_j \)

\[ \left( \frac{1}{k+1} \sum_{i=0}^{k} J_{-i}(x)w(S^{-i}x) \right) \cdot \left( \frac{1}{k+1} \sum_{i=0}^{k} [J_i(x)w(S^ix)]^{\frac{1}{p-1}} \right)^{p-1} \leq C \cdot 2^{2p} K_\infty^{2p}, \]
whence (34) holds on $D_n$, $n \in \mathbb{Z}$, and thus (b) has been established.

Let $p = 1$.

(b) $\Rightarrow$ (a). By (29), (b) is equivalent to

$$C = \sup_{n \geq 0} \|\Phi_{0,n}\|_{L^1(wd\mu)} < \infty.$$  

Since (3) implies

$$\int (\Phi_{0,n}f) \cdot w \, d\mu = \int f \cdot \left( \frac{1}{n+1} \sum_{i=0}^{n} J_{-i} \Phi^{-i}w \right) \, d\mu$$

for $0 \leq f \in M(\mu)$, (41) is equivalent to

$$\sup_{n \geq 0} \frac{1}{n+1} \sum_{i=0}^{n} J_{-i}(x)w(S^{-i}x) \leq Cw(x) \quad \text{a.e.}$$

on $X$. Hence, using (3) again, for a.e. $x \in X$ and all $j \in \mathbb{Z}$ we have

$$\sup_{n \geq 0} \frac{1}{n+1} \sum_{i=0}^{n} J_{j-i}(x)w(S^{-i}x) \leq CJ_{j}(x)w(S^{i}x).$$

Let $0 \leq f \in L^1(wd\mu)$. For an $N \geq 0$ we then define

$$f_{\Phi,N}^* = \max_{0 \leq n \leq N} \Phi_{0,n}f = \max_{0 \leq n \leq N} \frac{1}{n+1} \sum_{i=0}^{n} \Phi^{i}f$$

It follows that $f_{\Phi,N}^* \uparrow M^+(\Phi)f$ a.e. on $X$ as $N \to \infty$; and for any $L \geq 0$ we have, by (3),

$$(L+1) \int_{\{x : f_{\Phi,N}(x) > \lambda\}} w \, d\mu = \sum_{i=0}^{L} \int_{\{x : f_{\Phi,N}(S^{i}x) > \lambda\}} J_{i}(x)w(S^{i}x) \, d\mu$$

$$= \int_{\{0 \leq i \leq L : f_{\Phi,N}(S^{i}x) > \lambda\}} J_{i}(x)w(S^{i}x) \, d\mu.$$  

We then apply Lemma A together with (43) to infer that there exists a positive constant $C$ independent of $N$, $L \geq 0$ such that for a.e. $x \in X$

$$\sum_{\{0 \leq i \leq L : f_{\Phi,N}(S^{i}x) > \lambda\}} J_{i}(x)w(S^{i}x) \leq \frac{C}{\lambda} \sum_{i=0}^{L+N} f(S^{i}x)J_{i}(x)w(S^{i}x).$$

Hence

$$\int_{\{x : f_{\Phi,N}(x) > \lambda\}} w \, d\mu \leq \frac{C}{\lambda} \cdot \frac{1}{L+1} \int \sum_{i=0}^{L+N} f(S^{i}x)J_{i}(x)w(S^{i}x) \, d\mu$$

$$= \frac{C}{\lambda} \cdot \frac{L + N + 1}{L + 1} \int fw \, d\mu.$$
By letting $L \uparrow \infty$, and then $N \uparrow \infty$, we see that (a) holds.

(a) $\Rightarrow$ (b). Since $\Phi$ has no periodic part, if $n \geq 0$ is an integer then $X$ has the form

$$X = \bigcup_{j=1}^{\infty} B_j,$$

where the $B_j$ satisfy

$$B_j \cap S^\ell B_j = \emptyset \quad \text{for} \quad 0 \leq \ell \leq n. \quad (45)$$

For the moment let us fix $B_j$, and let $A$ be a measurable subset of $B_j$. If we set

$$F(-1) := \bigcup_{i=0}^{n} S^{-i} A$$

and if $x \in F(-1)$ then by (2), (28) and (45) we have

$$\max_{0 \leq k \leq n} \frac{1}{k+1} \left| \sum_{i=0}^{k} T^i \chi_A(x) \right| \geq \frac{1}{n+1} \cdot \frac{1}{K_\infty}. \quad \text{Therefore by (a)}$$

$$\int_{F(-1)} w \, d\mu \leq C \cdot (n+1) K_\infty \int_A w \, d\mu.$$

Since

$$\int_{F(-1)} w \, d\mu = \sum_{i=0}^{n} \int_{S^{-i} A} w \, d\mu = \sum_{i=0}^{n} \int_A J_{-i}(x) w(S^{-i} x) \, d\mu,$$

we then have

$$\int_A \frac{1}{n+1} \sum_{i=0}^{n} J_{-i}(x) w(S^{-i} x) \, d\mu \leq C \cdot K_\infty \int_A w \, d\mu,$$

which implies, as before, that

$$\frac{1}{n+1} \sum_{i=0}^{n} J_{-i}(x) w(S^{-i} x) \leq CK_\infty w(x) \quad \text{a.e.}$$

on $B_j$ and hence on $X$. Since the constant $CK_\infty$ is independent of $n \geq 0$, this establishes (42) and hence (b).

The proof is complete. \qed

Proof of Theorem 5. Let $1 < p < \infty$.

(c) $\Rightarrow$ (a) is obvious.

(a) $\Rightarrow$ (b). As in the proof of Theorem 4, it suffices to prove that there exists a positive constant $C$ such that

$$\left( \sum_{i=0}^{k} J_i(x) w(S^i x) \right) \cdot \left( \sum_{i=0}^{k} [J_i(x) w(S^i x)]^{p-1} \right)^{p-1} \leq C(k+1)^p \quad (46)$$
for a.e. \( x \in X \) and all \( k \geq 0 \).

To do so, let \( d_k, D_{-\infty}, D_{\infty} \) and \( D_n (n \in \mathbb{Z}) \) be the same as in the proof of Theorem 4 (cf. (35), (36)). Since \( \Phi \) has no periodic part by hypothesis, (a) implies that \( \{ x : w(Sx) = \infty \} = \{ x : w(x) = \infty \} \). Indeed if this is not true, then we can choose an \( E \in \mathcal{F} \), with \( \mu E > 0 \) and \( \int_E wd\mu < \infty \), such that

\[
SE \subset \{ x : w(x) = \infty \} \quad \text{and} \quad S^2(E) \cap (E \cup SE) = \emptyset.
\]

Then the function \( f = \chi_E \) (\( \in L^p(\mu) \)) satisfies \( H^*(T)f(x) \geq 1/K_\infty \) on \( SE \), whence

\[
\int_{\{ x : H^*f(x) > \lambda \}} w d\mu = \infty \quad \text{for all} \quad 0 < \lambda < \frac{1}{K_\infty}.
\]

This is a contradiction. Similarly (a) implies that \( \{ x : w(Sx) = 0 \} = \{ x : w(x) = 0 \} \). Therefore we have

\[
D_{-\infty} = \{ x : w(x) = \infty \}, \quad D_{\infty} = \{ x : w(x) = 0 \}, \quad SD_{-\infty} = D_{-\infty} \quad \text{and} \quad SD_{\infty} = D_{\infty}.
\]

Thus (46) holds clearly on \( D_{-\infty} \cup D_{\infty} \). To prove (46) on each \( D_n, n \in \mathbb{Z} \), we represent \( D_n \) as

\[
D_n = \bigcup_{j=1}^{\infty} B_j,
\]

where the \( B_j \) satisfy

\[
S^\ell B_j \cap B_j = \emptyset \quad \text{for} \quad 1 \leq \ell \leq 4(k + 1).
\]

If \( A \) is a measurable subset of \( B_j \) with \( 0 < \mu A < \infty \), then let

\[
E(1) := \bigcup_{i=0}^{k} S^i A \quad \text{and} \quad E(2) := \bigcup_{i=k+1}^{2k+1} S^i A.
\]

If \( 0 \leq f \in M(\mu) \) and \( \{ x : f(x) \neq 0 \} \subset E(1) \), then define a function \( f^\sim \) on \( X \) by the relation

\[
\begin{cases}
  f^\sim(S^{k+1-i}x) = |\text{sgn } h_{-i}(S^{k+1}x)|^{-1} \cdot f(S^{k+1-i}x) & \text{for } x \in A \text{ and } 1 \leq i \leq k + 1, \\
  f^\sim = 0 & \text{on } X \setminus \bigcup_{i=0}^{k} S^i A,
\end{cases}
\]

where \( \text{sgn } \alpha = \alpha/|\alpha| \) for a complex number \( \alpha \neq 0 \), and \( \text{sgn } 0 = 0 \).

Then for \( x \in A \) and \( k + 1 \leq j \leq 2k + 1 \) we have

\[
H^*(T)f^\sim(S^j x) \geq \left| \sum_{i=1}^{k+1} h_{-i-(j-k-1)}(S^i x) \cdot \frac{f^\sim(S^{k+1-i}x)}{i + (j - k - 1)} \right|.
\]

Since \( h_{j-k-1}(S^{k+1}x) \cdot h_{-i-(j-k-1)}(S^{i-k-1}(S^{k+1}x)) = h_{-i}(S^{k+1}x) \) by (2),

\[
h_{-i-(j-k-1)}(S^i x) = \frac{h_{-i}(S^{k+1}x)}{h_{j-k-1}(S^{k+1}x)}.
\]
Therefore for \( x \in A \) and \( k + 1 \leq j \leq 2k + 1 \) we have

\[
H^*(T)f^\sim(S^j)x \geq \frac{1}{h_{j-k-1}(S^{k+1}x)} \sum_{i=1}^{k+1} \frac{|h_{-i}(S^{k+1}x)| \cdot f(S^{k+1-i}x)}{i + (j - k - 1)} \geq K^{-2}_\infty \cdot \frac{1}{2(k+1)} \sum_{i=0}^{k} f(S^i x) \quad \text{(by (28)).}
\]

In particular, if \( 0 \leq f \in M(\mu) \) is such that

\[
\begin{align*}
&f(S^i x) = [J_i(x)w(S^i x)]^{\frac{1}{p-1}} &\text{for } x \in A \text{ and } 0 \leq i \leq k, \\
&f = 0 &\text{on } X \setminus \bigcup_{i=0}^{k} S^i A,
\end{align*}
\]

then, by (36) and the fact \( A \subset B_j \subset D_n \), we have

\[
H^*(T)f^\sim(S^j)x \geq K^{-2}_\infty \cdot \frac{1}{2(k+1)} \sum_{i=0}^{k} [J_i(x)w(S^i x)]^{\frac{1}{p-1}}
\]

for \( x \in A \) and \( k + 1 \leq j \leq 2k + 1 \).

Thus by (a)

\[
\int_{E(2)} w \, d\mu \leq C \cdot K^{2p}_\infty \frac{1}{2np} \int_{E(1)} f^p w \, d\mu.
\]

Since (3) implies

\[
\int_{E(1)} f^p w \, d\mu = \sum_{i=0}^{k} \int_{S^i A} f^p w \, d\mu = \sum_{i=0}^{k} \int_{A} f^p(S^i x)w(S^i x)J_i(x) \, d\mu,
\]

we can apply the following equations

\[
\sum_{i=0}^{k} f^p(S^i x)w(S^i x)J_i(x) = \sum_{i=0}^{k} [J_i(x)w(S^i x)]^{\frac{1}{p-1}} = d_k(x),
\]

to obtain that

\[
(48) \quad \int_{E(2)} w \, d\mu \leq C \cdot K^{2p}_\infty \frac{1}{2np} \int_{A} d_k(x) \, d\mu.
\]

Next, if \( 0 \leq f \in M(\mu) \) and \( \{x : f(x) \neq 0\} \subset E(2) \), then define a function \( f^\sim \) on \( X \) by the relation

\[
\begin{align*}
&f^\sim(S^k+i)x = [\text{sgn } h_i(S^k x)]^{-1} \cdot f(S^{k+i}x) &\text{for } x \in A \text{ and } 1 \leq i \leq k + 1, \\
&f^\sim = 0 &\text{on } X \setminus \bigcup_{i=k+1}^{2k+1} S^i A.
\end{align*}
\]

Then for \( x \in A \) and \( 0 \leq j \leq k \) we have

\[
H^*(T)f^\sim(S^j)x \geq \sum_{i=1}^{k+1} \frac{h_{i+(k-j)}(S^i x) \cdot f^\sim(S^{k+i}x)}{i + (k - j)}.
\]
Since \( h_{j-k}(S^k x) \cdot h_{i+(k-j)}(S^{j-k}(S^k x)) = h_i(S^k x) \) by (2),
\[
h_{i+(k-j)}(S^j x) = \frac{h_i(S^k x)}{h_{j-k}(S^k x)}.
\]

Hence it follows that
\[
H^*(T)f_\sim(S^j x) \geq \frac{1}{|h_{j-k}(S^k x)|} \cdot \sum_{i=1}^{k+1} \frac{|h_i(S^k x)| \cdot f(S^{k+i} x)}{i + (k - j)} \geq K^{-2}_\infty \cdot \frac{1}{2(k + 1)} \sum_{i=k+1}^{2k+1} f(S^i x) \quad \text{(by (28))}
\]
for \( x \in A \) and \( 0 \leq j \leq k \). In particular, if \( f = \chi_{E(2)} \) then
\[
H^*(T)f_\sim(S^j x) \geq K^{-2}_\infty \cdot \frac{1}{2}
\]
for \( x \in A \) and \( 0 \leq j \leq k \). Thus by (a)
\[
\int_{E(1)} w \, d\mu \leq C \cdot K^{2p}_\infty \cdot 2^p \int_{E(2)} w \, d\mu.
\]
We then use the following equations
\[
\int_{E(1)} w \, d\mu = \sum_{i=0}^{k} \int_{S^i A} w \, d\mu = \sum_{i=0}^{k} \int_{A} w(S^i x) J_i(x) \, d\mu,
\]
to obtain that
\[
\int_{A} \sum_{i=0}^{k} J_i(x) w(S^i x) \, d\mu \leq C \cdot K^{2p}_\infty \cdot 2^p \int_{E(2)} w \, d\mu.
\]
Combining this with (48) yields
\[
\int_{A} \sum_{i=0}^{k} J_i(x) w(S^i x) \, d\mu \leq C^2 \cdot K^{4p}_\infty \cdot \frac{1}{2(n-1)p} \int_{A} d_k(x) \, d\mu.
\]
Since \( 2^n \leq d_k/(k + 1) < 2^{n+1} \) on \( D_n \) and \( A \subset B_j \subset D_n \), it follows that
\[
2^{(n+1)p} \leq \left( \frac{1}{\mu A} \int_{A} \frac{1}{k + 1} d_k(x) \, d\mu \right)^p \leq 2^{(n+2)p}.
\]
Thus we obtain
\[
\left( \frac{1}{\mu A} \int_{A} \frac{1}{k + 1} \sum_{i=0}^{k} J_i(x) w(S^i x) \, d\mu \right) \cdot \left( \frac{1}{\mu A} \int_{A} \frac{1}{k + 1} d_k(x) \, d\mu \right)^{p-1} \leq C^2 \cdot K^{4p}_\infty \cdot 2^{3p},
\]
and therefore
\[
\left( \frac{1}{k+1} \sum_{i=0}^{k} J_i(x)w(S^i x) \right) \cdot \left( \frac{1}{k+1} \sum_{i=0}^{k} [J_i(x)w(S^i x)]^{p-1} \right)^{p-1} \leq C^2 \cdot K_{\infty}^{4p} \cdot 2^{3p} \quad \text{a.e.}
\]
on $B_j$ (and hence on $D_n$). Since the constant $C^2 \cdot K_{\infty}^{4p} \cdot 2^{3p}$ is independent of $k \geq 0$, we have proved (46) and hence (b).

(b) $\Rightarrow$ (c). By Remark 1 (i), we may assume without loss of generality that $X = \{ x : 0 < w(x) < \infty \}$. Then $T$ and $\tau$ can be considered to be invertible Lamperti operators on $M(\omega \mu) = M(\mu)$, whence (b) $\Rightarrow$ (c) follows from Lemma of [19].

Let $p = 1$.

(a) $\Rightarrow$ (b). As in the proof of Theorem 4 (cf. (41), (42)), (b) is equivalent to the existence of a positive constant $C$ such that
\[
(51) \quad \sup_{n \geq 0} \frac{1}{2n+1} \sum_{i=-n}^{n} J_i(x)w(S^i x) \leq Cw(x) \quad \text{a.e.}
\]
on $X$. To prove (51), let $N \geq 1$ be fixed arbitrarily. Since $\Phi$ has no periodic part by hypothesis, $X$ has the form
\[
X = \bigcup_{j=0}^{\infty} B_j,
\]
where the $B_j$ satisfy
\[
B_j \cap S^\ell B_j = \emptyset \quad \text{for} \quad 1 \leq \ell \leq 2N.
\]
If $A$ is a measurable subset of $B_j$ such that $0 < \mu A < \infty$, and if $x \in S^i A$ for some $i$ with $1 < |i| \leq N$, then by (28) we have
\[
H^*(T)\chi_A(x) \geq \frac{1}{K_{\infty}} \cdot \frac{1}{N}.
\]
Hence (a) implies
\[
\sum_{|i|=1}^{N} \int_{S^i A} w \, d\mu \leq C \cdot K_{\infty} \cdot N \int_A w \, d\mu.
\]
We now apply (3) to infer that
\[
\int_A \frac{1}{2N+1} \sum_{i=-N}^{N} J_i(x)w(S^i x) \, d\mu \leq (CK_{\infty} + 1) \int_A w \, d\mu;
\]
therefore
\[
\frac{1}{2N+1} \sum_{i=-N}^{N} J_i(x)w(S^i x) \leq (CK_{\infty} + 1)w \quad \text{a.e.}
\]
on $B_j$ and hence on $X$, completing the proof of (51).
(b) $\Rightarrow$ (a). By (51) and (3), we have

$$
\frac{1}{2n+1} \sum_{i=-n}^{n} J_{j+i}(x)w(S^{j+i}x) \leq C J_j(x)w(S^jx)
$$

for a.e. $x \in X$ and all $j \in \mathbb{Z}$ and $n \geq 0$. For an $N \geq 1$ we then define the truncated maximal operator $H_N^*(T)$ on $M(\mu)$ by the relation

$$
H_N^*(T)f = \max_{1 \leq n \leq N} \left| \sum_{k=-n}^{n} T^k f \right|
$$

where the prime means that the term with zero denominator is omitted.

Clearly we have

$$
H_N^*(T)f(x) \uparrow H^*(T)f(x) \quad \text{a.e.}
$$
on $X$ as $N \to \infty$. If $j \in \mathbb{Z}$, then

$$
|h_j(x)|H_N^*(T)f(S^jx) = \max_{1 \leq n \leq N} \left| \sum_{k=-n}^{n} \frac{h_j(x)h_k(S^jx)f(S^{j+k}x)}{k} \right|
$$

$$
= \max_{1 \leq n \leq N} \left| \sum_{k=-n}^{n} \frac{h_{j+k}(x)f(S^{j+k}x)}{k} \right| \quad \text{(by (2))},
$$

so that

$$
H_N^*(T)f(S^jx) = \frac{1}{|h_j(x)|} \cdot \max_{1 \leq n \leq N} \left| \sum_{k=-n}^{n} \frac{h_{j+k}(x)f(S^{j+k}x)}{k} \right|
$$

$$
\leq K_\infty \cdot \max_{1 \leq n \leq N} \left| \sum_{k=-n}^{n} \frac{h_{j+k}(x)f(S^{j+k}x)}{k} \right|.
$$

By this together with (3) we observe that for $L \geq 1$ and $\lambda > 0$

$$
(2L+1) \int_{\{x : H_N^*(T)f(x) > \lambda \}} w \ d\mu = \sum_{j=-L}^{L} \int_{\{x : H_N^*(T)f(S^jx) > \lambda \}} J_j(x)w(S^jx) \ d\mu
$$

$$
= \int_{\{-L \leq j \leq L : H_N^*(T)f(S^jx) > \lambda \}} J_j(x)w(S^jx) \ d\mu
$$

$$
= \int_{\{-L \leq j \leq L : \max_{1 \leq n \leq N} \left| \sum_{k=-n}^{n} \frac{h_{j+k}(x)f(S^{j+k}x)}{k} \right| > \lambda/K_\infty \}} J_j(x)w(S^jx) \ d\mu.
$$

Next we apply (52) together with a known result about the classical discrete Hilbert transform (see e.g. Theorem 10 of [10]) to infer that there exists a
positive constant $C$ such that

$$\sum_{L \leq j \leq L: \max_{1 \leq n \leq N} \left| \sum_{t=-n}^{n} h_{j+k}(s_j x) \right| > \lambda/K_\infty} J_j(x) w(S^j x)$$

$$\leq C \frac{K_\infty}{\lambda} \sum_{j=-N-L}^{N+L} |h_j(x) f(S^j x)| \cdot J_j(x) w(S^j x)$$

for a.e. $x \in X$ and all $\lambda > 0$ and $N, L \geq 1$. Thus by (28) and (3)

$$(2L + 1) \int_{\{x: H^*_N(T)f(x) > \lambda\}} w \, d\mu$$

$$\leq \int_X C \cdot \frac{K_\infty}{\lambda} \cdot \left( \sum_{j=-N-L}^{N+L} |h_j(x) f(S^j x)| \cdot J_j(x) w(S^j x) \right) \, d\mu$$

$$\leq C \cdot \frac{K_\infty^2}{\lambda} \int_X \sum_{j=-N-L}^{N+L} |f(S^j x)| \cdot J_j(x) w(S^j x) \, d\mu$$

$$= C \cdot \frac{K_\infty^2}{\lambda} \cdot (2N + 2L + 1) \int_X |f| w \, d\mu.$$ 

Letting $L \uparrow \infty$ yields

$$\int_{\{x: H^*_N(T)f(x) > \lambda\}} w \, d\mu \leq C \cdot \frac{K_\infty^2}{\lambda} \int_X |f| w \, d\mu.$$ 

Hence (a) follows from (53), and this completes the proof of Theorem 5. □

**Remark 2.** The hypothesis that $\Phi$ has no periodic part was used only in the proof of implication (a) $\Rightarrow$ (b) of Theorems 4 and 5. Thus, without this hypothesis, implication (b) $\Rightarrow$ (a) of Theorem 4 and implications (b) $\Rightarrow$ (c) $\Rightarrow$ (a) of Theorem 5 are true.

In the remainder of the paper we investigate the a.e. convergence of the ergodic sequence $\{T^n f\}$ and the ergodic partial sums $\{\sum_{k=1}^n (T^k f - T^{-k} f)/k\}$ in the sense of Cesàro-$\alpha$ means. For the basic properties of Cesàro-$\alpha$ means we refer the reader to Zygmund [24].

Following [4], for a real number $\alpha$ with $-1 < \alpha \leq 0$ we write

$$R_{n,1+\alpha}(T)f = \frac{1}{A_{n+1+\alpha}} \sum_{k=0}^{n} A_{n-k}^\alpha T^k f$$

and

$$H_{n,\alpha}(T) = \frac{1}{A_{n+1}^\alpha} \sum_{k=1}^{n} A_{n+1-k}^\alpha \left( \frac{T^k f - T^{-k} f}{k} \right).$$
where the Cesàro numbers $A_n^\beta$ are given as

$$A_n^\beta = \frac{(\beta + 1) \ldots (\beta + n)}{n!} \quad \text{and} \quad A_0^\beta = 1.$$ 

Two maximal operators $M_{1+\alpha}^+(T)$ and $H_\alpha^*(T)$ on $M(\mu)$ are defined by the relations

$$M_{1+\alpha}^+(T)f = \sup_{n \geq 0} |R_{n,1+\alpha}(T)f|$$

and

$$H_\alpha^*(T)f = \sup_{n \geq 0} |H_{n,\alpha}(T)f|.$$ 

Note that $M_1^+(T)f = M^+(T)f$ and $H_0^*(T)f = H^*(T)f$. In the theorems below we use the Lorentz spaces $L_{r,1}(wd\mu)$ with $1 \leq r < \infty$. Recall that $f \in L_{r,1}(wd\mu)$ if and only if

$$\|f\|_{r,1;wd\mu} := \int_0^\infty \left( \int_{\{x : |f(x)| > t\}} wd\mu \right)^{1/r} dt < \infty,$$

that $\|\chi_E\|_{1;wd\mu} = \left( \int_E wd\mu \right)^{1/r}$ for $E \in \mathcal{F}$ with $\int_E wd\mu < \infty$, and that $L_{r,1}(wd\mu) \subset L_{r,r}(wd\mu) = L^r(wd\mu)$. These properties of Lorentz spaces are explained in Hunt [9].

**Theorem 6.** Let $0 \leq w \leq \infty$ on $X$ and let $1 \leq p < \infty$. If $T$ is an invertible Lamperti operator on $M(\mu)$ satisfying (27) and if the linear modulus $\tau$ of $T$ becomes an operator on $M(wd\mu)$ and satisfies

$$\sup_{n \geq 0} \|\tau_0,n\|_{L^p(wd\mu)} < \infty,$$  \hfill (55)

then the following statements hold.

(a) When $1 < p \leq r < \infty$, the limit

$$\lim_{n \to \infty} R_{n,p/r}(T)f$$

exists a.e. on the set $\{x : w(x) > 0\}$ for all $f \in L^r(wd\mu)$; further there exists a positive constant $C$ such that

$$\|M_{p/r}^+(T)f\|_{L^r(wd\mu)} \leq C \|f\|_{L^r(wd\mu)}$$  \hfill (56)

for all $f \in L^r(wd\mu)$.

(b) When $1 = p \leq r < \infty$, the limit

$$\lim_{n \to \infty} R_{n,1/r}(T)f$$

exists a.e. on the set $\{x : w(x) > 0\}$ for all $f \in L_{r,1}(wd\mu)$.

**Theorem 7.** Let $0 \leq w \leq \infty$ on $X$ and let $1 \leq p < \infty$. If $T$ is an invertible Lamperti operator on $M(\mu)$ satisfying (27) and if the linear modulus $\tau$ of $T$ becomes an invertible operator on $M(wd\mu)$ and satisfies

$$\sup_{n \geq 0} \|\tau_{n,n}\|_{L^p(wd\mu)} < \infty,$$  \hfill (57)
then the following statements hold.  
(a) When \(1 < p \leq r < \infty\), the limit
\[
\lim_{n \to \infty} H_{n,(p/r)-1}(T)f
\]
exists a.e. on the set \(\{x : w(x) > 0\}\) for all \(f \in L^r(w\mu)\); further there exists a positive constant \(C\) such that
\[
\|H^*_{(p/r)-1}(T)f\|_{L^r(w\mu)} \leq C \|f\|_{L^r(w\mu)}
\]
for all \(f \in L^r(w\mu)\).

(b) When \(1 = p \leq r < \infty\), the limit
\[
\lim_{n \to \infty} H_{n,(1/r)-1}(T)f
\]
exists a.e. on the set \(\{x : w(x) > 0\}\) for all \(f \in L_{r,1}(w\mu)\).

**Proof of Theorem 6.** (a) By (29), \(\Phi\) becomes an operator on \(M(w\mu)\) and satisfies
\[
\sup_{n \geq 0} \|\Phi_{0,n}\|_{L^p(w\mu)} < \infty,
\]
whence we can apply Theorem 1 together with (28) to infer that there exists a positive constant \(C\) such that
\[
\left( \sum_{i=0}^{k} |h_{-i}(x)|^{-r} J_{-i}(x) w(S^{-i}x) \right) \cdot \left( \sum_{i=0}^{k} |h_{i}(x)|^{-r} J_{i}(x) w(S^{i}x) \right)^{\frac{1}{r-1}} \leq C (k+1)^p
\]
for a.e. \(x \in X\) and all \(k \geq 0\). Since \(0 < p/r \leq 1\) and \(1 < p = (p/r)r\), it follows from [15] (cf. especially the proofs of Corollary 3.4 and Theorem 3.1 of [15]) that
(i) the limit \(\lim_{n \to \infty} R_{n,p/r}(\tau)f\) exists a.e. on the set \(\{x : w(x) > 0\}\) for all \(f \in L^r(w\mu)\), and
(ii) the maximal operator \(M^+_{p/r}(\tau)\) is bounded in \(L^r(w\mu)\).

Since \(0 \leq M^+_{p/r}(T)f \leq M^+_{p/r}(\tau)|f|\) for \(f \in L^r(w\mu)\), (56) holds. And the a.e. convergence of \(R_{n,p/r}(T)f\) on the set \(\{x : w(x) > 0\}\) follows from Banach’s convergence principle, because \(\{g + (f - Tf) : Tg = g, f \in L^r(w\mu)\}\) is a dense subspace of \(L^r(w\mu)\) by a mean ergodic theorem, and for \(f \in L^r(w\mu)\) we have
\[
\lim_{n \to \infty} R_{n,p/r}(T)[f - Tf] = 0 \quad \text{a.e.}
\]
on the set \(\{x : w(x) > 0\}\). Indeed (59) holds for \(f\) of the form \(f = \chi_E\) by the proof of Proposition 3.2 of [15], and thus an approximation argument together with (56) can be used to see that (59) holds for any \(f \in L^r(w\mu)\).

(b) Let \(r < s < \infty\), where \(1 = p \leq r < \infty\). Then \(p = 1 < s/r \leq s\), and the Marcinkiewicz interpolation theorem implies that
\[
\sup_{n \geq 0} \|\tau_{0,n}\|_{L^{s/r}(w\mu)} < \infty.
\]
Since \(1 < s/r \leq s\), we then apply (a) to infer that the limit \(\lim_{n \to \infty} R_{n,1/r}(T)f\) exists a.e. on the set \(\{x : w(x) > 0\}\) for all \(f \in L^s(w \mu)\).

Since the Lorentz space \(L_{r,1}(w \mu)\) is a Banach space and \(L^s(w \mu) \cap L_{r,1}(w \mu)\) is a dense subspace of \(L_{r,1}(w \mu)\), it is enough to prove by the Banach convergence principle that

\[
M_{1/r}^+(T)f < \infty \quad \text{a.e.}
\]

on the set \(\{x : w(x) > 0\}\) for all \(f \in L_{r,1}(w \mu)\). By (29) and (4) it suffices to prove the following weak type inequality:

(W) There exists a positive constant \(C\) such that

\[
\int_{\{x : M_{1/r}^+(\Phi)f(x) > \lambda\}} w \, d\mu \leq C \frac{1}{\lambda^r} \|f\|_{r,1; w \mu}^r
\]

for all \(f \in L_{r,1}(w \mu)\) and \(\lambda > 0\).

If \(r = 1\) then, since \(\Phi\) satisfies (41), (W) follows from Theorem 4 (cf. also Remark 2).

If \(1 < r < \infty\) then, by the proof of Theorem 3.13 of Chapter V of [22], it suffices to prove the existence of a positive constant \(C\) such that

\[
\int_{\{x : M_{1/r}^+(\Phi)\chi_E(x) > \lambda\}} w \, d\mu \leq C \frac{1}{\lambda^r} \int_{E} w \, d\mu
\]

for all \(E \in \mathcal{F}\) and \(\lambda > 0\). To do so, we adapt the argument of Bernardis and Martín-Reyes [4] as follows.

Let \(f = \chi_E\), where \(E \in \mathcal{F}\). If we define, for an \(N \geq 1\),

\[
M_{1/r}^+(\Phi)N \chi_E(x) = \sup_{0 \leq n \leq N} \left\{ \frac{1}{A_n^{1/r}} \sum_{k=0}^{n} A_{n-k}^{(1/r)-1} \chi_E(S^k x) \right\}
\]

then \(M_{1/r}^+(\Phi)N \chi_E \uparrow M_{1/r}^+(\Phi)\chi_E\) a.e. on \(X\) as \(N \to \infty\). For the moment let us fix an \(N \geq 1\). If we set

\[
A := \{x : M_{1/r}^+(\Phi)N \chi_E(x) > \lambda\},
\]

then by (3)

\[
(L + 1) \int_A w \, d\mu = \int \sum_{i=0}^{L} \chi_A(S^i x)w(S^i x)J_i(x) \, d\mu
\]

\[
= \int \sum_{\{0 \leq i \leq L : M_{1/r}^+(\Phi)N \chi_E(S^i x) > \lambda\}} J_i(x)w(S^i x) \, d\mu.
\]

On the other hand, we know (cf. (41), (42), (43)) that there exists a positive constant \(C\) such that

\[
\sup_{n \geq 0} \frac{1}{n+1} \sum_{i=0}^{n} J_{j-i}(x)w(S^{j-i} x) \leq C \cdot J_j(x)w(S^j x)
\]
for a.e. \( x \in X \) and all \( j \in \mathbb{Z} \). Thus by Lemma 2.6 and Theorem E of [4] there exists a positive constant \( C \) such that

\[
\sum_{\{0 \leq i \leq L: M_{1/(\mu)}(\Phi)_{N \chi_{E}}(S^i x) > \lambda\}} J_i(x) w(S^i x) < C \frac{1}{\lambda^r} \left( \int_0^\infty \left[ \sum_{\{0 \leq i \leq N+L: \chi_{E}(S^i x) > t\}} J_i(x) w(S^i x) \right]^{1/r} dt \right)^r.
\]

Therefore we have

\[
(L + 1) \int_A w \, d\mu \leq C \frac{1}{\lambda^r} \int_X \left( \int_0^1 \left[ \sum_{\{0 \leq i \leq N+L: \chi_{E}(S^i x) > t\}} J_i(x) w(S^i x) \right]^{1/r} dt \right)^r d\mu
\]

\[
\leq C \frac{1}{\lambda^r} \int_X \left( \sum_{i=0}^{N+L} J_i(X) w(S^i x) \chi_{E}(S^i x) \right) d\mu \quad \text{(by Hölder's inequality)}
\]

\[
= C \frac{1}{\lambda^r} (N + L + 1) \int_E w \, d\mu \quad \text{(by (3)).}
\]

Letting \( L \uparrow \infty \) and then \( N \uparrow \infty \), we see that (61) holds, and this completes the proof of Theorem 6.

**Proof of Theorem 7.** By (57) we may assume without loss of generality that \( X = \{ x : 0 < w(x) < \infty \} \). Then \( T \) and \( \tau \) can be regarded as invertible Lamperti operators on \( M(w \mu) = M(\mu) \).

Let \( p \leq r < \infty \). Then by the Marcinkiewicz interpolation theorem

\[
\sup_{n \geq 0} \| \tau_{n,n} \|_{L^r(w \mu)} < \infty.
\]

Hence \( T \) becomes a bounded and invertible operator on \( L^r(w \mu) \). Let \( \tau_{p/r} \) denote the invertible (positive) Lamperti operator on \( M(w \mu) = M(\mu) \) defined by the relation

\[
\tau_{p/r} f = |h_1|^{r/p} \cdot \Phi f.
\]

Then we have

\[
\tau_{p/r}^i f = |h_i|^{r/p} \cdot \Phi^i f = |h_i|^{(r-p)/p} \cdot \tau^i f \quad (i \in \mathbb{Z})
\]

and by (28)

\[
\tau_{p/r}^i \leq K_{\infty}^{(r-p)/p} \cdot \tau^i \quad (i \in \mathbb{Z}).
\]

Thus

\[
\sup_{n \geq 0} \left\| \frac{1}{2n + 1} \sum_{i=-n}^n \tau_{p/r}^i \right\|_{L^p(w \mu)} < \infty.
\]
Since $0 < p/r \leq 1$ and $p = (p/r)r$, (a) now follows from [5] when $1 < p < r < \infty$, and from [19] when $1 < p = r < \infty$. (b) is a consequence of Theorem 1.4 of [4]. \hfill \Box

Remark 3. (i) In statement (b) of Theorems 6 and 7, the function $f$ in $L_{\theta, 1}(wd\mu)$ cannot be replaced by a function in $L'(wd\mu)$ when $1 = p < r < \infty$. In fact, if we consider an ergodic invertible measure preserving transformation $\phi$ on a nonatomic probability measure space $(X, \mathcal{F}, \mu)$ and an operator $T$ on $M(\mu)$ of the form $Tf = f \circ \phi$, then clearly $\|T^n\|_{L_p(\mu)} = 1$ for all $n \in \mathbb{Z}$ and $1 \leq p \leq \infty$. Dénier proved in [6] that if $1 < r < \infty$ then there exists a function $f \in L'(\mu)$ for which the a.e. convergence of the sequence $\{R_{n, 1/r}Tf(x)\}_{n=0}^{\infty}$ fails to hold. Later, modifying the idea of Dénier [6], Bernardis, Martín-Reyes and Sarrión Gavilán proved in [5] that if $1 < r < \infty$ then there exists an $f \in L'(\mu)$ for which the a.e. convergence of the sequence $\{H_{n, 1/r-1}Tf(x)\}_{n=1}^{\infty}$ fails to hold.

(ii) Statement (b) of Theorem 6 is not true if the hypothesis (27) is omitted. A counterexample can be found in [4].

(iii) Statement (b) of Theorem 7 is not true at least for the case $1 = p = r$ if the hypothesis (27) is omitted. This can be seen from [19].

4. CONCLUDING REMARKS

The purpose of this section is to prove the following weighted ergodic theorem, without assuming that $T$ satisfies (27).

Theorem 8. Let $0 \leq w \leq \infty$ on $X$ and let $1 < p < \infty$. Then the following statements hold for an invertible Lamperti operator $T$ on $M(\mu)$.

(a) If $T$ is an operator on $M(wd\mu)$ and satisfies

\[ K^+(p) := \sup_{n \geq 0} \|T^n\|_{L_p(wd\mu)} < \infty, \]

then for any $r$ with $1/p < r \leq 1$ the limit

\[ \lim_{n \to \infty} R_{n, r}Tf \]

exists a.e. on the set $\{x : w(x) > 0\}$ for every $f \in L^p(wd\mu)$; and the maximal operator $M^+_r(T)$ is bounded in $L^p(wd\mu)$.

(b) If $T$ is an invertible operator on $M(wd\mu)$ and satisfies

\[ K(p) := \sup_{n \in \mathbb{Z}} \|T^n\|_{L_p(wd\mu)} < \infty, \]

then for any $r$ with $1/p < r \leq 1$ the limit

\[ \lim_{n \to \infty} H_{n, r-1}Tf \]

exists a.e. on the set $\{x : w(x) > 0\}$ for every $f \in L^p(wd\mu)$; and the maximal operator $H^*_{r-1}(T)$ is bounded in $L^p(wd\mu)$.

Remark 4. In the above theorem we cannot take $r = 1/p$. See Remark 3 (i).
Proof of Theorem 8. (a) If $\tau_r$ denotes the invertible Lamperti operator on $M(\mu)$ defined by
\[
\tau_r f = |h_1|^{1/r} \cdot \Phi f,
\]
then we have
\[
\tau_r^i f(x) = |h_i(x)|^{1/r} \cdot \Phi^i f(x) \quad (i \in \mathbb{Z}).
\]
If $0 \leq f \in M(\mu)$ then, since $rp > 1$, it follows from H"{o}lder's inequality that
\[
\left( \frac{1}{n+1} \sum_{i=0}^n \tau_r^i f \right)^{rp} \leq \frac{1}{n+1} \sum_{i=0}^n (\tau_r^i f)^{rp} = \frac{1}{n+1} \sum_{i=0}^n [h_i \cdot \Phi^i(f^r)]^p = \frac{1}{n+1} \sum_{i=0}^n [\tau_r^i(f^r)]^p,
\]
whence
\[
\int_X \left( \frac{1}{n+1} \sum_{i=0}^n \tau_r^i f \right)^{rp} \cdot w \, d\mu \leq \frac{1}{n+1} \sum_{i=0}^n \int_X [\tau_r^i(f^r)]^p \cdot w \, d\mu \leq (K^+(p))^p \cdot \int f^{rp} \cdot w \, d\mu.
\]
Therefore $\tau_r$ becomes an operator on $M(\mu)$ and satisfies
\[
\sup_{n \geq 0} \left\| \frac{1}{n+1} \sum_{i=0}^n \tau_r^i \right\|_{L^{rp}(\mu)} < \infty.
\]
Thus by Theorem 1 there exists a positive constant $C$ such that
\[
\left( \sum_{i=0}^k |h_{-i}(x)|^{-p} J_{-i}(x) w(S^{-i}x) \right) \cdot \left( \sum_{i=0}^k |h_i(x)|^{-p} J_i(x) w(S^i x) \right)^{1/(rp-1)} \leq C (k+1)^{rp},
\]
for a.e. $x \in X$ and all $k \geq 0$. Since $0 < r \leq 1$ and $1 < rp$, it follows from [15], as in the above proof of (a) of Theorem 6, that

(i) the limit $\lim_{n \to \infty} R_{n, r}(f)$ exists a.e. on the set $\{ x : w(x) > 0 \}$ for every $f$ in $L^p(\mu)$, where $\tau$ is the linear modulus of $T$, and

(ii) the maximal operator $M_r^+(T)$ is bounded in $L^p(\mu)$.

Thus (a) follows similarly, as in (a) of Theorem 6.

(b) We may assume as before that $X = \{ x : 0 < w(x) < \infty \}$, and hence $T$ can be considered to be an invertible Lamperti operator on $M(\mu) = M(\mu)$. As in (a), we observe that
\[
\sup_{n \geq 0} \left\| \frac{1}{2n+1} \sum_{i=-n}^n \tau_r^i \right\|_{L^{rp}(\mu)} \leq K(p)^{1/r}.
\]
Thus (b) follows from [5] when $1/p < r < 1$, and from [19] when $1/p < r = 1$.

This completes the proof of Theorem 8. \qed

The next proposition may be considered to be a supplementary result to Theorem 1.

**Proposition.** Let $0 \leq w \leq \infty$ on $X$ and let $1 \leq p < \infty$. Then the following statements hold for an invertible Lamperti operator $T$ on $M(\mu)$.

(a) $T$ becomes an operator on $M(\mu w)$ and satisfies the norm condition

$$K^+(p) := \sup_{n \geq 0} \| T^n \|_{L^p(\mu w)} < \infty$$

if and only if there exists a positive constant $C$ such that for a.e. $x \in X$ and all $n \geq 0$

$$|h_{-n}(x)|^{-p} J_{-n}(x) \Phi^{-n} w(x) \leq C w(x). \tag{64}$$

(b) The linear modulus $\tau$ of $T$ becomes an operator on $M(\mu w)$ and satisfies the norm condition

$$\sup_{n \geq 0} \| \tau_0, n \|_{L^1(\mu w)} < \infty$$

if and only if there exists a positive constant $C$ such that for a.e. $x \in X$ and all $n \geq 0$

$$\frac{1}{n+1} \sum_{i=0}^{n} |h_{-i}(x)|^{-1} J_{-i}(x) \Phi^{-i} w(x) \leq C w(x). \tag{65}$$

**Proof.** (a) By (4) we may assume without loss of generality that $T$ is positive. Then for $0 \leq f \in M(\mu)$ and $n \geq 0$ we have, by (2) and (3),

$$\| T^n f \|_{L^p(\mu w)}^p = \int (T^n f)^p \cdot w \, d\mu = \int f^p \cdot (|h_{-n}|^{-p} J_{-n} \Phi^{-n} w) \, d\mu. \tag{66}$$

Thus (64) implies that $T$ becomes an operator on $M(\mu w)$ and satisfies the norm condition: $K^+(p) < \infty$. Conversely if $T$ is an operator on $M(\mu w)$ and satisfies the norm condition: $K^+(p) < \infty$, then for $f = \chi_A$ with $A \in \mathcal{F}$ we have by (66)

$$\int_A (|h_{-n}|^{-p} J_{-n} \Phi^{-n} w) \, d\mu = \int (T^n \chi_A)^p w \, d\mu$$

$$\leq \| T^n \|_{L^p(\mu w)}^p \cdot \int_A w \, d\mu \leq (K^+(p))^p \cdot \int_A w \, d\mu.$$

This completes the proof of (a).

(b) We may assume, as above, that $\tau = T$. Then for $0 \leq f \in M(\mu)$ and
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$n \geq 0$ we have, using (66) with $p = 1$, that

\begin{equation}
\|\tau_0, nf\|_{L^1(\omega d\mu)} = \int (\tau_0, nf) \cdot w \, d\mu = \int f \cdot \left( \frac{1}{n + 1} \sum_{i=0}^{n} |h_{-i}|^{-1} J_{-i} \Phi^{-i} w \right) \, d\mu.
\end{equation}

Thus (65) implies that $\tau$ becomes an operator on $M(\omega d\mu)$ and satisfies the norm condition.

Conversely if $\tau$ is an operator on $M(\omega d\mu)$ and satisfies

$$C := \sup_{n \geq 0} \|\tau_0, n\|_{L^1(\omega d\mu)} < \infty,$$

then for $f = \chi_A$ with $A \in \mathcal{F}$ we have

\begin{equation}
\int_A \left( \frac{1}{n + 1} \sum_{i=0}^{n} |h_{-i}|^{-1} J_{-i} \Phi^{-i} w \right) \, d\mu = \int (\tau_0, n\chi_A) \cdot w \, d\mu \\
\leq \|\tau_0, n\|_{L^1(\omega d\mu)} \cdot \int_A w \, d\mu \leq C \int_A \omega d\mu.
\end{equation}

Hence (65) follows, and the proof is complete. \hfill \Box

REFERENCES