On Morava K-Groups of Stunted Projective Spaces

Y. M. Yang

*Kyoto University

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1. INTRODUCTION

Let \( k(n)^*(-) \) be the connective Morava \( K \)-theory with coefficient ring \( k(n)^* = \mathbb{Z}/2[v_n] \) and let \( K(n)^*(-) \) be the Morava \( K \)-theory with coefficient ring \( K(n)^* = \mathbb{Z}/2[v_n, v_n^{-1}] \), where \( |v_n| = -2(2^n - 1) \). In this paper we determine the module structure of \( k(n)^*(RP^l_{m+1}) \) and the algebra structure of \( k(n)^*(RP^l) \) over \( k(n)^* \) at the prime 2. Here, the symbol \( RP^l_{m+1} \) denotes a stunted real projective space \( RP^l/RP^m \) (\( 0 \leq m < l \leq \infty \)).

Our principal tool for computing \( k(n)^*(RP^l_{m+1}) \) is the Atiyah-Hirzebruch spectral sequence (AHSS)

\[
E_2^{s,t} = H^s(RP^l_{m+1}; k(n)^*) \Rightarrow k(n)^*(RP^l_{m+1})
\]

and we use the result of Yagita([1], Lemma 2.1) for further computation.

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2. STATEMENT OF RESULTS AND PROOFS

In this section, we determine the module structure of the Morava \( K \)-theory of the stunted real projective space and the algebra structure of the Morava \( K \)-theory of the real projective space over \( k(n)^* \) at the prime 2.

The following lemma is needed for computation of the differentials of AHSS for \( k(n)^*(RP^\infty_{m+1}) \).

**Lemma 2.1 [Yagita]** Let \( X \) be a CW-complex. Let \( E_r \) be the Atiyah-Hirzebruch spectral sequence for \( k(n)^*(X) \)

\[
E_2^{s,t} = H^s(X; k(n)^t) \Rightarrow k(n)^{s+t}(X)
\]

and its differential be \( d_r : E_r^{s,t} \to E_r^{s+r, t-r+1} \). Let \( u \in H^s(X) \), then \( d_r(u) = 0 \) for \( r < 2^{n+1} - 1 \) and

\[
d_{2^{n+1}-1}(u) = Q_n(u) \otimes v_n.
\]
where \( Q_0 = S^q, Q_n = S^q Q^{n-1} + Q^{n-1} S^q \) (\( n \geq 1 \)).

First we determine differentials of AHSS for \( k(n)^* (R(P^\infty_{m+1})_{m+1}) \)

\[ E_2^{p,q}(R(P^\infty_{m+1})_{m+1}) = H^p(R(P^\infty_{m+1})_{m+1}; k(n)^q) \rightarrow k(n)^{p+q}(R(P^\infty_{m+1})_{m+1}). \]

Let us consider the following cofibration

\[ R^m \rightarrowtail R^\infty \rightarrowtail \frac{R^\infty}{R^m} = R^\infty_{m+1} \]

from which we get the following exact sequence

\[ 0 \rightarrowtail \tilde{H}^*(R^\infty_{m+1}) \rightarrowtail \tilde{H}^*(R^\infty) \rightarrowtail \tilde{H}^*(R^\infty_m) \rightarrowtail 0. \]

Throughout this paper all cohomology groups have \( Z/2 \) coefficients unless otherwise stated. Here we recall that \( H^*(R^\infty) = Z/2[u] \) where \( u \in H^1(R^\infty) \) and, via \( \pi^* \), we may identify \( \tilde{H}^*(R^\infty_{m+1}) \) with the submodule of \( \tilde{H}^*(R^\infty) \) generated by \( u^{m+1}, u^{m+2}, \ldots \).

**Proposition 2.2** In the AHSS for \( k(n)^* (R(P^\infty_{m+1})_{m+1}) \),

1. \( d_{2^{n+1}-1}(u^{2i}) = 0 \) for all \( 2i \geq m+1 \).
2. \( d_{2^{n+1}-1}(u^{2j+1}) = v_n u^{2j+2^{n+1}} \) for all \( 2j+1 \geq m+1 \).
3. There are isomorphisms

\[ \tilde{E}_2^{s,*} \cong \cdots \cong \tilde{E}_\infty^{s,*} = k(n)^* \{u^{2i} \mid m+1 \leq 2i \leq m+2^{n+1} - 1\} \]

\[ \oplus k(n)^* \{u^{2k} \mid m+2^{n+1} \leq 2k\}/(v_n). \]

**Proof.** By induction on \( n \) we can easily compute the action of \( Q_n \) on \( H^*(R^\infty) \) and hence on \( H^*(R(P^\infty_{m+1})_{m+1}) \). In fact we have \( Q_n(u^{2i}) = 0 \) and \( Q_n(u^{2i+1}) = u^{2i+2^{n+1}} \). Thus by the lemma 2.1 we get

\[ d_{2^{n+1}-1}(u^{2i}) = 0 \quad \text{for any} \quad 2i \geq m+1 \]

and

\[ d_{2^{n+1}-1}(u^{2j+1}) = v_n u^{2j+2^{n+1}} \quad \text{for any} \quad 2j+1 \geq m+1. \]

This proves 1 and 2.

Next we prove 3. From (1), \( v^a u^{2i} \) is cycle for any \( a \geq 0 \) and \( 2i \geq m+1 \). If \( a = 0 \), then \( u^{2i} \) can not be boundary. If \( a \geq 1 \) and \( m+1 \leq 2i \leq m+2^{n+1} - 1 \), then for dimensional reason, \( v^a u^{2i} \) can not be boundary. If \( a \geq 1 \) and \( 2i \geq m+2^{n+1} \), then since \( v^a u^{2i} = d_{2^{n+1}-1}(v^{a-1} u^{2i-2^{n+1}+1}) \), from (2), \( v^a u^{2i} \) is boundary. Therefore \( \tilde{E}_{2^{n+1}} \cong k(n)^* \{u^{2i} \mid m+1 \leq 2i \leq m+2^{n+1} - 1\} \oplus k(n)^* \{u^{2k} \mid m+2^{n+1} \leq 2k\}/(v_n) \). Since \( E_{2^{n+1}} \)-term is concentrated in even degree we have \( d_r = 0 \) for any \( r \geq 2^{n+1} \). Thus \( \tilde{E}_{2^{n+1}} \cong \cdots \cong \tilde{E}_\infty \).
Next we consider $\text{RP}_m^l$. Let $m < l$ and recall that
\begin{equation}
H^*(\text{RP}_m^l) = \mathbb{Z}/2\{u^{m+1}, \ldots, u^l\}.
\end{equation}
Let $i: \text{RP}_m^l \hookrightarrow \text{RP}_m^\infty$ be the standard inclusion. Then induced homomorphism
\begin{equation}
i^*: H^*(\text{RP}_m^\infty) \to H^*(\text{RP}_m^l)
\end{equation}
is epimorphism.

**Proposition 2.3** $\tilde{E}_\infty$-term of the Atiyah-Hirzebruch spectral sequence for $\tilde{k}(n)^*(\text{RP}_m^l)$ is given by:
\begin{equation}
\tilde{E}_\infty \cong k(n)^*\{u_{2i}, u_{2j+1}|m+1 \leq 2i \leq M, L \leq 2j+1 \leq l\}
\oplus k(n)^*\{v_{2k}|m+2n+1 \leq 2k \leq l\}/(v_n)
\end{equation}
as $k(n)^*$-module where $M = \text{Min}(l, m+2n+1 - 1)$, $L = \text{Max}(m+1, l - 2n+1 + 2)$ and $u_i$ denotes the class represented by $u^i$. 

**Proof.** For similarity, we only prove the proposition for the case: $l$ is odd and $m$ is even.

(i) If $l < m + 2n+1$, then for dimensional reason, all differentials are zero. Therefore $\tilde{E}_2 = \cdots = \tilde{E}_\infty \cong k(n)^*\{u^{m+1}, \ldots, u^l\}$.

(ii) If $l \geq m + 2n+1$, then since $i^*$ is epimorphism, the generators $u^{m+2k}$ are permanent cycle in AHSS for $\tilde{k}(n)^*(\text{RP}_m^l)$ for $k = 1, 2, \ldots, l - 1$. For dimensional reason, the generators $u^{m+2k+1}$ are cycles for $l - 2n+1 + 2 \leq m + 2k + 1 \leq l$. Also since $v_{2k}u^{m+2k+1}$ can not be boundary, from (1) and (2), $v_{2k}u^{m+2k+1}$ is cycle and is not boundary for $l - 2n+1 + 2 \leq m + 2k + 1 \leq l$. Therefore
\begin{equation}
\tilde{E}_{2n+1} \cong k(n)^*\{u^{m+2}, \ldots, u^{m+2n+1-2}, u^{l-2n+1+2}, \ldots, u^l\}
\oplus k(n)^*\{u^{m+2n+1}, \ldots, u^{l-1}\}/(v_n).
\end{equation}
For dimensional reason and the fact that $u^{m+2k}$ are all permanent cycles, we have $d_r = 0$ for all $r \geq 2n+1$. Thus $\tilde{E}_{2n+1} = \cdots = \tilde{E}_\infty$ and we get the proposition.

Let us consider the following filtration of $\tilde{k}(n)^*(\text{RP}_m^l)$
\begin{equation}
\tilde{k}(n)^*(\text{RP}_m^l) = \tilde{F}_m^0 \supset \cdots \supset \tilde{F}_m^{s-1} \supset \tilde{F}_m^{s+1} \supset \cdots
\end{equation}
where $\tilde{F}_m^{s-1} = \ker(\tilde{k}(n)^*(\text{RP}_m^l) \to \tilde{k}(n)^*((\text{RP}_m^l)^{s-1}))$ is a $k(n)^*$-submodule of $\tilde{k}(n)^*(\text{RP}_m^l)$. Then since $\tilde{F}_m^{l+1, s-l-1} = 0$, the filtration (4) is finite. Note that $\tilde{E}_\infty^{s}s \cong \tilde{F}_m^{s, s}/\tilde{F}_m^{s+1, s-1}$.

**Lemma 2.4** $\tilde{F}_m^{s, s} \cong \tilde{F}_m^{s+1, s-1} \oplus \tilde{E}_\infty^{s, s}$ as $k(n)^*$-module.
PROOF. We prove that the following exact sequence splits as $k(n)^*$-module;

$$0 \rightarrow \mathcal{E}_{s+1,*} \rightarrow \mathcal{E}_{s,*} \rightarrow \mathcal{E}_{\infty,*} \rightarrow 0.$$ 

By Proposition 2.3,

\begin{align*}
\mathcal{E}_{\infty,*} &\cong \begin{cases} 
  k(n)^* & \text{if } s \text{ is even and } m+1 \leq s \leq \text{Min}(l, m+2^{n+1}-1) \\
  k(n)^* & \text{if } s \text{ is odd and } \text{Max}(m+1, l-2^{n+1}+2) \leq s \leq l \\
  k(n)^*/(v_n) & \text{if } s \text{ is even and } m+2^{n+1} \leq s \leq l \\
  0 & \text{otherwise}
\end{cases}
\end{align*}

If $s$ is not in the third case, the exact sequence clearly splits. Suppose that $s$ is even and $m+2^{n+1} \leq s \leq l$. Let $u_s \in \mathcal{E}_{\infty,*}$ be as in Proposition 2.3. Then since $\pi$ is surjective, there exists an element $x \in \mathcal{E}_{n,*}$ such that $\pi(x) = u_s$. To prove the lemma it suffices to show that $v_n x = 0$. Since $\pi(v_n x) = v_n \pi(x) = v_n u_s = 0$, $v_n x$ is in $\mathcal{E}_{s+1,-2(2^{n}-1)-1}$. By Proposition 2.3, it is easy to see that $\mathcal{E}_{s+1,-2(2^{n}-1)-1} = 0$ for all $k \geq 1$. Thus $\mathcal{E}_{s+1,-2(2^{n}-1)-1} = 0$ and hence $v_n x = 0$. \hfill \Box

From Lemma 2.4, we have the following Theorem.

**Theorem 2.5** $\mathcal{E}_{\infty,*}(RP_{m+1}) \cong \bigoplus_s \mathcal{E}_{\infty,*,-s}$ as graded $k(n)^*$-module.

**Corollary 2.6** As $k(n)^*$-module

$$\mathcal{E}_{\infty,*}(RP_{m+1}) \cong k(n)^* \{x_{2i}, x_{2j+1} | m+1 \leq 2i \leq M, L \leq 2j+1 \leq l\}$$

$$\oplus k(n)^* \{x_{2k} | m+2^{n+1} \leq 2k \leq l\}/(v_n)$$

where $M = \text{Min}(l, m+2^{n+1}-1)$, $L = \text{Max}(m+1, l-2^{n+1}+2)$ and $x_i$ corresponds to $u_i$ under the above isomorphism.

**Theorem 2.7** There exists a ring isomorphism $E_{\infty,*} \cong k(n)^*(RP^I)$.

**Proof.** $E_{\infty,*}$ is generated by 1 and $u_i$ as $k(n)^*$-algebra and

$$u_i \cdot u_j = \begin{cases} u_{i+j} & \text{if } i+j \text{ is in the range given in Prop2.3.} \\
0 & \text{if } i+j \text{ is out of the range.}
\end{cases}$$

Let $x_i \in k(n)^*(RP^I)$ be as in Cor2.6. It is clear that both $x_i \cdot x_j$ and $x_{i+j}$ represent $u_{i+j}$ if $i+j$ is in the range given in Prop2.3 and $x_i \cdot x_j$ represents 0 if $i+j$ is out of the range. First we consider first case. In this case, $x_i \cdot x_j - x_{i+j}$ is in the higher filtration. But we can prove that the higher filtration is zero in the degree of $x_i \cdot x_j$ by using Prop2.3 as in the proof of Lem2.4 and hence $x_i \cdot x_j - x_{i+j} = 0$. Therefore $x_i \cdot x_j = x_{i+j}$. Similarly we have $x_i \cdot x_j = 0$ in the second case. This proves the theorem. \hfill \Box

**Theorem 2.8**
1. Let $l < 2^{n+1}$, then
   
   $$k(n)^*(RP^l) = k(n)^*[u]/(u^{l+1})$$

   where the class $u$ represents $x_1$.

2. Let $l$ be an odd integer such that $l > 2^{n+1}$, then
   
   $$k(n)^*(RP^l) = k(n)^*[x,y]/(v_n x^{2^n}, y^2 - x^{l-2^{n+1}+2}, x^{l+1}, y x^{2^n})$$

   where $x = x_2$, $y = x_{l-2^{n+1}+2}$.

3. Let $l$ be an even integer such that $l \geq 2^{n+1}$, then
   
   $$k(n)^*(RP^l) = k(n)^*[x,z]/(v_n x^{2^n}, z^2 - x^{l-2^{n+1}+3}, x^{l+2}, z \cdot x^{2^n-1})$$

   where $z = x_{l-2^{n+1}+3}$.

**Proof.** 1. Trivial. 2. Let $\psi : k(n)^*[x,y] \to k(n)^*(RP^l)$ be a $k(n)^*$-algebra homomorphism given by $\psi(x) = x_2$ and $\psi(y) = x_{l-2^{n+1}+2}$. Then it is easy to see that $\psi$ is surjective and $(v_n x^{2^n}, y^2 - x^{l-2^{n+1}+2}, x^{l+1}, y x^{2^n}) \subset \ker \psi$. By comparing graded dimension over $\mathbb{F}_2$, we see that $\psi$ induces an isomorphism $k(n)^*[x,y]/(v_n x^{2^n}, y^2 - x^{l-2^{n+1}+2}, x^{l+1}, y x^{2^n}) \cong k(n)^*(RP^l)$. 3. The proof is similar to 2. \qed

Using $K(n)^*(X) = k(n)^*(X)[v_n^{-1}]$, $K(n)^*(RP^l)$ and $K(n)^*(RP^m_{m+1})$ will immediately be deduced.

**Theorem 2.9** As $K(n)^*$-module $\tilde{K}(n)^*(RP^l_{m+1})$ is given by:

$$\tilde{K}(n)^*(RP^l_{m+1}) \cong K(n)^*[x_2i, x_{2j+1}| m + 1 \leq 2i \leq M, L \leq 2j + 1 \leq l$$

where $M = \text{Min}(l, m + 2^{n+1} - 1)$ and $L = \text{Max}(m + 1, l - 2^{n+1} + 2)$.

**Theorem 2.10**

1. Let $l < 2^{n+1}$, then
   
   $$K(n)^*(RP^l) = K(n)^*[u]/(u^{l+1})$$

   where the class $u$ represents $x_1$.

2. Let $l$ be an odd integer such that $l > 2^{n+1}$, then
   
   $$K(n)^*(RP^l) = K(n)^*[x,y]/(x^{2^n}, y^2 - x^{l-2^{n+1}+2})$$

   where $x = x_2$, $y = x_{l-2^{n+1}+2}$.

3. Let $l$ be an even integer such that $l \geq 2^{n+1}$, then
   
   $$K(n)^*(RP^l) = K(n)^*[x,z]/(x^{2^n}, z \cdot x^{2^n-1}, z^2 - x^{l-2^{n+1}+3})$$

   where $z = x_{l-2^{n+1}+3}$.

**References**


Y.M. YANG

Y. M. Yang
Department of Mathematics
Faculty of Science
Kyoto University
Kyoto, 310-0056

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