A Generation of the Hopf Construction

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A GENERATION OF THE HOPF CONSTRUCTION

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Introduction. Let $\Gamma$ be a co-Hopf space. For any space $X$, we define the $\Gamma$-suspension space of $X$ by $\Gamma X = \Gamma \wedge X$. For any map $f : X \to Y$, a map $\Gamma f : \Gamma X \to \Gamma Y$ is induced. It is called the $\Gamma$-suspension map of $f$. If $\Gamma = S^1(1$-sphere), then $\Gamma X = \Sigma X$ is the usual suspension space and $\Gamma f = \Sigma f : \Sigma X \to \Sigma Y$ is the usual suspension map.

It is known that there are various definitions of the Hopf construction. For example I.M. James provides one of the definitions by using the difference element $d(\cdot, \cdot)$ in [4] and M. Arkowitz and P. Silberbush study six different types of elements which are called the Hopf-type constructions in [1]. K.A. Hardie and A.V. Jansen define the Hopf construction $c(f, g) \in [\Sigma^{m+n+1}W, \Sigma Z]$ (we call it the Hopf construction with a space $W$) based on the definition by James in [2] when there is a pairing $\mu : S^m \times S^n \to Z^W$ with axes $f : S^m \to Z^W$ and $g : S^n \to Z^W$. On the other hand, N. Oda defines the $\Gamma$-Hopf construction in [6]. If $\Gamma = S^1$, then it is one of the Hopf-type constructions. In this paper, we introduce a concept of the skew pairing $\mu_W : (X \times Y) \wedge W \to Z$ with axes $f : X \wedge W \to Z$ and $g : Y \wedge W \to Z$ and we define the $\Gamma$-Hopf construction with a space $W$ by

$$J^W_\Gamma (\mu_W) = \Gamma \mu_W \circ (v \wedge 1W) \in [\Gamma (X \wedge Y) \wedge W, \Gamma Z]$$

for any skew pairing $\mu_W$, where $v$ is the element in Proposition 1.1. This generalizes the $\Gamma$-Hopf construction by Oda. Throughout this paper, the space $W$ is any space except otherwise stated explicitly.

In §1, we begin by studying fundamental properties of the $\Gamma$-Hopf construction with a space $W$. We next study the Hopf invariant. Let $Z$ be a connected CW-complex and $H : [\Sigma A, \Sigma Z] \to [\Sigma A, \Sigma (Z \wedge Z)]$ the Hopf invariant. Then we have the following results.

**Theorem 1.11** Let $v \in [\Sigma (X \wedge Y), \Sigma (X \times Y)]$ be the element of Proposition 1.1 and $\beta : \Sigma (X \times Y) \wedge W \to \Sigma Z$ any map. If $W$ is a co-Hopf space, then we have

$$H(\beta \circ (v \wedge 1W)) = H(\beta) \circ (v \wedge 1W).$$

**Theorem 1.13** Suppose that there is a skew pairing $\mu_W : (X \times Y) \wedge W \to Z$. If $W$ is a co-Hopf space, then $H(J^W_\Sigma (\mu_W)) = 0$.

In §2, we define an element $c(\alpha)$ for any skew pairing $\alpha : (X \times Y) \wedge W \to Z_\infty$ as follows

$$c(\alpha) = \phi(\alpha) \circ (v \wedge 1W) \in [\Sigma (X \wedge Y) \wedge W, \Sigma Z].$$
The element $c(\alpha)$ is a generalization of the Hopf construction with a space $W$. Let $Z_\infty$ be the reduced product space of a connected CW-complex $Z$. Then we have the following result.

**Theorem 2.2** If we are given two skew pairings $\alpha, \beta : (X \times Y) \wedge W \to Z_\infty$ with the same axes $f : X \wedge W \to Z_\infty$ and $g : Y \wedge W \to Z_\infty$, then the following relation holds.

$$c(\alpha) \cdot c(\beta) = c(\alpha \cdot \beta).$$

For any maps $\alpha : X \wedge W \to Z_\infty$ and $\beta : Y \wedge W \to Z_\infty$, we define two skew pairings $M, \overline{M} : (X \times Y) \wedge W \to Z_\infty$ by

$$M = (\alpha \circ (p_1 \wedge 1_W)) \circ (\beta \circ (p_2 \wedge 1_W)),$$

and

$$\overline{M} = (\beta \circ (p_2 \wedge 1_W)) \circ (\alpha \circ (p_1 \wedge 1_W)).$$

Using the isomorphism $\phi : [A, Z_\infty] \to [\Sigma A, \Sigma Z]$ (cf. (2.3) of [9]), we have the following results.

**Theorem 2.5** For any maps $\alpha : X \wedge W \to Z_\infty$ and $\beta : Y \wedge W \to Z_\infty$, we have

$$c(M) = 0 \quad \text{and} \quad c(\overline{M}) = c([\phi(\alpha), \phi(\beta)])^W_\Sigma.$$

Here $[\cdot, \cdot]^W_\Sigma$ denotes the generalized Hardie-Jansen product (cf. [8]).

Let

$$\tau : X \wedge Y \wedge W \wedge W \to X \wedge W \wedge Y \wedge W$$

be the natural homeomorphism interchanging the second and third factors of the smash products. Let $q : X \times Y \to X \wedge Y$ be the identification map. Let $\chi : W \to W \wedge W$ be the reduced diagonal map.

**Theorem 2.6** We define a skew pairing $M = (i \circ f \circ (p_1 \wedge 1_W)) \circ (i \circ g \circ (p_2 \wedge 1_W))$ for any maps $f : X \wedge W \to Z$ and $g : Y \wedge W \to Z$. Then we have

$$H(\phi(M)) = \Sigma(f \wedge g) \circ \Sigma(1_{\Sigma X \wedge Y} \wedge \chi) \circ (\Sigma q \wedge 1_W)$$

and hence

$$H(\phi(M)) \circ (v \wedge 1_W) = \Sigma(f \wedge g) \circ \Sigma(1_{\Sigma X \wedge Y} \wedge \chi).$$

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1. $\Gamma$-Hopf Construction with a Space $W$ for a Skew Pairing

We work in the category of compactly generated Hausdorff spaces with nondegenerate base point $*$ [10]. All maps and homotopies are base point preserving. For any spaces $X$ and $Y$, the space $X \times Y$ is the product space and $X \vee Y$ is the wedge sum (one-point union). The space $X \wedge Y$ is the subset of the space $X \times Y$. Let $X \wedge Y$ be the smash product which is the identification space $X \times Y / X \vee Y$. Let $[A, Z]$ be the set of homotopy classes of maps from $A$ to $Z$. For any elements $\alpha$ and $\beta$ in $[A, Z]$, we define the following maps: if $A$ is a co-Hopf space with co-multiplication $\nu : A \to A \vee A$, then we define

$$\alpha \circ \beta = \nabla_Z \circ (\alpha \vee \beta) \circ \nu : A \to Z$$

where $\nabla_Z : Z \vee Z \to Z$ is the folding map; or if $Z$ is a Hopf space with multiplication $\mu : Z \times Z \to Z$, then we define

$$\alpha \circ \beta = \mu \circ (\alpha \times \beta) \circ \Delta_A : A \to Z$$

where $\Delta_A : A \to A \times A$ is the diagonal map.

We shall recall the definition of pairings in [5]. A map $\mu : X \times Y \to Z$ is said to be a pairing with axes $f : X \to Z$ and $g : Y \to Z$ if the following diagram is homotopy commutative:

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{\mu} & Z \\
\downarrow{\jmath} & & \downarrow{\nabla_Z} \\
X \vee Y & \xrightarrow{f \vee g} & Z \vee Z
\end{array}
$$

where $\jmath : X \vee Y \to X \times Y$ is the inclusion map.

Now let $\Gamma$ be a co-group-like space, namely an associative co-Hopf space with an inverse. Then $\Gamma X$ is also a co-group-like space. Let $j_1 : X \to X \vee Y$ and $j_2 : Y \to X \vee Y$ be the inclusions, and let $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ the projections. We define a map

$$\rho = \Gamma(j_1 \circ p_1) \circ \Gamma(j_2 \circ p_2) : \Gamma(X \times Y) \to \Gamma(X \vee Y).$$

Let $q : X \times Y \to X \wedge Y$ be the identification map. Then we have the following results.

**Proposition 1.1.** (Proposition 2.1, Theorem 2.4 and Proposition 2.6 in [6]) There is a unique element $v \in [\Gamma(X \wedge Y), \Gamma(X \times Y)]$ such that

(I) $v \circ \Gamma q + \Gamma j \circ \rho = 1_{\Gamma(X \times Y)}$ in $[\Gamma(X \times Y), \Gamma(X \times Y)]$.

And these $\rho$ and $v$ satisfy the following relations:

(II) $\rho \circ \Gamma j = 1_{\Gamma(X \vee Y)}$ in $[\Gamma(X \vee Y), \Gamma(X \vee Y)]$.

(III) $\Gamma q \circ v = 1_{\Gamma(X \wedge Y)}$ in $[\Gamma(X \wedge Y), \Gamma(X \wedge Y)]$.

(IV) $\rho \circ v = *$ in $[\Gamma(X \wedge Y), \Gamma(X \vee Y)]$. 

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Suppose that there is a pairing $\mu : X \times Y \to Z$. Then the $\Gamma$-Hopf construction is the element

$$J_\Gamma(\mu) \in [\Gamma(X \wedge Y), \Gamma Z]$$

defined by $J_\Gamma(\mu) = \Gamma \mu \circ v$ in [6]. If $\Gamma = S^1$ (1-sphere), then the $\Gamma$-Hopf construction is one of the Hopf-type constructions in [1].

Let $t_{X \wedge W} : W \wedge X \to X \wedge W$ be the switching map defined by $t_{X \wedge W}(w \wedge x) = x \wedge w$ for any elements $x \in X$ and $w \in W$, where $x \wedge w$ is an elements in $X \wedge W$. It is a natural homeomorphism. The inverse map of $t_{X \wedge W}$ is denoted by $(t_{X \wedge W})^{-1} = t_{W \wedge X}$. For any map $f : X \to Y$ and the $\Gamma$-suspenion map $\Gamma t_{X \wedge W} : \Gamma(W \wedge X) \to \Gamma(X \wedge W)$, we have

$$\Gamma t_{Y \wedge W} \circ (\Gamma Wf) = (\Gamma f \wedge 1_W) \circ \Gamma t_{X \wedge W}.$$

Let us define $\hat{\rho} = \Gamma W(j_1 \circ p_1) + \Gamma W(j_2 \circ p_2) : \Gamma W(X \times Y) \to \Gamma W(X \vee Y)$. Then we have a unique element $\hat{v} \in [\Gamma W(X \wedge Y), \Gamma W(X \times Y)]$ such that

$$\hat{v} \circ \Gamma W q + \Gamma W j \circ \hat{\rho} = 1_{\Gamma W(X \times Y)}$$

by Proposition 1.1. (We use $\Gamma W$ instead of $\Gamma$.)

**Lemma 1.2.** For the elements

$$v \in [\Gamma(X \wedge Y), \Gamma(X \times Y)] \quad \text{and} \quad \hat{v} \in [\Gamma W(X \wedge Y), \Gamma W(X \times Y)],$$

the following diagram is homotopy commutative.

$$\begin{array}{ccc}
\Gamma W \wedge (X \wedge Y) & \xrightarrow{\hat{v}} & \Gamma W \wedge (X \times Y) \\
\Gamma t_{(X \wedge Y) \wedge W} \downarrow & & \uparrow \Gamma t_{(X \wedge Y) \wedge W} \\
\Gamma(X \wedge Y) \wedge W & \xrightarrow{n \wedge 1_W} & \Gamma(X \times Y) \wedge W
\end{array}$$

**Proof.** We have $\Gamma t_{(X \wedge Y) \wedge W} \circ \Gamma W q = (\Gamma q \wedge 1_W) \circ (\Gamma t_{(X \wedge Y) \wedge W})$ and

$$\Gamma t_{(X \wedge Y) \wedge W} \circ \Gamma W j \circ \hat{\rho}$$

$$= (\Gamma j \wedge 1_W) \circ \Gamma t_{(X \times Y) \wedge W} \circ (\Gamma W(j_1 \circ p_1) + \Gamma W(j_2 \circ p_2))$$

$$= (\Gamma j \wedge 1_W) \circ \{ (\Gamma t_{(X \times Y) \wedge W} \circ \Gamma W(j_1 \circ p_1))$$

$$+ (\Gamma t_{(X \times Y) \wedge W} \circ \Gamma W(j_2 \circ p_2)) \}$$

$$= (\Gamma j \wedge 1_W) \circ \{ (\Gamma((j_1 \circ p_1) \wedge 1_W) \circ \Gamma t_{(X \times Y) \wedge W})$$

$$+ (\Gamma((j_2 \circ p_2) \wedge 1_W) \circ \Gamma t_{(X \times Y) \wedge W}) \}$$

$$= (\Gamma j \wedge 1_W) \circ \{ \Gamma((j_1 \circ p_1) \wedge 1_W) + \Gamma((j_2 \circ p_2) \wedge 1_W) \} \circ \Gamma t_{(X \times Y) \wedge W}$$

$$= (\Gamma j \wedge 1_W) \circ (\rho \wedge 1_W) \circ \Gamma t_{(X \times Y) \wedge W}.$$
Therefore by the relation
\((\Gamma t_{(X \wedge Y) \wedge W})^{-1} \circ (v \wedge 1_W) \circ (\Gamma q \wedge 1_W) \circ (\Gamma t_{(X \wedge Y) \wedge W}) \circ (\Gamma j \wedge 1_W) \circ (\rho \wedge 1_W) \circ (\Gamma t_{(X \wedge Y) \wedge W})\)
\[\vdash (\Gamma t_{(X \wedge Y) \wedge W})^{-1} \circ (v \wedge 1_W) \circ (\Gamma t_{(X \wedge Y) \wedge W}) \circ (\Gamma j \wedge 1_W) \circ (\rho \wedge 1_W) \circ (\Gamma t_{(X \wedge Y) \wedge W}) = (\Gamma t_{(X \wedge Y) \wedge W})^{-1} \circ (v \wedge 1_W) \circ (\Gamma t_{(X \wedge Y) \wedge W}).\]
we have
\[\{(\Gamma t_{(X \wedge Y) \wedge W})^{-1} \circ (v \wedge 1_W) \circ (\Gamma t_{(X \wedge Y) \wedge W})\} \circ (\Gamma W q \vdash \Gamma W j \circ \rho = 1_{\Gamma_W(X \wedge Y)}.\]
Since \(\hat{v}\) is a unique element which satisfies
\[\hat{v} \circ (\Gamma W q \vdash \Gamma W j \circ \rho = 1_{\Gamma_W(X \wedge Y)},\]
we have \((\Gamma t_{(X \wedge Y) \wedge W})^{-1} \circ (v \wedge 1_W) \circ (\Gamma t_{(X \wedge Y) \wedge W}) = \hat{v}\). Hence we have the result.

**Proposition 1.3.** If we are given a pairing \(\mu : X \times Y \rightarrow Z\), then we have
\[\Gamma t_{Z \wedge W} \circ J_{\Gamma_W}(\mu) = \{J_{\Gamma}(\mu) \wedge 1_W\} \circ \Gamma t_{(X \wedge Y) \wedge W}.\]

**Proof.** By Lemma 1.2, we have
\[\Gamma t_{Z \wedge W} \circ J_{\Gamma_W}(\mu) = \Gamma t_{Z \wedge W} \circ (\Gamma W \mu \circ \hat{v}) = \Gamma (\mu \wedge 1_W) \circ \Gamma t_{(X \wedge Y) \wedge W} \circ \hat{v} = \Gamma (\mu \wedge 1_W) \circ (v \wedge 1_W) \circ \Gamma t_{(X \wedge Y) \wedge W} = \{(\Gamma (\mu \circ v) \wedge 1_W) \circ \Gamma t_{(X \wedge Y) \wedge W}\} = \{J_{\Gamma}(\mu) \wedge 1_W\} \circ \Gamma t_{(X \wedge Y) \wedge W}.\]

Now we call a map \(\mu_W : (X \times Y) \wedge W \rightarrow Z\) a skew pairing with axes \(f : X \wedge W \rightarrow Z\) and \(g : Y \wedge W \rightarrow Z\) when the following diagram is homotopy commutative:

\[
\begin{array}{ccc}
(X \times Y) \wedge W & \xrightarrow{\mu_W} & Z \\
| j \wedge 1_W \downarrow & & \downarrow \nabla_Z \\
(X \vee Y) \wedge W \approx (X \wedge W) \vee (Y \wedge W) & \xrightarrow{f \vee g} & Z \vee Z
\end{array}
\]

If \(W = S^0 = \{-1, 1\}\), then a skew pairing \(\mu_W\) is an ordinary pairing \(\mu : X \times Y \rightarrow Z\).

Let \(Z^W = \text{map}^*(W, Z)\) be the space of base point preserving maps from \(W\) to \(Z\) with compact-open topology. Let \(W\) be a fixed space. For any space \(A\) and \(Z\), let
\[\theta_W : [A \wedge W, Z] \longrightarrow [A, Z^W]\]
be the adjoint map. A map \(\mu_W : (X \times Y) \wedge W \rightarrow Z\) is a skew pairing with axes \(f : X \wedge W \rightarrow Z\) and \(g : Y \wedge W \rightarrow Z\) if and only if a map \(\theta_W(\mu_W) : X \times Y \rightarrow Z^W\) is an ordinary pairing with axes \(\theta_W(f) : X \rightarrow Z^W\) and \(\theta_W(g) : Y \rightarrow Z^W\).

Let \(\Omega X\) denote the loop space, that is, \(X^{S^1} = \text{map}^*(S^1, X)\). Let us assume that \(Z\) is a connected CW-complex. Let \(Z_{\infty}\) be the reduced product space and
$\zeta : Z_\infty \rightarrow \Omega \Sigma Z$ the homotopy equivalence proved by James [3]. We define an isomorphism $\phi$ by

$$\phi : [A, Z_\infty] \xrightarrow{\zeta^*} [A, \Omega \Sigma Z] \xrightarrow{\theta_{\Sigma}^{-1}} [\Sigma A, \Sigma Z].$$

Suppose that there is a skew pairing $\mu_W : (S^n \times S^n) \wedge W \rightarrow Z$ with axes $f : \Sigma^m W \rightarrow Z$ and $g : \Sigma^n W \rightarrow Z$. Let $M = (i \circ f \circ (p_1 \wedge 1_W)) + (i \circ g \circ (p_2 \wedge 1_W))$. Then in [2], Hardie and Jansen define a Hopf construction $c(f, g) \in [\Sigma^{m+n+1} W, \Sigma X]$ (we call it the Hopf construction with a space $W$) by

$$c(f, g) = \left\{ \phi^{-1} \left( d(\theta_W(M), \theta_W(i \circ \mu_W)) \right) \left| \mu_W : (S^n \times S^n) \wedge W \rightarrow Z \right. \right. \text{with axes } f \text{ and } g \right\}$$

where $d(\theta_W(M), \theta_W(i \circ \mu_W))$ is the difference element of $\theta_W(M)$ and $\theta_W(i \circ \mu_W)$ defined by James in [4].

When we are given a skew pairing $\mu_W : (X \times Y) \wedge W \rightarrow Z$, we give the following definition, using the element $v \in [\Gamma(X \wedge Y), \Gamma(X \times Y)]$ in Proposition 1.1.

**Definition 1.4.** If we are given a skew pairing $\mu_W : (X \times Y) \wedge W \rightarrow Z$, we define a $\Gamma$-Hopf construction with a space $W$ by

$$J^W_{\Gamma}(\mu_W) = \Gamma \mu_W \circ (v \wedge 1_W) \in [\Gamma(X \wedge Y) \wedge W, \Gamma Z].$$

If $W = S^0$, then $J^W_{\Gamma}(\mu_W)$ is the $\Gamma$-Hopf construction $J_{\Gamma}(\mu)$. If we are given a pairing $\mu : X \times Y \rightarrow Z$, then we have $J^W_{\Gamma}(\mu \wedge 1_W) = J_{\Gamma}(\mu) \wedge 1_W$.

**Lemma 1.5.** (Theorem (8.7) and (8.8) of Chapter X in [10]) Let $X$ be a co-Hopf space. For any elements $\alpha$ and $\beta$ in $[X, Y]$, and $\gamma$ in $[A, B]$, we have

$$\gamma \wedge (\alpha \wedge \beta) = \gamma \wedge \alpha \wedge \gamma \wedge \beta, \quad (\alpha \wedge \beta) \wedge \gamma = \alpha \wedge \gamma \wedge \beta \wedge \gamma.$$

**Proposition 1.6.** If we are given a skew pairing $\mu_W : (X \times Y) \wedge W \rightarrow Z$, then we have

(i) $J^W_{\Gamma}(\beta \circ \mu_W) = \Gamma \beta \circ J^W_{\Gamma}(\mu_W)$

(ii) $J^W_{\Gamma}(\mu_W \circ ((\gamma \times \delta) \wedge 1_W)) = J^W_{\Gamma}(\mu_W) \circ \Gamma(\gamma \wedge \delta) \wedge 1_W$

where $\beta : Z \rightarrow Z'$, $\gamma : X' \rightarrow X$ and $\delta : Y' \rightarrow Y$ are any maps.

**Proof.** (i) We have

$$J^W_{\Gamma}(\beta \circ \mu_W) = \Gamma(\beta \circ \mu_W) \circ (v \wedge 1_W) = \Gamma \beta \circ \Gamma \mu_W \circ (v \wedge 1_W) = \Gamma \beta \circ J^W_{\Gamma}(\mu_W).$$

(ii) Let $\rho' = \Gamma(j'_1 \circ p'_1) + \Gamma(j'_2 \circ p'_2)$ where $j'_1 : X' \rightarrow X' \vee Y'$, $j'_2 : Y' \rightarrow X' \vee Y'$ are the inclusions and $p'_1 : X' \times Y' \rightarrow X'$, $p'_2 : X' \times Y' \rightarrow Y'$ are the projections. Then there exists an element $\nu' \in [\Gamma(X' \times Y'), \Gamma(X' \times Y')]$ such that $\nu' \circ \Gamma q' + \Gamma j' \circ \rho' = 1_{\Gamma(X' \times Y')} \circ \gamma$ by Proposition 1.1, where $j' : X' \vee Y' \rightarrow X' \times Y'$ is the inclusion map and...
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\( q' : X' \times Y' \to X' \wedge Y' \) is the identification map. By Proposition 2.5 of [6], we have

\[
J_W^W(\mu_W \circ ((\gamma \times \delta) \wedge 1_W)) = \Gamma(\mu_W \circ ((\gamma \times \delta) \wedge 1_W)) \circ (v' \wedge 1_W) \\
= \Gamma \mu_W \circ (\Gamma(\gamma \times \delta) \wedge 1_W) \circ (v' \wedge 1_W) \\
= \Gamma \mu_W \circ (v \wedge 1_W) \circ (\Gamma(\gamma \wedge \delta) \wedge 1_W) \\
= J_W^W(\mu_W) \circ \Gamma(\gamma \wedge \delta) \wedge 1_W.
\]

**Proposition 1.7.** If we are given a skew pairing \( \mu_W : (X \times Y) \wedge W \to Z \) with axes \( f : X \wedge W \to Z \) and \( g : Y \wedge W \to Z \), then we have

\[
J_W^W(\mu_W) \circ (\Gamma q \wedge 1_W) = \Gamma \mu_W \circ (\Gamma f \circ (\Gamma p_1 \wedge 1_W)) \circ (\Gamma g \circ (\Gamma p_2 \wedge 1_W)).
\]

**Proof.** We have

\[
J_W^W(\mu_W) \circ (\Gamma q \wedge 1_W) \\
= \Gamma \mu_W \circ (v \wedge 1_W) \circ (\Gamma q \wedge 1_W) \\
= \Gamma \mu_W \circ (1_{\Gamma(\Delta X \times Y) \wedge W} \circ (\Gamma f \circ \rho) \wedge 1_W) \\
= \Gamma \mu_W \circ \Gamma(\Delta Z \circ (f \vee g)) \circ (\Gamma(j \circ f \circ p_1) \circ (\Gamma j_2 \circ p_2) \wedge 1_W) \\
= \Gamma \mu_W \circ \{ \Gamma(\Delta Z \circ (f \vee g) \circ (j_1 \wedge 1_W) \circ (p_1 \wedge 1_W) \\
+ \Gamma(\Delta Z \circ (f \vee g) \circ (j_2 \wedge 1_W) \circ (p_2 \wedge 1_W)) \} \\
= \Gamma \mu_W \circ (\Gamma f \circ (\Gamma p_1 \wedge 1_W) \circ (\Gamma p_2 \wedge 1_W)).
\]

Let \( \rho_W = \Gamma((j_1 \wedge 1_W) \circ (p_1 \wedge 1_W)) \circ (\Gamma j_2 \circ p_2 \wedge 1_W) \). We see \( \rho_W = \rho \wedge 1_W \) by the definitions of \( \rho \) and \( \rho_W \). Then by the same reason as Proposition 1.1, we have a unique element \( v_W \in [\Gamma(X \wedge Y) \wedge W, \Gamma(X \times Y) \wedge W] \) which satisfies

\[ v_W \circ (\Gamma q \wedge 1_W) \circ (\Gamma j \wedge 1_W) \circ \rho_W = 1_{\Gamma(\Delta X \times Y) \wedge W}. \]

**Lemma 1.8.** For the elements \( v \in [\Gamma(X \wedge Y), \Gamma(X \times Y)] \) and \( v_W \in [\Gamma(X \wedge Y) \wedge W, \Gamma(X \times Y) \wedge W] \), we have the following relation;

\[ v \wedge 1_W = v_W. \]

**Proof.** From (I) of Proposition 1.1, we have

\[ (v \circ \Gamma q \circ (\Gamma f \circ \rho) \wedge 1_W) = 1_{\Gamma(\Delta X \times Y) \wedge W} \]
\[ (v \circ \Gamma q) \wedge 1_W \circ (\Gamma f \circ \rho) \wedge 1_W = 1_{\Gamma(\Delta X \times Y) \wedge W} \]
\[ (v \wedge 1_W) \circ (\Gamma q \wedge 1_W) \circ (\Gamma f \circ \rho) \wedge 1_W = 1_{\Gamma(\Delta X \times Y) \wedge W} \]

Hence, \( v_W \circ (\Gamma q \wedge 1_W) = (v \wedge 1_W) \circ (\Gamma q \wedge 1_W) \) in the homotopy set \([\Gamma(\Delta X \times Y) \wedge W, \Gamma(X \times Y) \wedge W]\). Since \( (\Gamma q \wedge 1_W)^* \) is a monomorphism, we have the result.

**Remark 1.9.** If \( W \) is a co-grouplike space, then \( v_W = \Gamma v' \) for some \( v' \in [(X \wedge Y) \wedge W, (X \times Y) \wedge W] \).
Let \( h_2' : (Z_2, Z) \rightarrow (Z \wedge Z, \ast) \) be the shrinking map and \( h_2 : Z_\infty \rightarrow (Z \wedge Z)_\infty \) the combinatorial extension of \( h_2' \) in [3]. Then we define the Hopf invariant by
\[
\phi \circ (h_2) \circ \phi^{-1} : [A, Z_\infty] \rightarrow [A, (Z \wedge Z)_\infty] \\
\downarrow \phi \downarrow \phi \\
[A, (Z \wedge Z)_\infty] \rightarrow [A, (Z \wedge Z)_\infty].
\]

We see this in (2.7) of [9] if \( Z \) is a sphere. In general, \( H : [\Sigma A, \Sigma Z] \rightarrow [\Sigma A, \Sigma (Z \wedge Z)] \) is not a homomorphism, since \( h_2 \) is not a Hopf map. If \( A \) is a co-Hopf space, then the Hopf invariant \( H \) is a homomorphism. The following results are known.

**Proposition 1.10.** (cf. Proposition 2.2 in [9]) Let \( \alpha : A \rightarrow B, \beta : \Sigma B \rightarrow \Sigma Z, \gamma : \Sigma A \rightarrow \Sigma Z \) and \( \delta : Z \rightarrow Z' \) be any maps. Then we have
\[(i) H(\beta \circ \Sigma \alpha) = H(\beta) \circ \Sigma \alpha.
(ii) H(\Sigma \delta \circ \gamma) = \Sigma(\delta \wedge \delta) \circ H(\gamma).
\]

**Theorem 1.11.** Let \( v \in [\Sigma (X \wedge Y), \Sigma (X \times Y)] \) be the element of Proposition 1.1 and \( \beta : \Sigma (X \times Y) \wedge W \rightarrow \Sigma Z \) any map. If \( W \) is a co-Hopf space, then we have
\[H(\beta \circ (v \wedge 1_W)) = H(\beta) \circ (v \wedge 1_W).
\]

**Proof.** Since \( W \) is a co-Hopf space, the Hopf invariant
\[H : [\Sigma (X \times Y) \wedge W, \Sigma Z] \rightarrow [\Sigma (X \times Y) \wedge W, \Sigma (Z \wedge Z)]
\]
is a homomorphism. Then we have
\[H(\beta \circ (v \wedge 1_W)) \circ (\Sigma q \wedge 1_W)
= H(\beta \circ (v \wedge 1_W) \circ (\Sigma q \wedge 1_W))
= H(\beta \circ (1_{\Sigma (X \times Y)} \wedge \Sigma q \circ \rho \wedge 1_W)) \quad \text{(by (I) of Proposition 1.1)}
= H(\beta \circ 1_{\Sigma (X \times Y) \wedge W}) \wedge H(\beta \circ ((\Sigma q \circ \rho) \wedge 1_W))
= H(\beta) \circ (\Sigma q \circ (\Sigma (j \circ j_1 \circ p_1) \wedge 1_W))
= H(\beta) \circ ((\Sigma (j \circ j_1 \circ p_1) \wedge 1_W) \wedge (\Sigma q \circ \rho \wedge 1_W))
= H(\beta) \circ (1_{\Sigma (X \times Y) \wedge W} \wedge ((\Sigma q \circ \rho \wedge 1_W) \wedge (\Sigma (j \circ j_1 \circ p_1) \wedge 1_W)))
= H(\beta) \circ (1_{\Sigma (X \times Y) \wedge W} \wedge (\Sigma q \circ \rho \wedge 1_W) \circ (\Sigma (j \circ j_1 \circ p_1) \wedge 1_W))
= H(\beta) \circ (v \wedge 1_W) \circ (\Sigma q \wedge 1_W).
\]
Since \( (\Sigma q \wedge 1_W)^* \) is a monomorphism, we have the result.

**Remark 1.12.** If \( W \) is a co-grouplike space, then the proof is simplified making use of (i) of Proposition 1.10, since \( v \wedge 1_W \) is a suspension element by Remark 1.9.
Theorem 1.13. Suppose that there is a skew pairing $\mu_W : (X \times Y) \wedge W \to Z$. If $W$ is a co-Hopf space, then $H(J^W_\Sigma (\mu_W)) = 0$.

Proof. By Theorem 1.11, we have

$$H(J^W_\Sigma (\mu_W)) = H(\Sigma \mu_W \circ (v \wedge 1_W)) = H(\Sigma \mu_W \circ (v \wedge 1_W)) = 0.$$ 

Hence, we have the result.

2. A Hopf Construction Induced by a Skew Pairing

In this section we define a generalized Hopf construction $c(\alpha)$ induced by a skew pairing of the type $\alpha : (X \times Y) \wedge W \to Z_\infty$. Suppose that there is a skew pairing $\alpha : (X \times Y) \wedge W \to Z_\infty$. Then we define an element

$$c(\alpha) = \phi(\alpha) \circ (v \wedge 1_W) \in [\Sigma(X \wedge Y) \wedge W, \Sigma Z].$$

For a skew pairing $\mu_W : (X \times Y) \wedge W \to Z$ and the inclusion map $i : Z \to Z_\infty$, we have $c(i \circ \mu_W) = J^W_\Sigma (\mu_W)$. The following proposition is proved by the method similar to the proof of Proposition 1.7.

Proposition 2.1. If we are given a skew pairing $\alpha : (X \times Y) \wedge W \to Z_\infty$ with axes $f : X \wedge W \to Z_\infty$ and $g : Y \wedge W \to Z_\infty$, then we have

$$c(\alpha) \circ (\Sigma q \wedge 1_W) = \phi(\alpha) - \{\phi(f) \circ (\Sigma p_1 \wedge 1_W) \pm \phi(g) \circ (\Sigma p_2 \wedge 1_W)\}.$$

Theorem 2.2. If we are given two skew pairings $\alpha, \beta : (X \times Y) \wedge W \to Z_\infty$ with the same axes $f : X \wedge W \to Z_\infty$ and $g : Y \wedge W \to Z_\infty$, then the following relation holds.

$$c(\alpha) - c(\beta) = c(\alpha - \beta).$$

Proof. Since the space $Z_\infty$ is a Hopf space, we have an exact sequence:

$$0 \to [(X \wedge Y) \wedge W, Z_\infty] \xrightarrow{(q \wedge 1_W)^*} [(X \times Y) \wedge W, Z_\infty] \xrightarrow{(j \wedge 1_Y)^*} [(X \cap Y) \wedge W, Z_\infty] \to 0$$

by Lemma 1.3.5. in [11]. Then we have an element $\gamma \in [(X \wedge Y) \wedge W, Z_\infty]$ such that $\alpha - \beta = \gamma \circ (q \wedge 1_W)$. Then by Proposition 2.1, we have

$$(c(\alpha) - c(\beta)) \circ (\Sigma q \wedge 1_W)$$

$$= \{\phi(\alpha) - (\phi(f) \circ (\Sigma p_1 \wedge 1_W) \pm \phi(g) \circ (\Sigma p_2 \wedge 1_W))\}$$

$$\circ \{\phi(\beta) - (\phi(f) \circ (\Sigma p_1 \wedge 1_W) \pm \phi(g) \circ (\Sigma p_2 \wedge 1_W))\}$$

$$= \phi(\alpha) - \phi(\beta)$$

$$= \phi(\alpha - \beta)$$

$$= \phi(\gamma \circ (q \wedge 1_W)) = \phi(\gamma) \circ (\Sigma q \wedge 1_W)$$
and
\[
\begin{align*}
c(\alpha - \beta) \circ (\Sigma q \wedge 1_W) &= \phi(\alpha - \beta) \circ (v \wedge 1_W) \circ (\Sigma q \wedge 1_W) \\
&= \phi(\gamma \circ (q \wedge 1_W)) \circ (v \wedge 1_W) \circ (\Sigma q \wedge 1_W) \\
&= \phi(\gamma) \circ ((\Sigma q \circ v) \wedge 1_W) \circ (\Sigma q \wedge 1_W) \\
&= \phi(\gamma) \circ (\Sigma q \wedge 1_W) \quad \text{by (III) of Proposition 1.1.}
\end{align*}
\]

Since \((\Sigma q \wedge 1_W)^*\) is a monomorphism, we have the result.

**Corollary 2.3.** If we are given two skew pairings \(\mu_W, \mu'_W : (X \times Y) \wedge W \to Z\) with the same axes, then we have the following relation.
\[
J^W_\Sigma (\mu_W) = J^W_\Sigma (\mu'_W).
\]

**Proof.** Since \(c(i \circ \mu_W) = J^W_\Sigma (\mu_W)\) and \(c(i \circ \mu'_W) = J^W_\Sigma (\mu'_W)\), we have the result.

**Corollary 2.4.** If we are given two pairings \(\mu, \mu' : X \times Y \to Z\) with the same axes, then we have the following relation.
\[
J_\Sigma(\mu) = J_\Sigma(\mu').
\]

For any elements \(\alpha \in [\Gamma X, Z]\) and \(\beta \in [\Gamma Y, Z]\), \(\Gamma\)-Whitehead product \([\alpha, \beta]_\Gamma\) is defined in [7]. Now we recall that for any elements \(\alpha \in [\Gamma X \wedge W, Z]\) and \(\beta \in [\Gamma Y \wedge W, Z]\), the generalized Hardie-Jansen product \([\alpha, \beta]^W_\Gamma \in [\Gamma (X \wedge Y) \wedge W, Z]\) is defined by \(\theta^1_W([\theta_W(\alpha), \theta_W(\beta)]_\Gamma)\) in [8]. It is characterized by a relation
\[
[\alpha, \beta]^W_\Gamma \circ (\Sigma q \wedge 1_W) = \alpha \circ (\Gamma p_1 \wedge 1_W) + \beta \circ (\Gamma p_2 \wedge 1_W) - \alpha \circ (\Gamma p_1 \wedge 1_W) - \beta \circ (\Gamma p_2 \wedge 1_W)
\]
which is a commutator of \(\alpha \circ (\Gamma p_1 \wedge 1_W)\) and \(\beta \circ (\Gamma p_2 \wedge 1_W)\) by Corollary 1.14 in [8]. If \(W = S^0\), then this product \([\alpha, \beta]^W_\Gamma\) is the \(\Gamma\)-Whitehead product.

For any maps \(\alpha : X \wedge W \to Z_\infty\) and \(\beta : Y \wedge W \to Z_\infty\), we define two skew pairings \(M, M : (X \times Y) \wedge W \to Z_\infty\) by
\[
M = (\alpha \circ (p_1 \wedge 1_W)) \uplus (\beta \circ (p_2 \wedge 1_W)),
\]
and
\[
M = (\beta \circ (p_2 \wedge 1_W)) \uplus (\alpha \circ (p_1 \wedge 1_W)).
\]

Then the skew pairings \(M\) and \(\overline{M}\) have the axes \(\alpha : X \wedge W \to Z_\infty\) and \(\beta : Y \wedge W \to Z_\infty\).

**Theorem 2.5.** For any maps \(\alpha : X \wedge W \to Z_\infty\) and \(\beta : Y \wedge W \to Z_\infty\), we have
\[
c(M) = 0 \quad \text{and} \quad c(\overline{M}) = [\phi(\alpha), \phi(\beta)]^W_\Sigma.
\]
\textbf{Proof.} By Proposition 2.1, we have
\[
c(M) \circ (\Sigma q \wedge 1_W) = \phi(M) - (\phi(\alpha) \circ (\Sigma p_1 \wedge 1_W) + \phi(\beta) \circ (\Sigma p_2 \wedge 1_W))
\]
\[
= \phi(M) - \{\phi(\alpha \circ (p_1 \wedge 1_W)) + \phi(\beta \circ (p_2 \wedge 1_W))\}
\]
\[
= \phi(M) - \phi((\alpha \circ (p_1 \wedge 1_W)) + (\beta \circ (p_2 \wedge 1_W)))
\]
\[
= \phi(M) - \phi(M) = 0.
\]
\[
c(M) \circ (\Sigma q \wedge 1_W) = \phi(M) - (\phi(\alpha) \circ (\Sigma p_1 \wedge 1_W) + \phi(\beta) \circ (\Sigma p_2 \wedge 1_W))
\]
\[
= (\phi(\beta) \circ (\Sigma p_2 \wedge 1_W) + \phi(\alpha) \circ (\Sigma p_1 \wedge 1_W))
\]
\[
- (\phi(\alpha) \circ (\Sigma p_1 \wedge 1_W) + \phi(\beta) \circ (\Sigma p_2 \wedge 1_W))
\]
\[
= \phi(\beta) \circ (\Sigma p_2 \wedge 1_W) + \phi(\alpha) \circ (\Sigma p_1 \wedge 1_W)
\]
\[
- \phi(\beta) \circ (\Sigma p_2 \wedge 1_W) - \phi(\alpha) \circ (\Sigma p_1 \wedge 1_W)
\]
\[
= [\phi(\alpha), \phi(\beta)]_{\Sigma W} \circ (\Sigma q \wedge 1_W).
\]

Since \((\Sigma q \wedge 1_W)^*\) is a monomorphism, we have the results.

We remark that for any maps \(\alpha : S^m \to Z_\infty\) and \(\beta : S^n \to Z_\infty\) (and hence \(W = S^0\) in this case), I.M. James proves the relation \(\phi(d(M, M)) = (\Sigma \cdot 1)^m[\phi(\alpha), \phi(\beta)]_{\Sigma W}\) in Theorem 6.1 of [4]. Now let \(f : X \to Z\) and \(g : Y \to Z\) be any maps. By Theorem 2.5 \((W = S^0, \alpha = i \circ f, \beta = i \circ g),\) we have \(c(M) = \Sigma [\Sigma f, \Sigma g]_{\Sigma W}\) for a pairing \(M = (i \circ g \circ p_2) + (i \circ f \circ p_1)\). Hence \(c(M)\) is the generalization of \(\phi(d(M, M))\).

We define a natural map \(\sigma : (X \times Y) \wedge W \to (X \wedge W) \times (Y \wedge W)\) by
\[
\sigma((x, y) \wedge w) = (p_1(x, y) \wedge w, p_2(x, y) \wedge w) = (x \wedge w, y \wedge w) \quad (x \in X, y \in Y, w \in W).
\]
Then the skew pairing \(M = (\alpha \circ (p_1 \wedge 1_W)) + (\beta \circ (p_2 \wedge 1_W))\) is expressed by
\[
M = m \circ (\alpha \times \beta) \circ \sigma : (X \times Y) \wedge W \to Z_\infty.
\]

Let \(\chi : W \to W \wedge W\) be the reduced diagonal map and let
\[
\tau : X \wedge Y \wedge W \wedge W \to X \wedge W \wedge Y \wedge W
\]
be the natural homeomorphism interchanging the second and third factors of the smash products. We suppose that there is a skew pairing \(\mu_W : (S^m \times S^n) \wedge W \to Z\) with axes \(f : \Sigma^m W \to Z\) and \(g : \Sigma^n W \to Z\). For the Hopf construction by Hardie and Jansen in [2], they prove the following relation (Theorem 2.4 in [2]):
\[
H(c(f, g)) = \pm \Sigma (f \wedge g) \circ \Sigma^{m+n+1} \chi.
\]
Here we determine the Hopf invariant of \(\phi(M)\) for a skew pairing \(M = (i \circ f \circ (p_1 \wedge 1_W)) + (i \circ g \circ (p_2 \wedge 1_W))\).

**Theorem 2.6.** We define a skew pairing \(M = (i \circ f \circ (p_1 \wedge 1_W)) + (i \circ g \circ (p_2 \wedge 1_W))\) for any maps \(f : X \wedge W \to Z\) and \(g : Y \wedge W \to Z\). Then we have
\[
H(\phi(M)) = \Sigma (f \wedge g) \circ \Sigma \tau \circ (1_{\Sigma X \wedge Y} \wedge \chi) \circ (\Sigma q \wedge 1_W)
\]
and hence
\[
H(\phi(M)) \circ (v \wedge 1_W) = \Sigma (f \wedge g) \circ \Sigma \tau \circ (1_{\Sigma X \wedge Y} \wedge \chi).
\]
Proof. The following diagram is commutative.

\[
\begin{array}{ccccccccc}
(X \times Y) \land W & \xrightarrow{\sigma} & (X \land W) \times (Y \land W) & \xrightarrow{fxg} & Z \times Z & \xrightarrow{i \times i} & Z_\infty \times Z_\infty \\
q \land 1_W & \downarrow & q & \downarrow m & Z_2 & \xrightarrow{i} & Z_\infty \\
(X \land Y) \land W & \xrightarrow{\tau \circ (1_{X \land Y} \land \chi)} & (X \land W) \land (Y \land W) & \xrightarrow{f \land g} & (Z \land Z) & \xrightarrow{i} & (Z \land Z)_\infty \\
\end{array}
\]

Then by the diagram above, we have

\[
H(\phi(M)) = \phi(h_2 \circ M) = \phi(h_2 \circ ((i \circ f \circ (p_1 \land 1_W)) + (i \circ g \circ (p_2 \land 1_W)))) = \phi(h_2 \circ m \circ (i \times i) \circ (f \times g) \circ \sigma) = \phi(i \circ (f \land g) \circ \tau \circ (1_{X \land Y} \land \chi) \circ (q \land 1_W)) = \Sigma(f \land g) \circ \Sigma \circ (1_{\Sigma X \land Y} \land \chi) \circ (\Sigma q \land 1_W)
\]

Corollary 2.7. If \(W = S^0\) in Theorem 2.6, then for \(M = (i \circ f \circ p_1) + (i \circ g \circ p_2)\) we have

\[
H(\phi(M)) = \Sigma(f \land g) \circ \Sigma q, \text{ and hence } H(\phi(M)) \circ v = \Sigma(f \land g).
\]

Corollary 2.8. Let \(i_1 : X \to X \times Y\) and \(i_2 : Y \to X \times Y\) be the inclusions. Then we have

\[
H(\Sigma j \circ \rho) = H(\Sigma(i_1 \circ p_1) + \Sigma(i_2 \circ p_2)) = \Sigma(i_1 \land i_2) \circ \Sigma q
\]

Proof. We have

\[
\begin{align*}
\Sigma j \circ \rho &= \Sigma j \circ (\Sigma(j_1 \circ p_1) + \Sigma(j_2 \circ p_2)) \\
&= \Sigma(i_1 \circ p_1) + \Sigma(i_2 \circ p_2) \\
&= \phi(i \circ i_1 \circ p_1) + \phi(i \circ i_2 \circ p_2) \\
&= \phi((i \circ i_1 \circ p_1) + (i \circ i_2 \circ p_2)).
\end{align*}
\]

Then we see that \((i \circ i_1 \circ p_1) + (i \circ i_2 \circ p_2) : X \times Y \to (X \times Y)_\infty\) is a pairing. Here we put \(f = i_1\) and \(g = i_2\) in Corollary 2.7. Then we have the result.

References


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