A Remark on Localization of Injective Modules

U Syu*

*Chiba University

A REMARK ON LOCALIZATION OF INJECTIVE MODULES

U SYU

1. INTRODUCTION
Let $A$ be a commutative ring and $S$ a multiplicatively closed subset of $A$. In [D], Dade studies the conditions under which the localization $M[S^{-1}]$ of any injective $A$-module $M$ is an injective $A[S^{-1}]$-module. In this paper we shall investigate the Dade's result [D, Theorem 13] in case that $A$ is not necessarily commutative and a Gabriel topology $\mathcal{F}$ instead of $S$ respectively. That is, we shall study the conditions under which the localization $M_\mathcal{F}$ of any injective $A$-module $M$ is an injective $A_\mathcal{F}$-module. See [S, VI §5] for Gabriel topologies and [S, VI §6 p.151] for relations with commutative rings case.

Let $A$ be a ring can be non-commutative, $M$ a right $A$-module and $\mathcal{F}$ a Gabriel topology of right ideals on the ring $A$. Then we have the localization of $M$ at $\mathcal{F}$, that is, the direct limit

$$M_\mathcal{F} = \lim_{\longrightarrow} \text{Hom}_A \left( a, M/t(M) \right) \quad \text{for} \quad a \in \mathcal{F},$$

where $t(M)$ is the $\mathcal{F}$-torsion submodule of $M$. See [S, IX §1] for $M_\mathcal{F}$.

2. THE INJECTIVENESS OF LOCALIZED MODULES
As in the introduction, by $M_\mathcal{F}$ we denote $\lim_{\longrightarrow} \text{Hom}_A \left( a, M/t(M) \right)$ for $a \in \mathcal{F}$, and in this section we denote

$$\lim_{\longrightarrow} \text{Hom}_A (a, M) \quad \text{for} \quad a \in \mathcal{F}$$

by $M(\mathcal{F})$. Then we have two canonical homomorphisms:

$$\varphi_M : M \cong \text{Hom}_A (A, M) \rightarrow M(\mathcal{F})$$

and

$$\psi_M : M \cong \text{Hom}_A (A, M) \rightarrow M_\mathcal{F}.$$  

For these canonical homomorphisms, [S, Lemma 1.2, p.196] tells us that

Lemma 1. $\ker \psi_M = \ker \varphi_M = t(M)$.

By First Isomorphism Theorem and Lemma 1, we have $\psi_M (M) \cong M/\ker \psi_M = M/t(M)$. Therefore $\psi_M (M)$ is an $\mathcal{F}$-torsion-free $A$-module, and we have

$$M_\mathcal{F} = (M/t(M))(\mathcal{F}) \cong (\psi_M (M))(\mathcal{F}) \cong (\psi_M (M))_\mathcal{F}.$$
Hence by applying [S, Proposition 2.7, p.203] to $\psi_M(M)$ and using the isomorphism $(\psi_M(M))_F \cong M_F$, we have

**Proposition 2.** For every $A$-module $M$, the following properties are equivalent:

(a) $M_F$ is injective over $A_F$.

(b) $M_F$ is injective over $A$.

Proposition 2 is a generalization of [S, Proposition 2.7, p.203], but Proposition 2 does not need an assumption on $M$ in [S, Proposition 2.7, p.203], that is, $M$ is not needless to be torsion-free as an $A$-module.

**Lemma 3.** Let $A$ be a ring and $F$ a right Gabriel topology on $A$. Then for an injective right $A$-module $M$, the following are equivalent:

(a) $\text{Ext}^1_A(I, t(M)) = 0$ for any right $I$ ideal of $A$.

(b) $M/t(M)$ is an injective $A$-module.

When these equivalent conditions hold, $M_F$ is an injective $A_F$-module.

**Proof.** By the natural exact $A$-sequence $0 \to I \to A \to A/I \to 0$ we obtain a long exact sequence:

$$
\text{Hom}_A(I, t(M)) \to \text{Ext}^1_A(A/I, t(M)) \to \text{Ext}^1_A(A, t(M))
$$

$$
\to \text{Ext}^1_A(I, t(M)) \to \text{Ext}^2_A(A/I, t(M)) \to \text{Ext}^2_A(A, t(M)).
$$

Since $\text{Ext}^1_A(A, t(M)) = \text{Ext}^2_A(A, t(M)) = 0$, we have isomorphism $\text{Ext}^1_A(I, t(M)) \cong \text{Ext}^2_A(A/I, t(M))$. Similarly by natural exact $A$-sequence $0 \to t(M) \to M \to M/t(M) \to 0$ we obtain exact isomorphism $\text{Ext}^1_A(A/I, M/t(M)) \cong \text{Ext}^2_A(A/I, t(M))$ since $M$ is $A$-injective. Therefore $\text{Ext}^1_A(I, t(M)) \cong \text{Ext}^1_A(A/I, M/t(M))$ holds. Hence the conditions (a) and (b) are equivalent. Since $\psi_M(M)$ is an $\mathcal{F}$-torsion-free $A$-module, by [S, Proposition 2.7, p.203] $\psi_M(M)$ is an injective $A_F$-module when $\psi_M(M)$ is an injective $A$-module. Therefore when the conditions (a) and (b) hold, $M_F$ is an injective $A_F$-module since we have the isomorphism $\psi_M(M) \cong M/t(M)$.

As a particular case of Lemma 3, we get [S, Lemma 2.6, p.202], and obviously right hereditary rings satisfy the assumption in Lemma 3.

3. ON THE DADE’S RESULT

In this section, using lemmas in section 2 we have a result relating to Dade’s result [D, Theorem 13].

**Lemma 4.** Let $I$ be a right ideal of $A$. Then $\text{Ext}^1_A(I, t(M)) = 0$ for any injective $A$-module $M$ if and only if there exists an exact $A$-sequence $0 \to K \to P \to I \to 0$ ($P$ is projective) which satisfies the following condition:
A REMARK ON LOCALIZATION OF INJECTIVE MODULES

(1) If $L$ is any $A$-submodule of $K$ such that $K/L$ is an $\mathcal{F}$-torsion module, then there exists an $A$-submodule $N$ of $P$ such that $P/N$ is an $\mathcal{F}$-torsion module and $N \cap K = L$.

Proof. Assume that $\text{Ext}_A^1(I, t(M)) = 0$ for any injective $A$-module $M$. Let $0 \rightarrow K \rightarrow P \rightarrow I \rightarrow 0$ ($P$ is projective) be any exact $A$-sequence, $L$ be any $A$-submodule of $K$ such that $K/L$ is an $\mathcal{F}$-torsion module and take any injective $A$-module $M$ such that $M \supset K/L$. Considering natural homomorphism $f: K \rightarrow K/L$, since $K/L$ is a torsion module and $t(M)$ is a maximal torsion submodule, so $f \in \text{Hom}_A(K, t(M))$. By the exact $A$-sequence $0 \rightarrow K \rightarrow P \rightarrow I \rightarrow 0$ we have an exact sequence $\text{Hom}_A(P, t(M)) \rightarrow \text{Hom}_A(K, t(M)) \rightarrow \text{Ext}_A^1(I, t(M))$. Since $\text{Ext}_A^1(I, t(M)) = 0$ by the assumption, we have a homomorphism $g \in \text{Hom}_A(P, t(M))$ such that $g|_K = f$. Put $N = \ker g$. Then $g$ induces a monomorphism $P/N \rightarrow t(M)$. Since $t(M)$ is a torsion module, so $P/N$ is a torsion module. Since $L = \ker f, N = \ker g$ and $g|_K = f$, we have $L = \ker(g|_K) = K \cap N$.

Conversely, assume that there exists an exact $A$-sequence $0 \rightarrow K \rightarrow P \rightarrow I \rightarrow 0$ ($P$ is projective) which satisfies the condition (1). Then for any injective $A$-module $M$ we obtain an exact sequence

$$\text{Hom}_A(P, t(M)) \xrightarrow{h'} \text{Hom}_A(K, t(M)) \xrightarrow{h''} \text{Ext}_A^1(I, t(M)) \rightarrow 0.$$

For any $f \in \text{Ext}_A^1(I, t(M))$, there exists a homomorphism $g \in \text{Hom}_A(K, t(M))$ such that $h''(g) = f$. Put $L = \ker g$. Then $g$ induces a monomorphism $K/L \rightarrow t(M)$. Since $t(M)$ is a torsion module, so is $K/L$. Therefore by the condition (1) we have an $A$-submodule $N$ of $P$ such that $P/N$ is a torsion module and $N \cap K = L$. Thus $g: K \rightarrow t(M)$ induces a homomorphism: $K/N \cap K = K/L \rightarrow t(M)$.

By noting $K/N \cap K \cong (N + K)/N \subseteq P/N$ (see Second Isomorphism Theorem), that is, $g$ induces a homomorphism $g': (N + K)/N \rightarrow t(M)$. Since $M$ is an injective $A$-module, so $g'$ can be extended to $g'': P/N \rightarrow M$, and since $P/N$ is a torsion module, so $g''(P/N)$ is also a torsion module, that is, $g''(P/N) \subseteq t(M)$. Therefore composing $g''$ with the natural epimorphism $P \rightarrow P/N$, we have a homomorphism $e: P \rightarrow t(M)$, which satisfies $e|_K = g$, i.e. $h'(e) = g$. By the exactness we can obtain $f = h''(g) = h''h'(e) = 0$. Hence $\text{Ext}_A^1(I, t(M)) = 0$ holds. □
Notes that proof of Lemma 4 implies that one short exact $A$-sequence

$$(2) \ 0 \to K \to P \to I \to 0 \ (P \text{ is projective})$$

of $I$ satisfies (1) if and only if all such (2) do, too.

By Lemma 3 and Lemma 4, we have the following main result in this paper.

**Theorem 5** (See [D, Theorem 13]). *If a ring $A$ and a Gabriel topology $\mathcal{F}$ of right ideals of $A$ satisfy the following condition (3), then the localization $M_\mathcal{F}$ of any injective $A$-module $M$ is injective over $A_\mathcal{F}$.*

(3) *For any right ideal $I$ of $A$, there exists a short exact $A$-sequence (2) such that the condition (1) is satisfied.*

Note that there are differences between Theorem 5 and [D, Theorem 13]. For example, the condition (3) of Theorem 5 is stronger than the condition of [D, Theorem 13], and for commutative noetherian rings, every Gabriel topology satisfies the condition (3) by [S, Proposition 4.5, p.170]. We can give an example that the converse of Theorem 5 does not hold: Let $A$ be a quasi-Frobenius serial ring with Kupisch series $e_1A$, $e_2A$ and admissible sequence 2, 2. (See: [AF, Section 32]) Let $P_i = e_iA$ and $S_i = P_i/\text{rad}P_i$ for $i = 1, 2$. Then $P_i$ are projective and injective modules of composition length 2. Let $\mathcal{T}$ be the hereditary torsion class generated by $S_1$ and $\mathcal{F}$ the corresponding Gabriel topology on $A$. Then it is routine to check that the localization of any injective $A$-module is injective. However $P_2/t(P_2) \cong S_2$ is not injective. Therefore, by Lemma 3, the converse of Theorem 5 does not hold.

**REFERENCES**


A REMARK ON LOCALIZATION OF INJECTIVE MODULES

U SYU
DEPARTMENT OF MATHEMATICS
DIVISION OF MATHEMATICS AND PHYSICAL SCIENCE
GRADUATE SCHOOL OF SCIENCE AND TECHNOLOGY
CHIBA UNIVERSITY
1-33 YAYOI-CHO, INAGE-KU, CHIBA-CITY, JAPAN 263-8522
E-mail address: mshouyu@math.s.chiba-u.ac.jp

(Received October 22, 1998)