Group rings with nilpotent unit groups

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Dedicated to Professor Keizo Asano on his 60th birthday

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In their paper [1], J. M. Bateman and D. B. Coleman stated the following: Let $F$ be a field, and $G$ a finite group. (a) Let the group ring $FG$ be semi-simple. Then the unit group of $FG$ is nilpotent if and only if $G$ is abelian. (b) Let the characteristic of $F$ be a prime $p$ dividing the order of $G$. Then the unit group of $FG$ is nilpotent if and only if $G$ is a nilpotent group such that the $q$-Sylow subgroup is abelian for every prime $q 
eq p$. Unfortunately, they used there an incorrect lemma, which should be corrected as follows:

**Lemma 1.** Let $S$ be a ring with $1$, and $N$ a nilpotent ideal of $S$. If $S/N$ is commutative and $[N,S] = \{[x,y] = xy - yx | x \in N, y \in S\}$ is contained in $N^2$ then the unit group $S^*$ of $S$ is nilpotent. In particular, if $S/N^2$ is commutative then $S^*$ is nilpotent.

**Proof.** We define $(u,v) = u^{-1}v^{-1}uv$ for $u, v \in S^*$, and inductively $(u_1, \ldots, u_n) = ((u_1, \ldots, u_{n-1}), u_n)$ for $u_1, \ldots, u_n \in S^*$. Then, we see by induction that for $n \geq 1$

$(u_1, \ldots, u_n)^{-1} = (u_1, \ldots, u_{n-1})^{-1}u_n^{-1}[(u_1, \ldots, u_{n-1}) - 1, u_n] \in N^{n-1}$. Since $N$ is nilpotent, it follows that $S^*$ is nilpotent.

**Remark.** Let $D = Q + Qi + Qj + Qij$ be the quaternion division algebra over the rational number field $Q$. We consider the ring $S = \{ \begin{pmatrix} a & 0 \\ d & a \end{pmatrix} | d \in D, a \in C = Q + Qi \}$. Then, $N = \{ \begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix} | d \in D \}$ is an ideal of $S$ with $N^2 = 0$ and $S/N$ is isomorphic to the field $C$. For an arbitrary integer $n$, we have

$\begin{pmatrix} 1 & 0 \\ nj & 1 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -nj & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2nj & 1 \end{pmatrix}$, whence one will easily see that $S^*$ is not nilpotent. This example shows that the assumption $[N,S] \subseteq N^2$ is indispensable in Lemma 1. Next, we shall claim that the converse of Lemma 1 is not true. Evidently the radical $N$ of the ring $S = \{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} | a, b, c \in GF(2) \}$ coincides

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with \( \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \mid b \in GF(2) \) and \( S/N \) is isomorphic to \( GF(2) \oplus GF(2) \). Moreover, \( S' = \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \} \) is commutative and \([N,S] \neq 0 = N^3\).

Now, we shall prove the following:

**Proposition.** Let \( S \) be a semi-primary ring with 1 such that the radical \( R \) is nilpotent and \( S^* = S/R^a \) is commutative, and let \( G \) be a finite group. If (1) \( G \) is commutative or (2) \( S/R \) is of prime characteristic \( p \) and \( G \) is a nilpotent group such that the \( q \)-Sylow subgroup is commutative for every prime \( q \neq p \), then the unit group of the group ring \( SG \) is nilpotent.

**Proof.** We consider the ring homomorphism \( \lambda \) of \( \mathbb{S} = SG \) onto the group ring \( \mathbb{S}^* = S^* G \) defined by \( \sum_{s \in S} s \sigma \mapsto \sum_{s \in S} s^* \sigma \) where \( s^* \) is the residue class of \( s \in S \) modulo \( R^a \). Evidently, \( RG \) is nilpotent and \( \text{Ker} \lambda = R^a G = (RG)^a \). If \( G \) is commutative then \( \mathbb{S}/(RG)^a \) is isomorphic to the commutative ring \( \mathbb{S}^* \), and hence \( \mathbb{S}^* \) is nilpotent by Lemma 1. It remains therefore to prove the case (2). Let \( G = H \times P \) where \( P \) is a \( p \)-group and \( H \) an abelian group of order prime to \( p \). By [3; Corollary 1], the respective radicals \( \mathfrak{R} \) and \( \mathfrak{R}^* \) of \( SP \) and \( S^* P \) are \( \sum_{\rho \in P} S(\rho - 1) \) and \( \sum_{\rho \in P} S^*(\rho - 1) + (R/R^a)P \). Moreover, noting that \( (\mathfrak{R} H)^a \) contains \( \text{Ker} \lambda \) and \( \lambda((\mathfrak{R} H)^a) = (\mathfrak{R}^* H)^a \), we see that \( \mathbb{S}/(\mathfrak{R} H)^a \) is isomorphic to \( \mathbb{S}^*/(\mathfrak{R}^* H)^a \). As \( H \) is contained in the center of \( \mathbb{S}^* \) and \( [\sigma, \tau] = [\sigma - 1, \tau - 1] \in (\mathfrak{R}^* H)^a \) for every \( \sigma, \tau \in P \), it is easy to see that \( (\mathbb{S}^*/(\mathfrak{R}^* H)^a) \) and hence \( \mathbb{S}/(\mathfrak{R} H)^a \) is commutative. As was noted in the proof of [3; Corollary 1], \( \mathfrak{R}^k \) is contained in \( RP \) for some \( k \), which implies that \( \mathfrak{R} H \) is nilpotent. Hence, again by Lemma 1, \( \mathbb{S}^* \) is nilpotent.

As is well-known, the unit group of the complete \( n \times n \) matrix ring \( D_n \) over a division ring \( D \) is not nilpotent for \( n > 1 \). Moreover, it is known that the unit group of a division ring \( D \) is not commutative ([2] or [4]). Accordingly, we readily obtain

**Lemma 2.** If the unit group of an artinian semi-simple ring \( S \) is nilpotent then \( S \) is commutative.

Combining the proposition with Lemma 2, we can generalize somewhat the statement cited at the opening of this note.

**Theorem.** Let \( S \) be an artinian semi-simple ring, and \( G \) a finite group. Then, the unit group of the group ring \( SG \) is nilpotent if and

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only if $S$ is commutative and either (1) $G$ is abelian or (2) $S$ is of prime characteristic $p$ and $G$ is a nilpotent group such that the $q$-Sylow subgroup is commutative for every prime $q \neq p$.

Proof. By the validity of our proposition, it suffices to prove the only if part. If $S$ is simple and the characteristic of $S$ does not divide the order of $G$ then, as is well-known, $SG$ is artinian semisimple. Hence, $S$ and $G$ must be commutative by Lemma 2. Next, if $S$ is a simple ring of prime characteristic $p$ dividing the order of $G$ then by the fact noted just above $S$ and every $q$-Sylow subgroup of $G$ are commutative ($q \neq p$). Now, combining those above, we can readily complete our proof.

Although the converse of our proposition is not valid, we obtain the following:

Corollary. Let $S$ be a semi-primary ring with 1, and $G$ a finite group. If the unit group of $SG$ is nilpotent then the residue class ring $\bar{S}$ of $S$ modulo its radical $R$ is commutative and either (1) $G$ is commutative or (2) $\bar{S}$ is of prime characteristic $p$ and $G$ is a nilpotent group such that the $q$-Sylow subgroup is commutative for each prime $q \neq p$.

Proof. We consider the ring homomorphism $\mu$ of $SG$ onto $\bar{SG}$ defined by $\sum_{s \in S} s \sigma \mapsto \sum_{\bar{s} \in \bar{S}} \bar{s} \sigma$ where $\bar{s}$ is the residue class of $s$, modulo $R$. As is well known, $\text{Ker } \mu = RG$ is contained in the radical of $SG$, and so the unit group of $\bar{SG}$ is nilpotent. Hence, the corollary is evident by our theorem.

REFERENCES


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Added in proof. After the submission of this manuscript, the writers have learned that K. Eldridge has submitted a short paper that correct the error in [1]. Also P. B. Bhattacharya and S. K. Jain [Notices of Amer. Math. Soc. 16 (1969), 562] have presented a counterexample to the lemma of [1], provided another proof for the theorem of [1], and shown that if $S$ is an artinian ring with 1 and $G$ is a finite group such that the unit group of $SG$ is nilpotent then $SG$ satisfies a polynomial identity $(xy - yx)^n = 0$. Indeed, the last is an easy consequence of our theorem.