Continuity of additive functionals

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1. Let $E$ be a measure space with positive finite measure $\mu$, and $S$ be the space of all essentially finite measurable functions on $E$. A set $X$ of $S$ will be called normal if $x \in X, |x| \geq |y|$ imply $y \in X$.

In the following, we assume $X$ is a normal sublattice of $S$. A functional $T$ defined on $X$ is called additive if $x, y \in X; |x| \cap |y| = 0$ imply $T(x+y) = T(x) + T(y)$.

For the integral representation of $T$, the continuity condition is important.

Recently, L. Drewnowski and W. Orlicz have proved that if $T$ is $(uc_\infty)$ and $(uac)$, then $T$ can be represented by $f(r, t)$ such that

$$T(x) = \int_x f(x(t), t) d\mu$$

where $f(0, t) = 0$ a.e., $f(r, t)$ satisfies the generalized Carathéodory conditions and $f(x(t), t)$ is integrable on $E$ for each $x \in X$. [1]

In this note, we shall prove that the continuity conditions of Drewnowski and Orlicz (cf. also Friedman and Katz [2]) is replaced by apparently weaker condition: order continuity. Here we do not assume any "uniform" property.

The proof of Theorem 7 in [4] is not sufficient, so the assumption of Theorem must be replaced by order-continuity instead of $(CIII)$ (semi-continuity).

A sequence $x_n \in X(n = 1, 2, \ldots)$ is order-convergent to $x \in X$, if there exists $y \in X, |x_n| \leq |y|$ and $x_n(t) \to x(t)$ a.e. in $E$.

2. For an additive functional $T$ on $X$, we shall consider the following conditions:

$(uc_\infty)$ $T$ is uniformly continuous by $L_\infty$-norm (essential supremum norm) on every set $\{ y; |y| \leq |x| \}$ with $x \in X, ||x||_\infty < \infty$.

$(uac)$ $T$ is uniformly absolutely continuous on every set $\{ y; |y| \leq |x| \}, x \in X$; that is for each $\varepsilon > 0$, there is a positive number $\delta > 0$ such that $|T(yx_n)| \leq \varepsilon$ for $|y| \leq |x|$ and a measurable set $e$ with $\mu(e) < \delta$ where $x_n$ is a characteristic function of $e$.

$(o)$ If $x_n \in X(n = 1, 2, \ldots)$ is order-convergent to $x \in X$, then $\lim_{n \to \infty} T(x_n) = T(x)$.
Theorem (o) is equivalent to \((uc_w)\) and \((uac)\).

Proof. Since \((uc_w)\) and \((uac)\) imply (o) obviously, we shall prove the converse.

(o) \(\Rightarrow (uac)\) This can be proved by the same method used in the proof of theorem 1.3 of Drewnowski and Orlicz [1].

(o) \(\Rightarrow (uc_w)\) It is sufficient to prove the case \(X = \{x; |x(t)| \leq 1 ~a.e. t \in E\}\). Let denote by \(\lambda\) the set of all dyadic numbers in \([-1, 1]\) i.e. if \(\lambda \in \Lambda\), then \(\lambda = m/2^n\) for some integers \(n(>0)\), \(m\) with \(-2^n \leq m \leq 2^n\) and let \(h_\lambda(t) (\lambda \in \Lambda)\) be Radon-Nikodým's derivative of \(T(\lambda X_e)\) i.e. \(T(\lambda X_e) = \int e \leq h_\lambda(t) d\mu\) where \(X_e\) is a characteristic function of a measurable set \(e \subset E\). Now we take an arbitrary \(\delta > 0\) and fix it for a while. For integers \(n\) and \(m = -2^n + 1, \ldots, 0, 1, \ldots, 2^n\), we consider measurable sets:

\[
e_n^m = \{t; |h_\lambda - h_\mu(t)| \geq \delta \text{ for some } \lambda, \mu \in \Lambda \text{ with } (m-1)/2^n \leq \lambda, \mu \leq m/2^n\} (n = 1, 2, \cdots; m = -2^n + 1, \ldots, 0, 1, \ldots, 2^n)
\]

and

\[
b_n = \bigcup_{m=-2^n+1}^{2^n} e_n^m. \text{ Note that } e_n^m \text{ is written by } e_n^m = \bigcup_{\lambda, \mu = \Lambda} \{t; |h_\lambda(t) - h_\mu(t)| \geq \delta\}
\]

where \(\lambda, \mu (\in \Lambda)\) are countable. By definition of \(b_n\), we have \(b_1 \supset b_2 \supset \cdots \supset b_n \supset \cdots\), and let us denote their limit \(b = \bigcap_{n=1}^{\infty} b_n\).

(i) Let \(\mu(b) = 0\) for every \(\delta > 0\). For each \(\varepsilon > 0\), by \((uac)\), there exists \(\varepsilon' > 0\) such that \(|T(y X_e)| < \varepsilon\) for all \(y \in X\) and for \(\mu(e) < \varepsilon'\). At the same time we see that there is \(n\) with \(\mu(b_n) < \varepsilon'\) such that

\[
|h_\lambda(t) - h_\mu(t)| < \varepsilon \text{ for } |\lambda - \mu| \leq 1/2^n \text{ with } \lambda, \mu \in \Lambda, t \in E \sim b_n.
\]

This will be seen if we put \(\delta = \varepsilon/2\). Let \(||x(t) - x'(t)||_\infty < 1/2^{n+1}\). By (o) we can find \(\lambda_i, \mu_i (\in \Lambda) (i: \text{finite set})\) and mutually disjoint measurable sets \(e_i\) such that

\[
||x - \sum_{i}^\lambda \lambda_i X_{e_i}||_\infty \leq 1/2^{n+1}, \quad ||x' - \sum_{i}^\mu \mu_i X_{e_i}||_\infty \leq 1/2^{n+1}
\]

and

\[
|T(x) - T(\sum_{i}^\lambda \lambda_i X_{e_i})| < \varepsilon, \quad |T(x') - T(\sum_{i}^\mu \mu_i X_{e_i})| < \varepsilon.
\]

We have \(|\lambda_i - \mu_i| < 1/2^n\) for each \(i\) and

\[
|T(\sum_{i}^\lambda \lambda_i X_{e_i}) - T(\sum_{i}^\mu \mu_i X_{e_i})| \leq \sum_{i}^\infty |h_{\lambda_i} - h_{\mu_i}| d\mu
\]
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\[ + |T(\sum_{i}^{\lambda} X_{i} X_{\mu_{i}})| + |T(\sum_{i}^{\lambda} X_{i} X_{\mu_{i}})| \lesssim \varepsilon \mu(E) + 4\varepsilon. \]

Hence,
\[ |T(x) - T(x')| \lesssim |T(x) - T(\sum_{i}^{\lambda} X_{i})| + |T(\sum_{i}^{\lambda} X_{i}) - T(\sum_{i}^{\lambda} X_{i})| \]
\[ + |T(x') - T(\sum_{i}^{\lambda} X_{i})| \lesssim \varepsilon \mu(E) + 4\varepsilon. \]

(uv) follows from this.

(ii) Let \( \mu(b) > 0 \) for some \( \delta > 0 \). It will be shown that this case can not occur. For every \( n \), we shall define a partition of \( b \) by induction, that is
\[ b = b_{n} \cup \cdots \cup b_{n}, \]
\[ b_{n}(m = -2^{n} + 1, \ldots, 2^{n}) \] are mutually disjoint measurable sets and for every \( p \geq n \), \( b_{n} \subset e_{n}^{p} \) where \( e_{n}^{p} = \{ t \in e_{n}^{p} \mid m' \leq m' \mid 2^{n} \leq m' \leq m/2^{n} \} \). We take a note that the sequence \( e_{n}^{p} = \{ p = n, n + 1, \ldots \} \) is monotone decreasing with respect to \( p \) and that \( \bigcup_{m=-2^{n+1}}^{2^{n}} e_{n}^{p} = b_{n}. \)

Let us define \( b_{n}(m = -1, 0, 1, 2) \), at first we put
\[ a_{m} = b \cap e_{n}^{m} \cap e_{n+2}^{m} \cap \cdots, \quad (m = -1, 0, 1, 2). \]

Then, we have
\[ b = \bigcup_{m=-1}^{2} a_{m}, \quad b \sim \bigcup_{m=-1}^{2} a_{m} = \bigcup_{m=-1}^{2} a_{m} \subset \bigcup_{p=1}^{\infty} (b \sim \bigcup_{m=-1}^{2} e_{n}^{m}) \subset \bigcup_{p=1}^{\infty} (b \sim \bigcup_{m=-1}^{2} e_{n}^{m}) = \phi. \]

Hence we define, \( b_{1} = a_{1}, b_{1} = a_{1}, a_{1} = a_{1}, a_{1} = a_{1}, \ldots \), etc., then by definition \( b_{1} \subset a_{m} \subset e_{n}^{m} \) for all \( p \geq 1 \). Assuming we have defined \( b_{n} \), we shall define \( b_{n+1} \) of the form \( s = -2^{n+1} + 1, \ldots, 0, 1, \ldots, 2^{n+1} \). It suffices to define \( b_{n+1}^{b_{n}} \) of the form \( b_{n+1}^{b_{n}} \) with some \( m \in \{ -2^{n+1} + 1, \ldots, 0, 1, \ldots, 2^{n+1} \} \). We set
\[ b_{n+1}^{b_{n}} = b_{m} \cap e_{n+1}^{m} \cap e_{n+1}^{m} \cap \cdots, \]
\[ b_{n+1}^{b_{n}} = b_{m} \sim b_{n+1}^{b_{n}}. \]

We must check \( b_{n+1}^{b_{n}} \subset e_{n+1}^{b_{n+1}} \) for \( p \geq n+1 \). This will be done, since \( t \in b_{n+1}^{b_{n}} \Rightarrow t \in e_{n+1}^{b_{n+1}} \Rightarrow t \in e_{n+1, k}^{b_{n+1}} \) for some \( k \geq n+1 \) \( t \in e_{n+1, k}^{b_{n+1}} \) for all \( k \geq n+1 \) \( t \in e_{n+1, k}^{b_{n+1}} \) for all \( k \geq n+1 \). Since \( e_{n+1}^{b_{n+1}} \cap e_{n+1, k}^{b_{n+1}} = e_{n+1}^{b_{n+1}} \) by definition. For \( n \), we define measurable functions \( f_{n}, g_{n} \in X \) such that
\[ f_n(t) = \sum_{m=-2^n+1}^{2^n} \frac{m-1}{2^n} \chi_{a_m^n}(t), \]
\[ g_n(t) = \sum_{m=-2^n+1}^{2^n} \frac{m}{2^n} \chi_{a_m^n}(t). \]

We have by definition \( f_n(t) \leq f_{n+1}(t) \leq g_{n+1}(t) \leq g_n(t) \), and \( |f_n(t) - g_n(t)| \leq 1/2^n \). Hence, there exists \( x \in X \) with \( f_n \to x, \ g_n \to x \) (order convergence). For \( m = -2^n + 1, \ldots, 2^n \), we can decompose \( b_m^n \) into

\[ b_m^n = c_{m,1}^{n} \cup c_{m,2}^{n} \cup \cdots \quad \text{(mutually disjoint)} \]

where \( c_{m,t}^{n} \subset \{ t : h_{m,t}^{n}(t) - h_{m,t}^{n}(t) \geq \delta \} \) for every pair \( \lambda_{m,t}, \mu_{m,t} \in \Lambda \) with \( (m-1)/2^n \leq \lambda_{m,t}, \mu_{m,t} \leq m/2^n \), since \( b_m^n \subset e_m^n = \bigcup_{t} \{ t : h(t) - h(t) \geq \delta \} \) where \( (m-1)/2^n \leq \lambda, \mu \leq m/2^n \) with \( \lambda, \mu \in \Lambda \). If we define \( f'_{n}, g'_{n} \) as follows:

\[ f'_{n}(t) = \sum_{m=-2^n+1}^{2^n} \sum_{t} \chi_{e_{m,t}^{n}}(t) \]
\[ g'_{n}(t) = \sum_{m=-2^n+1}^{2^n} \sum_{t} \chi_{a_{m,t}^{n}}(t), \]

then \( f_n(t) \leq f'_{n}(t), \ g'_{n}(t) \leq g_n(t) \) and

\[ T(f'_{n}) - T(g'_{n}) \geq \delta_{\nu}(b) \]

with \( f'_{n} \to x \) and \( g'_{n} \to x \) (order convergence). This contradicts the condition (o).

Remark. In above Theorem, the condition (o) is replaced by the condition that \( T \) is continuous by measure convergence.

REFERENCES


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