Some generalizations of duality theorems in mathematical programming problems

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SOME GENERALIZATIONS OF DUALITY THEOREMS IN MATHEMATICAL PROGRAMMING PROBLEMS

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§ 1. Introduction and problem setting

Let $X$, $Z$ and $W$ be real linear spaces and suppose that $Z$ and $W$ are in duality with respect to a certain bilinear functional $((, ))$. Let $C$ and $D$ be nonempty sets in $X$ and $Z$ respectively, and let $f$ and $g$ be finite-valued real functions on $C$ and $D$ respectively. Assume that $g(z) = -\infty$ for every $z \in D$. Let $A$ be a transformation from $C$ into $Z$.

We shall be concerned with the following two problems:

(I) Determine $M = \inf \{ f(x) - g(Ax); x \in C \}$,

(II) Determine $M^* = \sup \{ g^*(w) - f^*_C(w); w \in W \}$,

where

$$g^*(w) = \inf \{ ((z, w)) - g(z); z \in D \}$$

and

$$f^*_C(w) = \sup \{ ((Ax, w)) - f(x); x \in C \}.$$

Here we define

$$r + \infty = \infty + r = \infty, \quad r - \infty = -\infty + r = -\infty$$

for all real numbers $r$, and set

$$\infty + \infty = \infty, \quad -\infty - \infty = -\infty, \quad -(\infty) = \infty.$$

More precisely, we shall study the problems

(i) the existence of $x$ or $w$ which attains the infimum or the supremum,

(ii) relations between the values $M$ and $M^*$.

An answer to problem (ii) is called a duality theorem.

R. T. Rockafellar [6] investigated these problems in the case where $A$ is linear and continuous, $C$ and $D$ are convex sets and $f$ and $-g$ are convex functions. Our problems (I) and (II) contain the problems discussed by U. Dieter [3], K. S. Kretschmer [4] and R. Van Slyke and R. Wets [7]. M. Yamasaki [8] studied the above problems in the case where $C$ is a convex set, $D$ is a convex cone, $f$ is a convex function, $g = 0$ and $A$ is convex with respect to $D$. 69
In the present paper, we shall generalize duality theorems given in [3], [4], [7] and [8] by making use of a well-known separation theorem. We shall introduce in § 5 a condition which was called the normality condition in [6] and [7]. By means of this condition, duality theorems in § 3 will be generalized.

§ 2. Preliminaries

For later use, we shall recall some notions and results in [1] and [2]. Let $X$ and $Y$ be real linear spaces in duality with respect to a certain bilinear functional $(\ , \ )$. Let us denote the weak topology on $X$ by $w(X, Y)$ and the Mackey topology by $s(X, Y)$. A locally convex Hausdorff topology $t(X, Y)$ on $X$ compatible with this duality is stronger than $w(X, Y)$ and weaker than $s(X, Y)$. If $X$ is assigned $t(X, Y)$, then every element of $Y$ is identified with a $t(X, Y)$-continuous linear functional on $X$.

Let $R$ be the set of real numbers and $R_0$ the set of non-negative real numbers.

We shall utilize the following separation theorem:

**Proposition 1.**  Let $K$ be a $w(X, Y)$-closed convex set in $X$ and $x_0$ be an element of $X$ such that $x_0 \not\in K$. Then there exist $y_0 \in Y$ and $\alpha \in R$ such that

\[
((x_0, y_0)) \geq \alpha \geq (x, y_0)
\]

for all $x \in K$.

Next we shall recall the conjugate operation of convex functions in [3], which will be used in § 4. For a finite-valued real convex function $p$ on $X$ with nonempty convex domain $P$, the conjugate function $p^*$ and the conjugate set $P^*$ are defined by

\[
p^*(y) = \sup \{ ((x, y)) - p(x) : x \in P \} ,
\]

\[
P^* = \{ y \in Y : p^*(y) < \infty \} .
\]

Then $p^*$ is a finite-valued real convex function with convex domain $P^*$.

Let us define

\[
[p, P] = \{ (x, r) : x \in P \text{ and } r \geq p(x) \} .
\]

For a finite-valued real concave function $q$ on $X$ with nonempty convex domain $Q$, there are similar definitions:

\[
q^*(y) = \inf \{ ((x, y)) - q(x) : x \in Q \} ,
\]

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1) [1], p. 73, Proposition 4 and [2], p. 50, Proposition 1.
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\[ Q^* = \{ y \in Y : q^*(y) \geq -\infty \}, \]
\[ [q, Q] = \{ (x, r) : x \in Q \text{ and } r \leq q(x) \}. \]

Then \( q^* \) is a finite-valued real concave function with convex domain \( Q^* \).

Dieter proved

Proposition 2. Let \( X \times R \) and \( Y \times R \) be in duality with respect to the bilinear functional \( < , > \) defined by

\[ <(x, r), (y, s)> = ((x, y)) + rs \]

for all \( (x, r) \in X \times R \) and \( (y, s) \in Y \times R \).

(1) If \( P \) is \( w(X, Y) \)-closed and \( p \) is lower semicontinuous with respect to \( w(X, Y) \), then \([p, P]\) is \( w(X \times R, Y \times R) \)-closed.

(2) If \([p, P]\) is \( w(X \times R, Y \times R) \)-closed, then \( p^{**} = (p^*)^* = p \) and \( P^{**} = (P^*)^* = P \).

§ 3. Duality theorems

Let \( Z \times R \) and \( W \times R \) be in duality with respect to the bilinear functional \( < , > \) defined by

\[ <(z, r), (w, s)> = ((z, w)) + rs \]

for every \( (z, r) \in Z \times R \) and \( (w, s) \in W \times R \). Let \( E, E_0 \) and \( L \) be the sets in \( Z \times R \) defined by

\[ E = \{ (Ax - z, r + f(x) - g(z)) : x \in C, z \in D \text{ and } r \in R_0 \}, \]
\[ L = \{ (0, r) : 0 \in Z \text{ and } r \in R \}, \]
\[ E_0 = E \cap L. \]

In case \( C \cap A^{-1} (D) \) is not empty, we have

\[ E_0 = \{ (0, r + f(x) - g(Ax)) : 0 \in Z, x \in C \cap A^{-1} (D) \text{ and } r \in R_0 \}. \]

First we shall study the existence of \( x \) which attains the value \( M \) of problem (I). We have

Theorem 1. Assume that the value \( M \) is finite. Then there exists \( x \in C \) such that \( Ax \in D \) and \( M = f(x) - g(Ax) \) if and only if the set \( E_0 \) is \( w(Z \times R, W \times R) \)-closed.

Proof. Since \( M \) is finite, we have

\[ \{ 0 \} \times (M, + \infty) \subset E_0 \subset \{ 0 \} \times [M, + \infty). \]

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2) [3], p. 98, Hilfssatz 5 and p. 99, Hilfssatz 7.
Therefore the set $E_n$ is $w(Z \times R, W \times R)$-closed if and only if $(0, M)$ belongs to $E_n$. We see easily that there exists $x \in C$ such that $Ax \in D$ and $M = f(x) - g(Ax)$ if and only if $(0, M) \in E_n$.

Observe that the set $E_n$ is $w(Z \times R, W \times R)$-closed whenever the set $E$ is $w(Z \times R, W \times R)$-closed, since the set $L$ is $w(Z \times R, W \times R)$-closed. However, the $w(Z \times R, W \times R)$-closedness of the set $E_n$ does not necessarily imply the $w(Z \times R, W \times R)$-closedness of the set $E$. This is shown by Example 5.1 in [4] or Example 3.5 in [7].

As for the $w(Z \times R, W \times R)$-closedness of the set $E_n$, we have

**Proposition 3.** Let $X$ be a topological linear space and let $Z$ be assigned $w(Z, W)$. Assume that the functions $f$ and $-g$ are lower semicontinuous and that the transformation $A$ is continuous. If $C \cap A^{-1}(D)$ is a nonempty and compact set, then the set $E_n$ is $w(Z \times R, W \times R)$-closed.

**Proof.** Let $(0, r_t; t \in T)$ be a net in $E_n$ which $w(Z \times R, W \times R)$-converges to $(z, r) \in Z \times R$. Then $z = 0$ and there exists $x_t \in C \cap A^{-1}(D)$ such that $r_t = f(x_t) - g(Ax_t)$. By the compactness of $C \cap A^{-1}(D)$, there exists a subnet $(x_t; t \in T')$ which converges to some $x \in C \cap A^{-1}(D)$. Then by the continuity of $A$ and the lower semicontinuity of $f$ and $-g$, we have

$$r = \lim_{t \in T'} r_t \geq \lim_{t \in T'} f(x_t) - g(Ax_t) \geq f(x) - g(Ax),$$

and hence $(0, r) \in E_n$. Therefore the set $E_n$ is $w(Z \times R, W \times R)$-closed.

Next we shall investigate some relations between the values $M$ and $M^*$. We have

**Theorem 2.** It is always valid that $M^* \leq M$.

**Proof.** In case $C \cap A^{-1}(D)$ is empty, we have $M = \infty$ and our assertion is obvious. In case $C \cap A^{-1}(D)$ is not empty, let $x$ and $w$ be arbitrary elements of $C \cap A^{-1}(D)$ and $W$ respectively. The inequalities

$$f(x) + f^*_w(w) \geq ((Ax, w)),$$
$$g(Ax) + g^*(w) \leq ((Ax, w))$$

follow from the definitions of $f^*_w$ and $g^*$ in §1. Thus we have

$$f(x) - g(Ax) \geq g^*(w) - f^*_w(w).$$

This completes the proof.

Before giving the converse relation $M^* \geq M$, we shall prepare
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Lemma 1. If \( w \in W \) and \( \alpha \in R \) satisfy the inequality
\[
\alpha \gtrless ((u, w)) - r
\]
for all \((u, r) \in E\), then
\[
\alpha \gtrless f^*_s(w) - g^*(w).
\]

Proof. Since \((Ax - z, f(x) - g(z)) \in E\) for any \( x \in C \) and \( z \in D\), we have
\[
\alpha \gtrless ((Ax - z, w)) - f(x) + g(z)
\]
\[
= \{(Ax, w)) - f(x)\} - ((z, w)) - g(z)\}.
\]
From the definitions of \( f^*_s \) and \( g^* \), it follows that
\[
\alpha \gtrless f^*_s(w) - g^*(w).
\]
Now we shall prove

Theorem 3. If the value \( M \) is finite and the set \( E \) is convex and \( w(Z \times R, W \times R) \)-closed, then \( M = M^* \) holds.

Proof. For an arbitrarily fixed \( \varepsilon > 0, (0, M - \varepsilon) \notin E \). Since \( E \) is a \( w(Z \times R, W \times R) \)-closed convex set, there exist \((w, s) \in W \times R \) and \( \alpha \in R \) such that
\[
(M - \varepsilon)s \gtrless (u, w) + rs
\]
for all \((u, r) \in E\) by Proposition 1. From the fact that \((0, M + \varepsilon) \in E\), it follows that \((M - \varepsilon)s \gtrless (M + \varepsilon)s\) and hence \( s \leq 0\). Writing \( \alpha_0 = \alpha / s \) and \( w_0 = -w / s\), we have
\[
M - \varepsilon \leq \alpha_0 \leq -((u, w_0)) + r
\]
for all \((u, r) \in E\). By means of Lemma 1, we see that
\[
\alpha_0 \leq g^*(w_0) - f^*_s(w_0) \leq M^*.
\]
Therefore \( M^* > M - \varepsilon\). By the arbitrariness of \( \varepsilon\), we conclude that \( M^* \geq M\). The converse inequality was given in Theorem 2. This completes the proof.

Theorem 4. If the value \( M^* \) is finite and the set \( E \) is convex and \( w(Z \times R, W \times R) \)-closed, then \( M = M^* \) holds.

Proof. Suppose \((0, M^*) \notin E\). By Proposition 1 there exist \((w, s) \in W \times R \) and \( \alpha \in R \) such that
\[
M^* s \gtrless (u, w) + rs
\]
for all \((u, r) \in E\). For a fixed \((u_t, r_t) \in E\), we have \((u_t, r_t + t) \in E\) for all \(t \in R_o\) and by (1)

\[ \alpha \geq ((u, w)) + r, s + ts. \]

Letting \(t \to \infty\), we see that \(s \leq 0\). First we shall consider the case where \(s < 0\). Writing \(\alpha_a = \alpha / s\) and \(w_a = -w / s\), we have

\[ M^* < \alpha_a \leq ((u, w_a)) + r \]

for all \((u, r) \in E\). It follows from Lemma 1 that

\[ \alpha_a \leq g^*(w_a) - f^*_a(w_a) \leq M^*. \]

This is a contradiction. Next we shall consider the case where \(s = 0\). Then we have

\[ 0 > \alpha \geq ((u, w)) \]

for all \((u, r) \in E\). On the other hand, there exist \(v \in W\) and \(\beta \in R\) such that

\[ \beta \geq ((u, v)) - r \]

for all \((u, r) \in E\). In fact, by our assumption that \(M^*\) is finite, we can find \(v \in W\) such that both \(f^*_a(v)\) and \(g^*(v)\) are finite. By the definitions of \(f^*_a\) and \(g^*\), we have

\[ \beta = f^*_a(v) - g^*(v) \geq ((Ax, v)) - f(x) - ((z, v)) + g(z) \]

\[ \geq ((Ax - z, v)) - (r + f(x) - g(z)) \]

for all \(x \in C, z \in D\) and \(r \in R_o\), which implies (4). On account of (3) and (4), we have

\[ \alpha + t \geq ((u, tw + v)) - r \]

for all \((u, r) \in E\) and \(t \in R_o\). We see by Lemma 1 that

\[ \alpha + t \geq f^*_a(tw + v) - g^*(tw + v) \geq -M^*. \]

Letting \(t \to \infty\), we have \(M^* = -\infty\), since \(\alpha < 0\). This is a contradiction. Thus \((0, M^*) \in E\). It follows that \(M^* \geq M\). On account of Theorem 2, we have \(M = M^*\).

With regard to the convexity of the set \(E\), we have

Theorem 5. Assume that \(C\) and \(D\) are convex sets and \(f\) and \(-g\) are convex functions. If any one of the following conditions (M.1) and (M.2) is fulfilled, then the set \(E\) is convex:

(M.1) \(A\) is linear,
(M. 2) $D$ is a cone, $A$ is convex with respect to $D^*$, i.e.,
\[ A(tx_i + (1-t)x_j) - tAx_i - (1-t)Ax_j \in D \]
for any $x_i, x_j \in C$ and $t \in R_0$ with $0 < t < 1$, and $g$ is increasing with respect to $D$, i.e., $g(z_i) \geq g(z_j)$ whenever $z_i - z_j \in D$.

**Proof.** Assume condition (M. 2). Let $(u_i, r_i) \in E (i = 1, 2)$ and $t \in R_0$, $0 < t < 1$. Then there exist $x_i \in C, z_i \in D$ and $s_i \in R_0$ such that $u_i = Ax_i - z_i$ and $r_i = s_i + f(x_i) - g(z_i)$. Let us denote $u_i = tu_i + (1-t)u_0$, $r_i = tr_i + (1-t)r_0$, $x_i = tx_i + (1-t)x_0$, $z_i = tz_i + (1-t)z_0$ and $s_i = ts_i + (1-t)s_0$. Then $x_i \in C, z_i \in D$ and $s_i \in R_0$. Since $A$ is convex with respect to $D$, we have $tAx_i + (1-t)Ax_2 = Ax_i - v$ for some $v \in D$. Thus $u_i = Ax_i - (v + z_i) \in A(C) - D$. On the other hand, by the convexity of $f$ and $-g$ and by the assumption that $g$ is increasing with respect to $D$, we have
\[ r_i = s_i + tf(x_i) + (1-t)f(x_2) - tg(z_i) - (1-t)g(z_2) \]
\[ \geq s_i + f(x_i) - g(z_i) \geq s_i + f(x_2) - g(v + z_i), \]
and hence $r_i = s_i + f(x_i) - g(v + z_i)$ for some $s_i \in R_0$. Therefore $(u_i, r_i) \in E$ and the set $E$ is convex. Similarly we can prove that condition (M. 1) implies the convexity of the set $E$.

By means of Theorem 5, we see that Theorems 3 and 4 are some generalizations of duality theorems in [3], [4], [7] and [8].

We shall study the $w(Z \times R, W \times R)$-closedness of the set $E$. In the rest of this section, we always assume that $X$ is a topological linear space, that $Z$ is assigned $w(Z, W)$, that the sets $C$ and $D$ are closed, that the functions $f$ and $-g$ are lower semicontinuous and that the transformation $A$ is continuous. Then we have

**Theorem 6.** Assume that, for any $w(Z \times R, W \times R)$-convergent net \{(u_i, r_i) ; t \in T\} in $E$, there exist \{x_i ; t \in T\} $\subset C$ and \{z_i ; t \in T\} $\subset D$ such that
\[ u_i = Ax_i - z_i, \quad r_i \geq f(x_i) - g(z_i) \]
and \{x_i ; t \in T\} contains a convergent subnet. Then the set $E$ is $w(Z \times R, W \times R)$-closed.

**Proof.** Let \{(u_i, r_i) ; t \in T\} be a net in $E$ which $w(Z \times R, W \times R)$-converges to $(u, r) \in Z \times R$. By our assumption, there exist \{x_i ; t \in T\} $\subset C$ and \{z_i ; t \in T\} $\subset D$ such that
\[ u_i = Ax_i - z_i, \quad r_i \geq f(x_i) - g(z_i) \]

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3) We correct the definition of this notion in [8], p. 332 in the present form.
and \( \{x_t; t \in T \} \) contains a subnet \( \{x_i; t \in T' \} \) which converges to some \( x \). Then \( \{z_t; t \in T' \} \) converges to \( Ax-u=z \), since \( A \) is continuous. Since \( C \) and \( D \) are closed, we have \( x \in C \) and \( z \in D \). By the lower semicontinuity of \( f \) and \( -g \), we have

\[
 r = \liminf_{t \in T'} r_t \geq \liminf_{t \in T'} f(x_t) - \liminf_{t \in T'} g(z_t) \geq f(x) - g(z).
\]

Therefore \( (u,r) \in E \) and the set \( E \) is \( w(Z \times R, W \times R) \)-closed.

**Corollary.** If the set \( C \) is compact, then the set \( E \) is \( w(Z \times R, W \times R) \)-closed.

Similarly we can prove

**Proposition 4.** Assume that \( A \) is homeomorphic and that the set \( D \) is compact. Then the set \( E \) is \( w(Z \times R, W \times R) \)-closed.

§ 4. The case where \( A \) is linear and continuous

We shall recall the convex programming problems studied by Rockafellar [6].

Let \( X \) and \( Y \) be real linear spaces which are in duality with respect to the bilinear functional \( (\cdot, \cdot) \), and let \( Z \) and \( W \) be real linear spaces which are in duality with respect to the bilinear functional \( (\cdot, \cdot)_\circ \). Let \( C \) and \( D \) be nonempty convex sets in \( X \) and \( Z \) respectively, and let \( f \) and \( -g \) be finite-valued real convex functions on \( C \) and \( D \) respectively. Let \( A \) be a linear transformation from \( X \) into \( Z \) which is \( w(X,Y) \)-continuous and let \( A^* \) be its adjoint. Thus \( A^* \) is a linear transformation from \( W \) into \( Y \) which is \( w(W,Z)-w(Y,X) \)-continuous and satisfies \( ((Ax,w))_\circ = ((x,A^*w))_\circ \) for all \( x \in X \) and \( w \in W \).

By virtue of the conjugate operations for convex sets and convex functions defined in § 2, we see that the function \( g^* \) defined in § 1 is the conjugate function of the concave function \( g \) and that \( f^*_g(w) = f^*(A^*w) \) holds, where \( f^* \) is the conjugate function of the convex function \( f \). Let us denote by \( C^* \) and \( D^* \) the conjugate sets of convex sets \( C \) and \( D \) respectively. The convex programming problems discussed in [6] are as follows:

(III) Determine \( N = \inf \{ f(x) - g(Ax); x \in C \text{ and } Ax \in D \} \),

(IV) Determine \( N^* = \sup \{ g^*(w) - f^*(A^*w); w \in D^* \text{ and } A^*w \in C^* \} \).

Here we use the convention that the infimum and the supremum on the empty set are equal to \(+ \infty\) and \(- \infty\) respectively.
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These problems contain the problems investigated by Dieter [3], Kretschmer [4]. Dieter discussed the case where \( X = Z \) and \( A \) is the identity transformation. Kretschmer discussed the case where

\[
\begin{align*}
    f(x) &= ((x, y_0)), \quad C = P, \\
    g(z) &= 0, \quad D = Q + z_0,
\end{align*}
\]

where \( P \) and \( Q \) are convex cones which are \( w(Z, W) \)-closed and \( w(Z, W) \)-closed respectively, and \( y_0 \in Y \) and \( z_0 \in Z \) are fixed elements. In this case, problems (III) and (IV) are called linear programming problems. Van Slyke and Wets [7] investigated problem (III) in the case where \( g = 0 \) and \( D = \{b\}, \quad (b \in Z) \).

Now we shall apply our results in §3 to problems (III) and (IV). On account of Theorems 3, 4 and 5, we have

**Proposition 5.** Let \( Z \times R \) and \( W \times R \) be in duality as in §3 and let \( E \) be the set in \( Z \times R \) defined by

\[
E = \{(Ax - z, r + f(x) - g(z)); x \in C, z \in D \text{ and } r \in R_o \}.
\]

If the set \( E \) is \( w(Z \times R, W \times R) \)-closed and either \( N \) or \( N^* \) is finite, then \( N = N^* \) holds.

Since problems (III) and (IV) have symmetry, we can derive a dual statement to the above result. Observing that

\[
-N^* = \inf \{-g^*(w) - (-f^*(A^*w)); w \in D^* \text{ and } A^*w \in C^*\},
\]

we shall consider the following problem:

(V) Determine \(-N^{**} = \sup \{-f^{**}(x) + g^{**}(A^{**}x); x \in C^{**} \text{ and } A^{**}x \in D^{**}\}\).

It is always valid that \( N^{**} \leq N \). If the sets \([f, C]\) and \([g, D]\) defined in §2 are \( w(X \times R, Y \times R) \)-closed and \( w(Z \times R, W \times R) \)-closed respectively, then \( f^{**} = f \), \( g^{**} = g \), \( C^{**} = C \) and \( D^{**} = D \) by Proposition 2. In this case, the set \( P \) in \( Y \times R \) defined by

\[
F = \{(A^*w - y, r - g^*(w) + f^*(y)); w \in D^*, y \in C^* \text{ and } r \in R_o \}
\]

plays the role of the set \( E \) in §3. Noting \( A^{**} = A \) and applying Theorems 3, 4 and 5, we have

**Proposition 6.** Assume that the sets \([f, C]\) and \([g, D]\) are \( w(X \times R, Y \times R) \)-closed and \( w(Z \times R, W \times R) \)-closed respectively. If the set \( F \) is \( w(Y \times R, X \times R) \)-closed and either \( N \) or \( N^* \) is finite, then \( N = N^* \) holds.
We shall give an application of Theorem 6.

**Proposition 7.** Let $C$ and $D$ be $w(X, Y)$-closed and $w(Z, W)$-closed respectively and let $f$ and $-g$ be lower semicontinuous with respect to $w(X, Y)$ and $w(Z, W)$ respectively. Assume that any $w(X, Y)$-bounded set in $X$ is relatively $w(X, Y)$-compact. If we further assume that $A^*(D^*) \cap (C^*)^\circ$ is not empty, then the set $E$ is $w(Z \times R, W \times R)$-closed, where $(C^*)^\circ$ denotes the $s(Y, X)$-interior of $C^*$.

**Proof.** Let $\{(u_t, r_t) : t \in T\}$ be a net in $E$ which $w(Z \times R, W \times R)$-converges to $(u, r) \in Z \times R$. Then there exist $x_t \in C$ and $z_t \in D$ such that $u_t = Ax_t - z_t$ and $r_t \leq f(x_t) - g(z_t)$. By the definitions of $f^*$ and $g^*$, we have

$$r_t \geq (x_t, y_t) - (z_t, w_t) - f^*(y_t) - g^*(w_t)$$

for all $y \in C^*$ and $w \in D^*$. By our assumption, there are $y_0$ and $w_0$ such that $w_0 \in D^*$ and $y_0 = A^* w_0 \in (C^*)^\circ$. For any $y \in Y$, there exists $\varepsilon > 0$ such that $y_0 \pm \varepsilon y \in (C^*)^\circ$. Consequently

$$r_t \geq ((x_t, y_0 \pm \varepsilon y), (z_t, w_0) - f^*(y_0 \pm \varepsilon y) + g^*(w_0)$$

$$=((Ax_t - z_t, w_0) + \varepsilon (x_t, y_0) - f^*(y_0 \pm \varepsilon y) + g^*(w_0)$$

$$=((u_t, w_0) + \varepsilon (x_t, y_0) - f^*(y_0 \pm \varepsilon y) + g^*(w_0).$$

Since $r_t - ((u_t, w_0)) : t \in T$ converges to $r - ((u, w_0))$, there is $t_0 \in T$ such that $\{r_t - ((u_t, w_0)) : t \in T, t > t_0\}$ is bounded. Consequently $\{(x_t, y_0) : t \in T, t > t_0\}$ is bounded for every $y \in Y$, and hence $\{x_t : t \in T, t > t_0\}$ is relatively $w(X, Y)$-compact by our assumption. Thus $\{x_t : t \in T\}$ contains a $w(X, Y)$-convergent subnet. Therefore the set $E$ is $w(Z \times R, W \times R)$-closed by Theorem 6.

Note that any $w(X, Y)$-bounded set in $X$ is relatively $w(X, Y)$-compact provided that $Y$ is a disk space (= espace tonnelé) and $X$ is the topological dual space of $Y$ ([2], p. 65, Théorème 1).

**§ 5. Normality condition**

We return to the general problem (I). Let $E$ be as defined in § 3 and denote by $\bar{E}$ the $w(Z \times R, W \times R)$-closure of $E$. We shall introduce another quantity $m$ defined by

$$m = \inf\{r : r \in R \text{ and } (0, r) \in \bar{E}\},$$

where we set $m = \infty$ in the case where $(0, r) \notin \bar{E}$ for any $r \in R$. This
quantity was called the subvalue in the case of linear programming problems (cf. [4]).

We have

**Theorem 7.** It is always valid that $M^* \leq m \leq M$.

*Proof.* The inequality $m \leq M$ follows immediately from the definitions of $m$ and $M$. To prove $M^* \leq m$, we may suppose that $m < \infty$. Let $(0, r) \in \overline{E}$. Then there exists a net $\{(u_t, r_t); t \in T\}$ in $E$ which $w(Z \times R, W \times R)$-converges to $(0, r)$. For every $t \in T$, there exist $x_t \in C$, $z_t \in D$ and $s_t \in R_0$ such that $u_t = Ax_t - z_t$ and $r_t = s_t + f(x_t) - g(z_t)$. By the definitions of $f^*_d$ and $g^*$, we have

$$r_t \geq f(x_t) - g(z_t) \equiv ((Ax_t, w)) - f^*_d(w) - ((z_t, w)) + g^*(w) \equiv ((u_t, w)) + g^*(w) - f^*_d(w)$$

for any $w \in W$ and hence $r \geq g^*(w) - f^*_d(w)$. Thus we have $M^* \leq m$.

**Theorem 8.** If the set $\overline{E}$ is convex and $M^* \geq -\infty$, then $M^* = m$ holds.

*Proof.* On account of Theorem 7, it suffices to show the inequality $M^* \geq m$ in the case where $M^*$ is finite. Suppose $(0, M^*) \not\in \overline{E}$. Applying Proposition 1 to $(0, M^*)$ and the $w(Z \times R, W \times R)$-closed convex set $E$, we can arrive at a contradiction by the same argument as in the proof of Theorem 4. Therefore $(0, M^*) \not\in \overline{E}$. Thus we have $M^* \geq m$.

**Theorem 9.** If the set $\overline{E}$ is convex and $m < \infty$, then $M^* = m$ holds.

*Proof.* By Theorem 7, it is enough to show the inequality $M^* \geq m$ in the case where $m$ is finite. For an arbitrarily fixed $\varepsilon > 0$, we have $(0, m - \varepsilon) \not\in \overline{E}$ by the definition of $m$. Applying Proposition 1 to $(0, m - \varepsilon)$ and the $w(Z \times R, W \times R)$-closed convex set $E$, we can prove the inequality $m - \varepsilon < M^*$ by the same argument as in the proof of Theorem 3. By the arbitrariness of $\varepsilon$, we have $m \leq M^*$.

Note that the set $\overline{E}$ is convex whenever the set $E$ is convex ([1], p. 50, Proposition 14).

By means of Theorem 5, we see that Theorems 8 and 9 are some generalizations of Theorem 2 in [4].

Now we introduce

**Definition.** Problem (I) is said to be normal if $\overline{E} \cap L = \overline{E}_0$, where $L$ and $E_0$ are the sets defined in § 3 and $\overline{E}_0$ is the $w(Z \times R, W \times R)$-closure of $E_0$. 

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The normality condition was first introduced in [6], cf. [7].
We shall prove

**Theorem 10.** Problem (I) is normal if and only if $M=m$.

**Proof.** Observe that $\bar{E}_0=\{0\} \times [M, +\infty)$ in case $M$ is finite, $\bar{E}_0=L$ in case $M=+\infty$ and $\bar{E}_0$ is empty in case $M=-\infty$. Similarly, $\bar{E} \cap L=\{0\} \times [m, +\infty)$ in case $m$ is finite, $\bar{E} \cap L=L$ in case $m=+\infty$ and $\bar{E} \cap L$ is empty in case $m=-\infty$. Our theorem follows from these observations.

From Theorems 8, 9 and 10, we obtain

**Corollary 1.** Assume that problem (I) is normal and $\bar{E}$ is convex. If $M<\infty$ or $-\infty< M^*$, then $M=M^*$.

From Theorems 7 and 10, we have

**Corollary 2.** If $M=M^*$, then problem (I) is normal.

These corollaries are a generalization of Theorem 7 of [6]. We easily have

**Proposition 8.** If the set $E$ is $w(Z \times R, W \times R)$-closed, then problem (I) is normal.

By this proposition, we see that Corollary 1 is a generalization of Theorems 3 and 4. However it seems difficult to verify the normality in the case where the set $E$ is not $w(Z \times R, W \times R)$-closed.

We have

**Proposition 9.** Assume that the set $E$ is convex. If $L$ intersects the $s(Z \times R, W \times R)$-interior $E^o$ of $E$, then problem (I) is normal.

**Proof.** Suppose $\bar{E}_0 \neq \bar{E} \cap L$. Then there exists $(0, r_0) \in \bar{E} \cap L$ such that $(0, r_0) \notin \bar{E}_0$. By our assumption, there is $(0, r_0) \in L$ such that $(0, r_0) \in E^o$. Let $U$ be a convex $s(Z \times R, W \times R)$-neighborhood of $(0, r_0)$ satisfying $U \subset E^o$. Since $(0, r_0) \in \bar{E}$ and $E$ is convex, we see that the set

$$V=\{(z, s) : z=tu, s=(1-t)r_0+tr \text{ for all } (u, r) \in U \text{ and } t \in R_0 \}$$

with $0 < t \leq 1$.

is contained in $E$ ([1], p. 51, Proposition 15). It is clear that $L \cap V \subset L \cap E = E_0$. Since $(0, (1-t)r_0+tr) \in L \cap V$ for all $t, 0 < t \leq 1$, we see that $(0, r_0) \in \bar{L} \cap \bar{V} \subset \bar{E}_0$. This is a contradiction. Therefore $\bar{E}_0=\bar{E} \cap L$.

This is a straightforward extension of Proposition 5.2 in [7].
SOME GENERALIZATIONS OF DUALITY THEOREMS

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