A note on conjugates

Hisao Tominaga*

*Okayama University
A NOTE ON CONJUGATES

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Recently in his paper [1] I. N. Herstein proved the following: \textit{If in a division ring }D\textit{ an element }d \in D\textit{ has only a finite number of conjugates then it has only one, that is, }d\textit{ is in the center of }D\textit{. On the other hand, relating to Galois theory of infinite degree, F. Kasch has obtained the following ([2, Satz 4]): \textit{Let }U\textit{ be an arbitrary ring, and }D\textit{ be an infinite division subring of }U\textit{. If }t\textit{ is an element of }U\textit{ not contained in }V_u(D)\textit{ then it has an infinite number of conjugates by non-zero elements in }D\textit{. Noting that any finite division ring is commutative by a well-known theorem of Wedderburn, one will readily see that Herstein's theorem is an easy consequence of Kasch's.}

In what follows, we shall prove a theorem which is a sharpening as well as a generalization of Kasch's theorem. And one will see also that Herstein's theorem can be shown without making use of Cartan's theorem.

We use the following conventions throughout: By a ring we mean a ring with an identity, and by a subring we mean one which contains this identity. By a simple ring we shall mean a two-sided simple ring with minimum condition for one-sided ideals. For any non-empty subset }B\textit{ in a ring }A\textit{, }V \mu (B)\textit{ will denote the centralizer of }B\textit{ in }A\textit{. If }K\textit{ is a division ring then }K^*\textit{ will be the group of its non-zero elements under the multiplication of }K\textit{. And for any set }S, \overline{S}\textit{ will signify the cardinal number of }S\textit{.}

Now we shall begin our course with the following sharpening of Kasch's theorem.

\textbf{Lemma.} \textit{Let }D\textit{ be an infinite division subring of a ring }U\textit{, and }T\textit{ be the set of conjugates of an element }t \in U\textit{ by all non-zero elements in }D\textit{. Then }\overline{T} = \overline{D} \textit{ or }1\textit{.}

\textbf{Proof.} Clearly }\overline{T}\textit{ coincides with the index of }V_u(t)^*\textit{ in }D^*\textit{. Hence we have }\overline{D} = \overline{T} \cdot \overline{V_u(t)}\textit{. Now we assume }V_u(t) \subseteq D\textit{. Then there exists some }d \in D\textit{ with }dt \neq td\textit{. And for any different }v, v'\textit{ in }V_u(t)\textit{, there holds }\overline{d} \cdot \overline{t} \cdot \overline{d + v} = \overline{d} \cdot \overline{t} \cdot \overline{d + v'}\textit{. For, if not, }\overline{(d + v) t (d + v)^{-1}} = \overline{t'} = \overline{(d + v') t (d + v')^{-1}}\textit{ implies }t = t', \textit{ from which we can readily obtain a contradiction }dt = td\textit{. The last fact shows evidently }\overline{V_u(t)} \subseteq \overline{T}\textit{, accordingly we have }\overline{D} = \overline{T} \cdot \overline{V_u(t)} = \overline{T}\textit{, as desired.}

Our principal theorem is stated as follows.
Theorem. Let $R$ be an infinite simple subring of a ring $U$, and $T$ be the set of conjugates of an element $t \in U$ by all regular elements in $R$. Then $\overline{T} = \overline{R}$ or $1$.

Proof. We set, throughout the proof, $R = \sum_{i,j=1}^{n} e_{ij} D e_{ij}$ where $e_{ij}$'s are matric units and $D = V_R(\{e_{ij}\})$ is a division ring. Then, as is well-known, $U = \sum_{i,j=1}^{n} V e_{ij}$, where $V = V_U(\{e_{ij}\})$. Since our assertion for $n = 1$ is the above lemma itself, we shall assume $n \geq 1$, and set $t = \sum_{i,j=1}^{n} c_{ij} e_{ij}$ with $c_{ij} \in V$. Now we shall prove that if $\overline{T} < \overline{R}$ then $t$ is in $V_r(R)$. By assumption $\{dt d^{-1}; d \in D^*\} \subseteq \overline{T} < \overline{R} = \overline{D}$, a fortiori, $\{dc_{ij} d^{-1}; d \in D^*\} < \overline{D}$ for any $i, j$. Hence all $c_{ij}$'s are contained in $V_r(D) = V_U(R)$ by the above lemma. If $c_{pq} \neq 0$ for some $p \neq q$ then there holds

$$(1 - d e_{qp}) t (1 - d e_{qp})^{-1} = (1 - d e_{qp}) t (1 + d e_{qp})$$

$$= \sum_{i,j=1}^{n} c_{ij} e_{ij} - \sum_{i,j=1}^{n} d c_{pq} e_{ij} + \sum_{i,j=1}^{n} d c_{iq} e_{ip} - d c_{pp} e_{qp}$$

for any $d \in D$. Noting that the coefficient of $e_{pp}$ in the last equation is $c_{pp} + d c_{pq}$, we have a contradiction $\overline{T} \supseteq \overline{D} = \overline{R}$. Hence $t = \sum_{i,j=1}^{n} c_{ij} e_{ij}$.

Again for any $p \neq q$ and $d \in D$, there holds

$$(1 - d e_{qp}) t (1 - d e_{qp})^{-1} = \sum_{i,j=1}^{n} c_{ij} e_{ij} - d c_{pp} e_{qp} + d c_{pq} e_{qp}.$$ 

As the coefficient of $e_{qp}$ is $d(c_{pq} - c_{pp})$, we must have $c_{pq} - c_{pp} = 0$, that is, $t = c_{pq} \in V_U(R)$. This completes the proof.

The next is only a restatement of our theorem.

Corollary. Let $R$ be an infinite simple subring of a ring $U$, and $T$ be a subset of $U$ which is transformed into itself by all regular elements in $R$. If $\overline{T} < \overline{R}$ then $T \subseteq V_U(R)$.

REFERENCES


DEPARTMENT OF MATHEMATICS,
OKAYAMA UNIVERSITY

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