A Note on Osofsky-Smith Theorem

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A NOTE ON OSOFSKY-SMITH THEOREM

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A famous result of B.Osofsky says that a ring $R$ is semisimple artinian if and only if every cyclic left $R$-module is injective. The crucial point of her proof was to show that such a ring has finite uniform dimension. In [7], B.Osofsky and P.F.Smith proved more generally that a cyclic module $M$ has finite uniform dimension if every cyclic subfactor of $M$ is an extending module. Extending modules have been studied extensively in recent years and many generalizations have been considered by many authors (see, for examples, [1-4, 6, 8, 9]). Lopez-Permouth, Oshiro and Tariq Rizvi in [6] introduced the concepts of extending modules and (quasi-)continuous modules relative a given left $R$-module $X$. Let $S$ be the class of all semisimple left $R$-modules and all singular left $R$-modules. We say a left $R$-module $N$ is $S$-extending if $N$ is $X$-extending for any $X \in S$. Every extending left $R$-module is $S$-extending but the converse is not true. Exploiting the techniques of [7] we prove the following result: Let $M$ be a cyclic left $R$-module. Assume that all cyclic subfactors of $M$ are $S$-extending. Then $M$ satisfies ACC on direct summands. As a corollary we show that if cyclic left $R$-module $M$ is extending and all cyclic subfactors of $M$ are $S$-extending, then $M$ has finite uniform dimension.

Throughout this note we write $A \leq_e B$ $(A|B)$ to denote that $A$ is an essential submodule (a direct summand) of $B$.

A left $R$-module $M$ is called singular if, for every $m \in M$, the annihilator $l(m)$ of $m$ is an essential left ideal of $R$.

Lemma 1 ([4, 4.6]). The following are equivalent for a left $R$-module $M$.
(1) $M$ is singular.
(2) $M \cong L/K$ for a left $R$-module $L$ and $K \leq_e L$.

Let $M, X$ be left $R$-modules. Define the family

$\mathcal{A}(X, M) = \{A \subseteq M|\exists Y \subseteq X, \exists f \in Hom(Y, M), f(Y) \leq_e A\}$.

Consider the properties

$\mathcal{A}(X, M)$-$\langle C_1 \rangle$: For all $A \in \mathcal{A}(X, M)$, $\exists A^*|M$, such that $A \leq_e A^*$.

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\(A(X, M)-(C_2):\) For all \(A \in A(X, M),\) if \(B|M\) is such that \(A \cong B,\) then \(A|M.\)

\(A(X, M)-(C_3):\) For all \(A \in A(X, M)\) and \(B|M,\) if \(A|M\) and \(A \cap B = 0\) then \(A \oplus B|M.\)

According to [6], \(M\) is said to be \(X\)-extending, \(X\)-quasi-continuous or \(X\)-continuous, respectively, if \(M\) satisfies \(A(X, M)-(C_1), A(X, M)-(C_1)\) and \(A(X, M)-(C_3), A(X, M)-(C_1)\) and \(A(X, M)-(C_2).\)

According to [8, 1, 2], a left \(R\)-module \(M\) is called a CESS-module if every complement with essential socle is a direct summand, equivalently, every submodule with essential socle is essential in a direct summand of \(M.\) Now the following result is clear.

**Proposition 2.** A left \(R\)-module \(M\) is a CESS-module if and only if \(M\) is \(X\)-extending for any semisimple left \(R\)-module \(X.\)

**Definition 3.** Let \(S\) be the class of all semisimple left \(R\)-modules and all singular left \(R\)-modules. A left \(R\)-module \(M\) is called \(S\)-extending if \(M\) is \(X\)-extending for any \(X \in S.\)

Note that every extending left \(R\)-module is clearly \(S\)-extending. But the following example shows that the converse is not true.

**Example 4.** Let \(M\) be a free \(\mathbb{Z}\)-module of infinite rank. Since \(M\) is non-singular and has no socle, \(M\) is clearly \(S\)-extending. But \(M\) is not extending by [5, Theorem 5].

Let \(S_1\) and \(S_2\) be the classes of all semisimple left \(R\)-modules, of all singular left \(R\)-modules, respectively. Then \(S_1 \oplus S_2\) is defined to be the class of left \(R\)-modules \(M\) such that \(M = A \oplus B\) is a direct sum of \(A \in S_1\) and \(B \in S_2.\)

**Proposition 5.** A left \(R\)-module \(M\) is \(S\)-extending if and only if it is \(X\)-extending for any \(X \in S_1 \oplus S_2.\)

**Proof.** It follows from the fact that if \(0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0\) is an exact sequence then \(M\) is \(X\)-extending if and only if it is both \(X'\)-extending and \(X''\)-extending by [6, Proposition 2.7].

**Proposition 6.** Let \(M\) be a cyclic left \(R\)-module. Assume that all cyclic subfactors of \(M\) are \(S\)-extending. Then \(M\) satisfies ACC on direct summands.

**Proof.** We prove this by adapting the proof of [7, Theorem 1 and 4, 7.12]. Suppose that \(M\) does not satisfy ACC on direct summands and that \(A_1 \subset A_2 \subset A_3 \subset \ldots \) is an infinite ascending chain of direct summands \(A_i (i \geq 1)\) of \(M.\) Then there exists a submodule \(B_1\) of \(M\) such
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that $M = A_1 \oplus B_1$. Thus $A_2 = A_2 \cap (A_1 \oplus B_1) = A_1 \oplus (A_2 \cap B_1)$ so that $A_2 \cap B_1$ is a direct summand of $B_1$. Let $B_2$ be a submodule of $B_1$ such that $B_1 = (A_2 \cap B_1) \oplus B_2$. Then $M = A_2 \oplus B_2$. Repeating this argument we can produce an infinite descending chain

$$B_1 \supset B_2 \supset B_3 \supset \ldots$$

of direct summands $B_i$ of $M$ such that $M = A_i \oplus B_i$. For each $i \geq 1$, there exists a nonzero submodule $C_{i+1}$ of $M$ such that $B_i = B_{i+1} \oplus C_{i+1}$. Put $C_1 = A_1$. Then

$$M = C_1 \oplus C_2 \oplus \cdots \oplus C_n \oplus B_n$$

and $\oplus_{i=n+1}^{\infty} C_i \subset B_n$ for all $n \geq 1$. Clearly $C_i$ is cyclic since $M$ is cyclic, and so $C_i$ contains a maximal submodule $W_i$. Put

$$P = M/(\oplus_{i=1}^{\infty} W_i), \quad Q = (\oplus_{i=1}^{\infty} C_i)/(\oplus_{i=1}^{\infty} W_i).$$

Then clearly $P$ is a cyclic subfactor of $M$ and $Q$ is a semisimple submodule of $P$. By the hypothesis, $P$ is $X$-extending for any $X \in \mathcal{S}$. Particularly $P$ is $Q$-extending. It is easy to see that $Q \in \mathcal{A}(Q, P)$, and so there exists a direct summand $Q^*$ of $P$ such that $Q \leq_e Q^*$.

Note that $Q = \oplus_{i=1}^{\infty} S_i$ is an infinite direct sum of simple left $R$-modules $S_i$ ($i \geq 1$). Let $\{1, 2, \ldots\}$ be a disjoint union of countable sets $\{I_j | j = 1, 2, \ldots\}$. Set $Q_j = \oplus_{i \in I_j} S_i$, $j = 1, 2, \ldots$. Then $Q_j$ is a non-finitely generated semisimple left $R$-module. Clearly $Q^*$ is a cyclic subfactor of $M$.

By the hypothesis, $Q^*$ is $X$-extending for any $X \in \mathcal{S}$. Particularly $Q^*$ is $Q_j$-extending. It is easy to see that $Q_j \in \mathcal{A}(Q_j, Q^*)$, and so there exists a direct summand $Q^*_j$ of $Q^*$ such that $Q_j \leq_e Q^*_j$. Clearly $Q^*_j$ is finitely generated, and thus $Q_j \neq Q^*_j$.

Let $D_j = (Q_j^* + Q)/Q$. Since $Q_j^* \cap (\oplus_{k \neq j} Q_k) = 0$ and $Q_j \neq Q_j^*$, it is easy to see that $D_j \neq 0$. Also $Q_j \leq Q \cap Q_j^* \leq Q_j^*$, so $Q \cap Q_j^* \leq_e Q_j^*$. This implies that $D_j \simeq Q_j^*/(Q_j^* \cap Q)$ is singular by Lemma 1. Hence

$$D = \sum_{j=1}^{\infty} D_j = \oplus_{j=1}^{\infty} D_j$$

is a singular submodule of $Q^*/Q$. Since $Q^*/Q$ is a cyclic subfactor of $M$, it follows that $Q^*/Q$ is $X$-extending for any $X \in \mathcal{S}$. Particularly $Q^*/Q$ is $D$-extending. It is easy to see that $D \in \mathcal{A}(D, Q^*/Q)$, and so there exists a direct summand $D^*$ of $Q^*/Q$ such that $D \leq_e D^*$.

Since $D^*$ is a cyclic submodule of $Q^*/Q$, there exists a cyclic submodule $H$ of $Q^*$ such that $D^* = (H + Q)/Q$. It is easy to see that $Q_j^* \cap H \neq 0$. Thus $Q_j \cap H = (Q_j^* \cap H) \cap Q_j \neq 0$. Hence there exists a non-zero simple submodule $V_j$ of $Q_j \cap H$. Let $V = \oplus_{j=1}^{\infty} V_j$. Then $V \leq H$. Since $H$ is a cyclic subfactor of $M$, it follows that $H$ is $X$-extending for any $X \in \mathcal{S}$. 

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Particularly $H$ is $V$-extending. Clearly $V \in \mathcal{A}(V, H)$, and so there exists a direct summand $V^*$ of $H$ such that $V \leq V^*$. It is easy to see that $V \neq V^*$ since $V^*$ is cyclic. If $(V^* + Q)/Q = 0$, then $V^* \leq Q$, and thus $V^*$ is semisimple. Hence $V$ is a direct summand of $V^*$. But $V \leq V^*$, it follows that $V = V^*$, a contradiction. Thus $(V^* + Q)/Q \neq 0$.

For any $n \geq 1$, we have $(V^* \cap \bigoplus_{j=1}^{n} Q_j) \cap Q = V^* \cap (\bigoplus_{j=1}^{n} Q_j) = V^* \cap (\bigoplus_{j=1}^{n} Q_j) \cap Q = V^* \cap (\bigoplus_{j=1}^{n} Q_j) = \bigoplus_{j=1}^{n} V_j$. Since $V^* \cap (\bigoplus_{j=1}^{n} Q_j)$ is semisimple, it follows that

$$\left(V^* \cap \bigoplus_{j=1}^{n} Q_j \right) \cap Q = \bigoplus_{j=1}^{n} V_j.$$ 

Clearly, $\bigoplus_{j=1}^{n} V_j$ is a finitely generated submodule of $Q$. Thus there exists a finitely generated submodule $N$ of $\bigoplus_{i=1}^{\infty} C_i$ such that $(N + \bigoplus_{i=1}^{\infty} W_i) / (\bigoplus_{i=1}^{\infty} W_i) = \bigoplus_{j=1}^{n} V_j$. Suppose that $N \leq \bigoplus_{i=1}^{m} C_i$. It is easy to see that

$$L = (\bigoplus_{i=1}^{m} C_i + \bigoplus_{i=1}^{\infty} W_i) / (\bigoplus_{i=1}^{\infty} W_i)$$

is semisimple. Thus $\bigoplus_{j=1}^{n} V_j$ is a direct summand of $L$. It is easy to see that $L$ is a direct summand of $P$. Thus $\bigoplus_{j=1}^{n} V_j$ is a direct summand of $P$. Let $P = (\bigoplus_{j=1}^{n} V_j) \oplus P_1$. By modularity, $V^* \cap (\bigoplus_{j=1}^{n} Q_j) = (V^* \cap (\bigoplus_{j=1}^{n} Q_j) \cap Q) \oplus (V^* \cap (\bigoplus_{j=1}^{n} Q_j) \cap P_1)$. But it is easy to see that $(V^* \cap (\bigoplus_{j=1}^{n} Q_j)) \cap Q \leq V^* \cap (\bigoplus_{j=1}^{n} Q_j)$. Thus $(V^* \cap (\bigoplus_{j=1}^{n} Q_j)) \cap Q = V^* \cap (\bigoplus_{j=1}^{n} Q_j)$, which implies that $V^* \cap (\bigoplus_{j=1}^{n} Q_j) \leq Q$. This holds for each $n \geq 1$, hence it follows that $V^* \cap (\bigoplus_{j=1}^{\infty} Q_j) \leq Q$. But $Q \leq \bigoplus_{j=1}^{\infty} Q_j$, it follows that

$$(\bigoplus_{j=1}^{\infty} (Q_j^* + Q)/Q) \cap ((V^* + Q)/Q) = 0.$$ 

Now it follows that $(V^* + Q)/Q = 0$, which is a contradiction, because $D \leq D^*$. This completes the proof of the proposition.

Now we have the main result of this paper, which generalizes Osofsky-Smith theorem ([7, Theorem 1]).

**Theorem 7.** Let $M$ be a cyclic extending left $R$-module. Assume that all cyclic subfactors of $M$ are $S$-extending. Then $M$ has finite uniform dimension.

**Proof.** By Proposition 6, $M$ is a finite direct sum of indecomposable submodules. Since every direct summand of an extending module is extending, the result follows by the fact that each indecomposable extending module is uniform. 

**References**


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