Mathematical Journal of Okayama University

Volume 47, *Issue* 1 2005 *Article* 2

JANUARY 2005

The Galois Action on the Torsor of Homotopy Classes of Paths on a Projective Line minus a Finite Number of Points

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Math. J. Okayama Univ. **47** (2005), 29–37

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Contents

0. Introduction.

0.1. Deligne on a conference in Schloss Ringberg considered the mixed Hodge structure on the fundamental group of $P^1\backslash\{0,1,-1,\infty\}$. He showed that the motivic Galois Lie algebra associated to this mixed Hodge structure contains a free Lie subalgebra on generators in degree $1,3,5,\ldots,2n+1,\ldots$ corresponding to $\log 2,\zeta(3),\zeta(5),\ldots,\zeta(2n+1),\ldots$.

In [W1] and [DW] we were studying actions of Galois groups on fundamental groups. In this note we are studying the action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the torsor of (ℓ-adic) paths from $\overrightarrow{01}$ to -1 on $P^1_{\overline{\mathbb{Q}}}\backslash\{0,1,\infty\}$. We show that the associated graded Lie algebra of the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_{\ell^\infty}))$ contains a free Lie subalgebra over $\mathbb{Q}_\ell$ on generators in degree $1,3,5,\ldots,2n+1,\ldots$. We use the idea working modulo 2 from Deligne's talk in Schloss Ringberg.

In [W1] section 5 we were studying some general aspects of actions of Galois groups on torsors of paths. To make this paper self contained we recall some definitions and results from [W1] in sections 1 and 2.

1. Torsors of paths.

1.1. Let K be a number field and let $a_1,\ldots,a_{n+1}$ be K-points of a projective line P^1_K. Let $V=P^1_K\backslash\{a_1,\ldots,a_n,a_{n+1}\}$. For simplicity we assume that $a_{n+1}=\infty$.

We denote by $\widehat{V}(K)$ the set of K-points of V and of tangential base points defined over K. Let $z,v\in\widehat{V}(K)$. Let $\pi_1(V_{\bar K},v)$ be the ℓ-completion of the etale fundamental group of $V_{\bar K}$ and let $\pi(V_{\bar K},z,v)$ be the set of ℓ-adic

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paths from v to z on $V_{\bar{K}}$. The set $\pi(V_{\bar{K}}, z, v)$ is a $\pi_1(V_{\bar{K}}, v)$-torsor. The Galois group $G_K := \mathrm{Gal}(\bar{K}/K)$ acts on $\pi_1(V_{\bar{K}}, v)$ and on $\pi(V_{\bar{K}}, z, v)$ in a compatible way, i.e., $\sigma(p \cdot S) = \sigma(p) \cdot \sigma(S)$, where $\sigma \in G_K,\ p \in \pi(V_{\bar{K}}, z, v)$ and $S \in \pi_1(V_{\bar{K}}, v)$.

Let us fix a path $p \in \pi(V_{\bar{K}}, z, v)$. We define a bijection of sets

$$t_p : \pi(V_{\bar{K}}, z, v) \to \pi_1(V_{\bar{K}}, v)$$

setting $t_p(q) := p^{-1} \cdot q$ (the composition of paths is from right to left). The bijection t_p is not G_K-equivariant. Using the bijection t_p we transport the action of G_K on $\pi(V_{\bar{K}}, z, v)$ into the action of G_K on $\pi_1(V_{\bar{K}}, v)$.

Let $\sigma \in G_K$. We set

$$f_p(\sigma) := p^{-1} \cdot \sigma(p).$$

The element $f_p(\sigma) \in \pi_1(V_{\bar{K}}, v)$. Let us define a new action of G_K on $\pi_1(V_{\bar{K}}, v)$ setting

$$\sigma_p(S) := f_p(\sigma) \cdot \sigma(S).$$

Observe that

$$(\tau \cdot \sigma)_p = \tau_p \cdot \sigma_p,$$

i.e., we have an action of G_K on $\pi_1(V_{\bar{K}}, v)$. We have

$$t_p(\sigma(q)) = \sigma_p(t_p(q)),$$

i.e., the bijection t_p is G_K-equivariant if we equip $\pi_1(V_{\bar{K}}, v)$ with the new action of G_K.

1.2. We fix generators of $\pi_1(V_{\bar{K}}, v)$ in the following way. At each missing point a_i we choose a tangential base point v_i defined over K. Let γ_i be a path from v to v_i. Then x_i is the composition of the path γ_i + a small loop around a_i in the opposit clockwise direction + the path γ_i^{-1}. We can assume that $x_{n+1} \cdot x_n \cdot \ldots \cdot x_1 = 1$.

To study the action of G_K on the torsor $\pi(V_{\bar{K}}, z, v)$, i.e., the action

$$(\ )_p : G_K \to \ \mathrm{Aut}_{set}(\pi_1(V_{\bar{K}}, v))$$

it is very convenient to embed $\pi_1(V_{\bar{K}}, v)$ into the ring of formal power series in non-commuting variables.

Let $\mathbb{Q}_\ell\{\{X_1, \ldots, X_n\}\}$ (resp. $\mathbb{Q}_\ell\{X_1, \ldots, X_n\}$) be a $\mathbb{Q}_\ell$-algebra of formal power series (resp. of polynomials) in non-commuting variables $X_1, \ldots, X_n$. Let

$$k : \pi_1(V_{\bar{K}}, v) \to \mathbb{Q}_\ell\{\{X_1, \ldots, X_n\}\}$$

be a continuous multiplicative embedding given by $k(x_i) = e^{X_i}$ for $i = i, \ldots, n$.

Let us set

$$\Lambda_p(\sigma) := k(f_p(\sigma)).$$

The action of G_K on $\pi_1(V_{\bar{K}}, v)$ induces a homomorphism

$$G_K \to \ \mathrm{Aut}_{\mathbb{Q}_\ell-\mathrm{algebra}}(\mathbb{Q}_\ell\{\{X_1, \ldots, X_n\}\}).$$

The action of G_K on $\pi(V_{\bar{K}}, z, v)$, i.e., the action $(\ )_p$ induces a homomorphism

$$\varphi_p : G_K \to \mathrm{Aut}_{\mathbb{Q}_\ell-\mathrm{linear}}(\mathbb{Q}_\ell\{\{X_1, \ldots, X_n\}\}).$$

Let $\omega \in \mathbb{Q}_\ell\{\{X_1, \ldots, X_n\}\}$ and let $\sigma \in G_K$. Then

$$\varphi_p(\sigma)(\omega) = \Lambda_p(\sigma) \cdot \sigma(\omega).$$

1.3. We shall study the Lie algebras of derivations of free Lie algebras.

Let Lie(V) be a free Lie algebra over $\mathbb{Q}_\ell$ on free generators $X_1, \ldots, X_n$. Let $L(V) := \varprojlim_n \mathrm{Lie}(V)/\Gamma^n\mathrm{Lie}(V)$. We identify $\mathrm{Lie}(V)$ (resp. $L(V)$) with the Lie algebra of Lie elements of $\mathbb{Q}_\ell\{X_1, \ldots, X_n\}$ (resp. of $\mathbb{Q}_\ell\{\{X_1, \ldots, X_n\}\}$).

If L is a Lie algebra then we denote by Der L the Lie algebra of derivations of L.

Let $\underline{n} := \{1, \ldots, n\}$. We set

$$\mathrm{Der}^*\mathrm{Lie}(V) := \{D \in \mathrm{Der\ Lie}(V) \mid \forall i \in \underline{n}\ \exists A_i \in \mathrm{Lie}(V), D(X_i) = [X_i, A_i]\}$$

and

$$\mathrm{Der}^*L(V) := \{D \in \mathrm{Der}L(V) \mid \forall i \in \underline{n}\ \exists A_i \in L(V), D(X_i) = [X_i, A_i]\}.$$

The derivation $D \in \mathrm{Der}^*\mathrm{Lie}(V)$ such that $D(X_i) = [X_i, A_i]$ for $i \in \underline{n}$ we denote by $D_{(A_1,\ldots,A_n)}$ or $D_{(A_i)_{i\in\underline{n}}}$. Let $< X_i >$ be a vector subspace of $\mathrm{Lie}(V)$ generated by X_i. Observe that we have an isomorphism of vector spaces

$$\mathrm{Der}^*\mathrm{Lie}(V) \approx \bigoplus_{i=1}^{n} (\mathrm{Lie}(V)/ < X_i >)$$

which maps $D_{(A_1,\ldots,A_n)}$ onto $(A_1, \ldots, A_n)$. We introduce on $\bigoplus_{i=1}^{n} (\mathrm{Lie}(V)/ < X_i >)$ a new bracket $\{\ \}$ defined in the following way

$$\{(A_i)_{i\in\underline{n}}, (B_i)_{i\in\underline{n}}\} := ([A_i, B_i] + D_{(A_j)_{j\in\underline{n}}}(B_i) - D_{(B_j)_{j\in\underline{n}}}(A_i))_{i\in\underline{n}}.$$

Lemma 1.3.1. The vector space $\bigoplus_{i=1}^{n} (\mathrm{Lie}(V)/ < X_i >)$ equip with the bracket $\{\ \}$ is a Lie algebra isomorphic to the Lie algebra $\mathrm{Der}^*\mathrm{Lie}(V)$. The isomorphism of Lie algebras maps $(A_i)_{i\in\underline{n}}$ onto $D_{(A_j)_{j\in\underline{n}}}$. □

The vector space $\bigoplus_{i=1}^{n} (\mathrm{Lie}(V)/ < X_i >)$ equip with the Lie bracket $\{\ \}$ we shall denote by $(\bigoplus_{i=1}^{n} (\mathrm{Lie}(V)/ < X_i >), \{\ \})$.

We define a semi-direct product of Lie algebras

$$\mathrm{Lie}(V)\tilde{\times}\mathrm{Der}^*\mathrm{Lie}(V)$$

defining a Lie bracket $\{\ \}$ on the product of vector spaces $\mathrm{Lie}(V)\times\mathrm{Der}^*\mathrm{Lie}(V)$ in the following way

$$\{(\lambda, D_\beta), (\lambda_1, D_{\beta_1})\} := ([\lambda, \lambda_1] + D_\beta(\lambda_1) - D_{\beta_1}(\lambda), [D_\beta, D_{\beta_1}]).$$

Hence the Lie bracket in a semi-direct product of Lie algebras

$$\mathrm{Lie}(V)\tilde{\times}(\bigoplus_{i=1}^{n} (\mathrm{Lie}(V)/ < X_i >), \{\ \})$$

is given by

$$\{(\lambda, \beta), (\lambda_1, \beta_1)\} := ([\lambda, \lambda_1] + D_\beta(\lambda_1) - D_{\beta_1}(\lambda), \{\beta, \beta_1\}).$$

We recall that $\mathbb{Q}_\ell\{X_1, \dots, X_n\}$ is a $\mathbb{Q}_\ell$-algebra of polynomials in non-commuting variables $X_1, \dots, X_n$. Observe that any derivation of the Lie algebra $\mathrm{Lie}(V)$ (resp. $L(V)$) induces a derivation of the $\mathbb{Q}_\ell$-algebra $\mathbb{Q}_\ell\{X_1, \dots, X_n\}$ (resp $\mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\}$). Let $\omega \in \mathbb{Q}_\ell\{X_1, \dots, X_n\}$ (resp. $\omega \in \mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\}$). We denote by L_ω the left multiplication by ω in the corresponding $\mathbb{Q}_\ell$-algebra. We denote by $L_{\mathrm{Lie}(V)}$ (resp. $L_{L(V)}$) the set of left multiplications by elements of $\mathrm{Lie}(V)$ (resp. $L(V)$). Observe that the semi-direct product

$$L_{\mathrm{Lie}(V)}\tilde{\times}\mathrm{Der}^*\mathrm{Lie}(V) \subset \mathrm{End}_{\mathbb{Q}_\ell-\mathrm{linear}}(\mathbb{Q}_\ell\{X_1, \dots, X_n\}).$$

Notice that the Lie algebras $\mathrm{Lie}(V)\tilde{\times}\mathrm{Der}^*\mathrm{Lie}(V)$ and $L_{\mathrm{Lie}(V)}\tilde{\times}\mathrm{Der}^*\mathrm{Lie}(V)$ are obviously isomorphic. The same is true if we replace $\mathrm{Lie}(V)$ by $L(V)$ and $\mathbb{Q}_\ell\{X_1, \dots X_n\}$ by $\mathbb{Q}_\ell\{\{X_1, \dots X_n\}\}$.

1.4. Using the representations

$$G_K \to \mathrm{Aut}_{\mathbb{Q}_\ell-\mathrm{algebra}}(\mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\}) \tag{1.4.1}$$

and

$$\varphi_p : G_K \to \mathrm{Aut}_{\mathbb{Q}_\ell-\mathrm{linear}}(\mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\})$$

we shall define filtrations of the Galois group G_K. We set

$$\begin{aligned} G_m &= G_m(V, v) \\ &:= \ker(\psi_m : G_K \to \mathrm{Aut}_{\mathbb{Q}_\ell-\mathrm{algebra}}(\mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\}/I^{m+1})), \end{aligned}$$

where I is the augmentation ideal of the $\mathbb{Q}_\ell$-algebra $\mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\}$ and ψ_m is induced by the action (1.4.1) of G_K on the $\mathbb{Q}_\ell$-algebra $\mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\}$.

We set

$$\begin{aligned} H_m &= H_m(V, z, v) \\ &:= \ker(\varphi_{p,m} : G_m \to \mathrm{Aut}_{\mathbb{Q}_\ell-\mathrm{linear}}(\mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\}/I^m)), \end{aligned}$$

where $\varphi_{p,m}$ is induced by φ_p.

We set

$$G_\infty := \bigcap_{m=1}^{\infty} G_m \text{ and } H_\infty := \bigcap_{m=1}^{\infty} H_m.$$

2. Lie algebras of actions of Galois groups on torsors.

2.1. We have seen in section 1 that the action of G_K on the torsor $\pi(V_{\bar{K}}, z, v)$ leads to the Galois representation

$$\varphi_p : G_K \to \operatorname{Aut}(\mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\}),$$

where $\varphi_p(\sigma)(\omega) = \Lambda_p(\sigma) \cdot \sigma(\omega)$. It is shown in [W1] Lemma 5.1.7 that for $\sigma \in \operatorname{Gal}(\bar{K}/K(\mu_\ell\infty))$.

$$\log \varphi_p(\sigma) = L_{\log \varphi_p(\sigma)(1)} + \log \sigma. \tag{2.1.1}$$

Moreover we have

$$(\log \sigma)(X_i) = [X_i, \log \varphi_{\gamma_i}(\sigma)(1)] \tag{2.1.2}$$

for $i = 1, \dots, n$ (see [W1] Proposition 5.1.8). Passing with the representation φ_p to Lie algebras we get a homomorphism of Lie algebras

$$\operatorname{Lie}\varphi_p : \operatorname{Lie}(H_1/H_\infty \otimes \mathbb{Q}) \to \operatorname{End}_{\mathbb{Q}_\ell-\text{linear}}(\mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\}).$$

It follows from (2.1.1) and (2.1.2) that $\operatorname{Lie}\varphi_p$ factors through

$$\operatorname{Lie}\varphi_p : \operatorname{Lie}(H_1/H_\infty \otimes \mathbb{Q}) \to L_{L(V)} \tilde{\times} \operatorname{Der}^* L(V).$$

We recall that we have a canonical isomorphism

$$L_{L(V)} \tilde{\times} \operatorname{Der}^* L(V) \approx L(V) \tilde{\times} (\overset{n}{\underset{i=1}{\oplus}} (L(V)/ < X_i >), \{\ \}).$$

Let $\sigma \in \operatorname{Gal}(\bar{K}/K(\mu_\ell\infty))$. We shall calculate coordinates of $(\operatorname{Lie} \varphi_p)(\sigma)$ in $L(V) \tilde{\times} (\overset{n}{\underset{i=1}{\oplus}} (L(V)/ < X_i >), \{\ \})$.

Lemma 2.1.3. Let $\sigma \in \operatorname{Gal}(\bar{K}/K(\mu_\ell\infty))$. Then

$$(\operatorname{Lie}\varphi_p)(\sigma) = (\log \varphi_p(\sigma)(1), (\log \varphi_{\gamma_i}(\sigma)(1))_{i \in \underline{n}}).$$

Proof. The lemma follows from (2.1.1) and (2.1.2). □

We pass with the morphism $\operatorname{Lie}\varphi_p$ to associated graded Lie algebras. Then we get a morphism

$$gr\operatorname{Lie}\varphi_p : gr\operatorname{Lie}(H_1/H_\infty \otimes \mathbb{Q}) \to L_{\operatorname{Lie}(V)} \tilde{\times} \operatorname{Der}^* \operatorname{Lie}(V).$$

Let us set

$$\phi_p := gr\operatorname{Lie}\varphi_p.$$

Lemma 2.1.4. Let $\sigma \in H_n$. Then

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i) $\log \varphi_p(\sigma)(1) \equiv \log \Lambda_p(\sigma) \bmod \Gamma^{n+1}\mathrm{Lie}(V)$,
ii) the class of $\log\Lambda_p(\sigma) \bmod \Gamma^{n+1}\mathrm{Lie}(V)$ does not depend on a choice of a path p from v to z.

Proof. The lemma is already proved in [W1]. □

Let $\sigma \in H_n$. We denote by $\mathcal{L}(z,v)(\sigma)$ the class of $\log\Lambda_p(\sigma) \bmod \Gamma^{n+1}\mathrm{Lie}(V)$.

Now we can calculate coordinates of $\phi_p(\sigma)$ in $L_{\mathrm{Lie}(V)}\tilde{\times}\mathrm{Der}^*\mathrm{Lie}(V) \approx \mathrm{Lie}(V)\tilde{\times}(\overset{n}{\underset{i=1}{\oplus}} (\mathrm{Lie}(V)/ < X_i >), \{\ \})$.

Lemma 2.1.5. Let $\sigma \in H_n$. Then

$$\phi_p(\sigma) = (\mathcal{L}(z,v)(\sigma), (\mathcal{L}(v_i,v)(\sigma))_{i\in\underline{n}})$$
$$\text{in Lie } (V)\tilde{\times}(\overset{n}{\underset{i=1}{\oplus}} (\mathrm{Lie}(V)/< X_i >), \{\ \}).$$

Proof. The lemma follows from Lemmas 2.1.3 and 2.1.4. □

It follows from Lemma 2.1.5 that the morphism of Lie algebras

$$\phi_p : gr\mathrm{Lie}(H_1/H_\infty \otimes \mathbb{Q}) \to L_{\mathrm{Lie}(V)}\tilde{\times}\mathrm{Der}^*\mathrm{Lie}(V).$$

does not depend on a choice of a path p from v to z, hence we shall denote it by $\phi_{z,v}$.

We set

$$t_V(z,v) := \mathrm{image}(\phi_{z,v}).$$

Observe that the Lie algebra $t_V(v,v)$ is the associated graded Lie algebra of the image of $\mathrm{Gal}(\bar{K}/K(\mu_{\ell^\infty}))$ in $\mathrm{Aut}(\pi_1(V_{\bar{K}},v))$. This Lie algebra was studied in [W1] section 15. To indicate the importance of the Lie algebra $t_V(v,v)$ we set

$$\delta_V(v) := t_V(v,v).$$

3. Examples.

Let $V = P^1_{\mathbb{Q}}\setminus\{0,1,\infty\}$. In the fundamental group $\pi_1(V_{\overline{\mathbb{Q}}}, \overrightarrow{01})$ we have two generators x - loop around 0 and y - loop around 1. We embed $\pi_1(V_{\overline{\mathbb{Q}}}, \overrightarrow{01})$ into $\mathbb{Q}_\ell\{\{X,Y\}\}$ mapping x onto e^X and y onto e^Y.

Proposition 3.1. The Lie algebras $\delta_V(\overrightarrow{01})$ and $t_V(\overrightarrow{10}, \overrightarrow{01})$ are isomorphic.

Proof. It follows from Lemma 2.1.5 that

$$\phi_{\overrightarrow{01},\overrightarrow{01}}(\sigma) = (0, (0, \mathcal{L}(\overrightarrow{10}, \overrightarrow{01})(\sigma)))$$

and

$$\phi_{\overrightarrow{10},\overrightarrow{01}}(\sigma) = (\mathcal{L}(\overrightarrow{10},\overrightarrow{01})(\sigma), (0, \mathcal{L}(\overrightarrow{10},\overrightarrow{01})(\sigma)))$$

in $\mathrm{Lie}(V)\tilde{\times}((\mathrm{Lie}(V)/<X>) \oplus (\mathrm{Lie}(V)/<Y>), \{\ \})$. It is clear that the map $\delta_V(\overrightarrow{01}) \to t_V(\overrightarrow{10},\overrightarrow{01})$ sending $(0,(0,\mathcal{L}))$ to $(\mathcal{L},(0,\mathcal{L}))$ is an isomorphism of the corresponding Lie algebras. □

Proposition 3.2. The Lie algebra $t_V(-1,\overrightarrow{01})$ contains a free Lie subalgebra on free generators in degree $1, 3, 5, \dots, 2n+1, \dots$.

Proof. The proof is based on Deligne's ideas indicated in [D]. It follows from Lemma 2.1.5 that

(3.2.1) $$\phi_{-1,\overrightarrow{01}}(\sigma) = \big(\mathcal{L}(-1,\overrightarrow{01})(\sigma), (0, \mathcal{L}(\overrightarrow{10},\overrightarrow{01})(\sigma))\big)$$

in $\mathrm{Lie}(V)\tilde{\times}((\mathrm{Lie}(V)/<X>) \oplus (\mathrm{Lie}(V)/<Y>), \{\ \})$. Let I_n be a vector subspace of $\mathrm{Lie}(V)$ generated by Lie brackets of the Lie algebra $\mathrm{Lie}(V)$ which contain at least n Y's. Let us set

$$\mathcal{I}_n := I_n \oplus (I_n \oplus I_n).$$

Observe that $\mathcal{I}_n$ is a Lie ideal of the Lie algebra $\mathrm{Lie}(V)\tilde{\times}((\mathrm{Lie}(V)/<X>) \oplus (\mathrm{Lie}(V)/<Y>), \{\ \})$.

Let $n > 1$ and let $\sigma \in H_n$. It follows from the definition of ℓ-adic polylogarithms in [W1] section 11 and from the definition of the filtration $\{H_k\}_{k\in\mathbb{N}}$ of $G_{\mathbb{Q}}$ that

(3.2.2) $$\mathcal{L}(\overrightarrow{10},\overrightarrow{01})(\sigma) \equiv \ell_n(\overrightarrow{10})(\sigma)[..[Y,X],X^{n-2}] \bmod I_2 + \Gamma^{n+1}L(V)$$

and

(3.2.3) $$\mathcal{L}(-1,\overrightarrow{01})(\sigma) \equiv \ell_n(-1)(\sigma)[..[Y,X],X^{n-2}] \bmod I_2 + \Gamma^{n+1}L(V).$$

It follows from the work of Soulé (see [S1] and [S2]) and the relation between ℓ-adic polylogarithms and classes of Soulé (see [W1] Corollary 14.3.3 and also [NW] Remark 2 and [W2] Proposition 3.4) that $\ell_{2n+1}(\overrightarrow{10}) \neq 0$ and $\ell_{2n}(\overrightarrow{10}) = 0$. In [W1] Corollary 11.2.3 and also in [W2] Theorem 2.1 we have proved the identity

$$2^{n-1}(\ell_n(-1) + \ell_n(1)) = \ell_n(1)$$

after the restriction to H_n. ($\ell_n(1)$ denotes $\ell_n(\overrightarrow{10})$.) Hence we get that

$$\ell_n(1) = \frac{2^{n-1}}{1-2^{n-1}}\ell_n(-1) \tag{3.2.4}$$

for $n > 1$. This implies that $\ell_{2n}(-1) = 0$.

Let $n > 1$ and let $\sigma \in H_n$. It follows from (3.2.1) - (3.2.4) that in the Lie algebra $t_V(-1, \overrightarrow{01})$ there is an element of the form

$$(\ell_n(-1)(\sigma)[..[Y,X],X^{n-2}]+u_n,(0,\frac{2^{n-1}}{1-2^{n-1}}\ell_n(-1)(\sigma)[..[Y,X],X^{n-2}]+\omega_n))$$

where $u_n, \omega_n \in I_2$. Let us take $\sigma \in H_{2n+1}$ such that $\ell_{2n+1}(-1)(\sigma) \neq 0$. Multiplying by $(1-2^{2n})$ and dividing by $\ell_{2n+1}(-1)(\sigma)$ we get an element of the form

$$z_{2n+1} := ((1-2^{2n})[..[Y,X],X^{2n-1}]+u_{2n+1},(0,2^{2n}[..[Y,X],X^{2n-1}]+w_{2n+1}))$$

($u_{2n+1}, w_{2n+1} \in I_2$) in the Lie algebra $t_V(-1, \overrightarrow{01})$.

Let $n = 1$. It follows from [W1] Proposition 11.0.8 that $\ell_1(-1) = \ell(2)$. The ℓ-adic logarithm $\ell(2)$ is the Kummer character associated to 2 (see [W1] Proposition 14.1.0.). Hence there is an element $\sigma \in H_1$ such that $\ell(2)(\sigma) \neq 0$. Therefore we get that $\mathcal{L}(\overrightarrow{10}, \overrightarrow{01})(\sigma) = 0$ and $\mathcal{L}(-1, \overrightarrow{01})(\sigma) = \ell(2)(\sigma)Y$. Hence the element

$$z_1 := (Y,(0,0))$$

belongs to $t_V(-1, \overrightarrow{01})$.

Let us set $t_{2n+1} = ((1-2^{2n})[..[Y,X],X^{2n-1}],(0,2^{2n}[..[Y,X],X^{2n-1}]))$ for $n > 1$ and $t_1 = (Y,(0,0))$. Observe that for any Lie bracket of length r in the Lie algebra $\mathrm{Lie}(V)\tilde{\times}((\mathrm{Lie}(V)/<X>)\oplus(\mathrm{Lie}(V)/<Y>),\{\ \})$ we have

$$\{\ldots\{z_{i_1},z_{i_2}\}\ldots z_{i_r}\} \equiv \{\ldots\{t_{i_1},t_{i_2}\}\ldots,t_{i_r}\} \bmod \mathcal{I}_{r+1}\,.$$

Let us set $s_{2n+1} = [..[Y,X],X^{2n-1}]$ for $n > 0$ and $s_1 = Y$. Notice that the elements $t_1, t_3, \ldots$ and $s_1, s_2, \ldots$ have integer coefficients. Observe that

$$\{\ldots\{t_{i_1},t_{i_2}\}\ldots,t_{i_r}\} \equiv ([\ldots[s_{i_1},s_{i_2}]\ldots,s_{i_r}],(0,0)) \bmod 2,$$

where $[\ ,\ ]$ is the standard Lie bracket in the free Lie algebra $\mathrm{Lie}(V)$.

The Hall basic Lie elements in $s_1, s_3, \ldots, s_{2n+1}, \ldots$ in the free Lie algebra $\mathrm{Lie}(V)$ are linearly independent. Hence the Hall basic Lie elements $z_1, z_3, \ldots, z_{2n+1}, \ldots$ in the Lie algebra $\mathrm{Lie}(V)\tilde{\times}((\mathrm{Lie}(V)/<X>)\oplus(\mathrm{Lie}(V)/<Y>),\{\ \})$ are linearly independent. Hence the elements $z_1, z_3, \ldots, z_{2n+1}, \ldots$ are free generators of a free Lie subalgebra of $t_V(-1, \overrightarrow{01})$. $\square$

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(*Received July 13, 2004*)