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Ramanujan's Mock Theta Functions

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RAMANUJAN'S MOCK THETA FUNCTIONS

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1. Introduction

In his last letter to G. H. Hardy, S. Ramanujan wrote, " I discovered very interesting functions recently which I call mock theta functions". He also gave a long list of 'third order', 'fifth order' and 'seventh order' mock theta functions together with identities satisfied by them. All the results on third and fifth order mock theta functions were proved by G. N. Watson in his two celebrated papers [8,9].

Andrews rediscovered Ramanujan's notebook while visiting Cambridge University and called it Ramanujan's 'lost' notebook. The 'lost' notebook contains several further results on mock theta functions which were proved by Andrews [1,2], Andrews and Garvan [3], Andrews and Hickerson [4]. Lastly the 'lost' notebook contains eight identities for the tenth order mock theta function. Choi [5] has proved the first two of Ramanujan's tenth order mock theta function identities and said that further identities will be proved in subsequent papers.

In this paper we have given relations and expansions of partial mock theta functions, mock theta functions of tenth, third, fifth and sixth order. We have also given two Continued Fractions for tenth order mock theta functions.

In section 4, we give a proof of a simple identity. In section 5, using this identity we connect the tenth order mock theta functions, partial tenth order mock theta functions with mock theta functions, partial mock theta functions of third, fifth, and sixth order. In section 6, we give relations between fifth order mock theta functions and third order mock theta functions and their partial sums. In section 7, we have given expansions of a tenth order mock theta functions in terms of partial mock theta function of tenth order.

In section 8, we have proved two lemmas and with the help of these lemmas expressed the tenth order mock theta functions as a Continued Fraction.

2. Notation

The following q-notations have been used.

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For $|q^k| < 1$,

$$
\begin{aligned}
(a;q^k)_n &= \prod_{j=0}^{n-1}(1-aq^{kj}), \quad n \geq 1 \\
(a;q^k)_0 &= 1, \\
(a;q^k)_\infty &= \prod_{j=0}^{\infty}(1-aq^{kj}), \\
(a)_n &= (a;q)_n, \\
(a_1,a_2,\ldots,a_m;q^k)_n &= (a_1;q^k)_n(a_2;q^k)_n\cdots(a_m;q^k)_n.
\end{aligned}
$$

A generalized basic hypergeometric series with base q_1 is defined as

$$
\begin{aligned}
&{}_A\phi_{A-1}\left[a_1,a_2,\ldots,a_A;b_1,b_2,\ldots,b_{A-1};q_1,z\right] \\
&\qquad = \sum_{n=0}^{\infty}\frac{(a_1;q_1)_n\cdots(a_A;q_1)_n z^n}{(b_1;q_1)_n\cdots(b_{A-1};q_1)_n(q;q_1)_n}, \quad |z|<1.
\end{aligned}
$$

3. Definition of Partial Mock Theta Functions

Third order mock theta functions, (Watson [8])

$$
\begin{aligned}
f(q) &:= \sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q)_n^2}, \\
\Phi(q) &:= \sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q^2;q^2)_n}, \\
\Psi(q) &:= \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q^2)_n}, \\
\gamma(q) &:= \sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}.
\end{aligned}
$$

Fifth order mock theta functions, (Watson [9])

$$
\begin{aligned}
f_0(q) &:= \sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n}, \\
F_0(q) &:= \sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q;q^2)_n}.
\end{aligned}
$$

Sixth order mock theta functions, (Andrews and Hickerson [4])

$$\Phi_L(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q)_{2n}},$$

$$\Psi_L(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q)_{2n+1}},$$

$$\rho_L(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q)_n}{(q;q^2)_{n+1}},$$

$$\sigma_L(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}(-q)_n}{(q;q^2)_{n+1}}.$$

To distinguish these functions from the others in notation, we have put a suffix L in their symbols.

Tenth order mock theta functions, (Choi [5])

$$\Phi_R(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q^2)_{n+1}},$$

$$\Psi_R(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q;q^2)_{n+1}},$$

$$X_R(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}},$$

$$\chi_R(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}}.$$

To distinguish these functions from the others in notation, we have put a suffix R in their symbols.

By taking the partial sums of the series defining the mock theta functions from 0 to N we have partial mock theta functions. We put N in the suffix to denote the partial sum from 0 to N. Thus partial mock theta functions of tenth order will be

$$\Phi_{RN}(q) := \sum_{n=0}^{N} \frac{q^{n(n+1)/2}}{(q;q^2)_{n+1}}.$$

Similar notation will be used for the other tenth order mock theta functions and mock theta functions of third order, fifth order and sixth order.

4. Main Summation Formula

We give a proof of the following identity

$$\sum_{\gamma=0}^{p}\alpha_\gamma\beta_\gamma=\beta_{p+1}\sum_{\gamma=0}^{p}\alpha_\gamma+\sum_{m=0}^{p}(\beta_m-\beta_{m+1})\sum_{\gamma=0}^{m}\alpha_\gamma. \tag{4.1}$$

Proof. The proof is a simple rearrangement of series

$$\begin{aligned}\sum_{m=0}^{p}(\beta_m-\beta_{m+1})\sum_{\gamma=0}^{m}\alpha_\gamma &= (\beta_0-\beta_1)\alpha_0+(\beta_1-\beta_2)\sum_{\gamma=0}^{1}\alpha_\gamma\\ &\quad+(\beta_2-\beta_3)\sum_{\gamma=0}^{2}\alpha_\gamma+\cdots+(\beta_p-\beta_{p+1})\sum_{\gamma=0}^{p}\alpha_\gamma\\ &=\alpha_0\beta_0+\alpha_1\beta_1+\cdots+\alpha_p\beta_p\\ &\quad-\beta_{p+1}(\alpha_0+\alpha_1+\cdots+\alpha_p)\\ &=\sum_{\gamma=0}^{p}\alpha_\gamma\beta_\gamma-\beta_{p+1}\sum_{\gamma=0}^{p}\alpha_\gamma,\end{aligned}$$

which proves (4.1). □

5(a). Relation between Tenth Order Partial Mock Theta Functions and Tenth Order Mock Theta Functions

(i) Taking $\alpha_\gamma=\frac{q^{\gamma(\gamma+1)/2}}{(q;q^2)_{\gamma+1}}$, $\beta_\gamma=q^{\gamma+1}$ in (4.1), we have

$$\Psi_{Rp}(q)=q^{p+2}\Phi_{Rp}(q)+(1-q)\sum_{m=0}^{p}q^{m+1}\Phi_{Rm}(q). \tag{5.1}$$

(ii) Letting $p\to\infty$ in (i), we have

$$\Psi_R(q)=(1-q)\sum_{m=0}^{\infty}q^{m+1}\Phi_{Rm}(q). \tag{5.2}$$

(iii) Taking $\alpha_\gamma=\frac{q^{\gamma(\gamma+1)/2}}{(q;q^2)_{\gamma+1}}$, $\beta_\gamma=(q;q^2)_{\gamma+1}$ and letting $p\to\infty$ in (4.1), we have

$$\Psi(q)=(q;q^2)_\infty\Phi_R(q)+\sum_{m=0}^{\infty}(q;q^2)_{m+1}q^{2m+3}\Phi_{Rm}(q), \tag{5.3}$$

where

$$\Psi(q)=\sum_{n=0}^{\infty}q^{n(n+1)/2}.$$

(iv) Taking $\alpha_\gamma = \frac{q^{(\gamma+1)(\gamma+2)/2}}{(q;q^2)_{\gamma+1}}$, $\beta_\gamma = \frac{1}{q^{\gamma+1}}$ in (4.1), we have

$$q^{p+2}\Phi_{Rp}(q) = \Psi_{Rp}(q) - (1-q)\sum_{m=0}^{p} q^{p-m}\Psi_{Rm}(q), \tag{5.4}$$

(v) Now

$$\begin{aligned} X_{Rm}(q) &= \sum_{\gamma=0}^{m} \frac{(-1)^\gamma q^{\gamma^2}}{(-q;q)_{2\gamma}} \\ &= \sum_{\gamma=0}^{m} \frac{(-1)^\gamma q^{\gamma^2}(1+q^{2\gamma+1})}{(-q;q)_{2\gamma+1}} \\ &= \sum_{\gamma=0}^{m} \frac{(-1)^\gamma q^{\gamma^2}}{(-q;q)_{2\gamma+1}} + \chi_{Rm}(q). \end{aligned} \tag{5.5}$$

Taking $\alpha_\gamma = \frac{(-1)^\gamma q^{\gamma^2}}{(-q;q)_{2\gamma+1}}$, $\beta_\gamma = q^{2\gamma+1}$ in (4.1) and using (5.5), we have

$$\begin{aligned} \sum_{m=0}^{p} q^{2m}(X_{Rm}(q) - \chi_{Rm}(q)) = &-\frac{q^{2p+2}}{1-q^2}(X_{Rp}(q) - \chi_{Rp}(q)) \\ &+ \frac{1}{q(1-q^2)}\chi_{Rp}(q). \end{aligned} \tag{5.6}$$

(vi) Making $p \to \infty$ in (v), we have

$$\chi_R(q) = (1-q^2)\sum_{m=0}^{\infty} q^{2m+1}(X_{Rm}(q) - \chi_{Rm}(q)). \tag{5.7}$$

5(b). Relation between Tenth Order Mock Theta Functions and Sixth Order Mock Theta Functions

In this section we connect all the four tenth order mock theta functions with sixth order mock theta functions.

(i) Taking $\alpha_\gamma = \frac{(-1)^\gamma q^{\gamma^2}(q;q^2)_\gamma}{(-q;q)_{2\gamma}}$, $\beta_\gamma = \frac{1}{(q;q^2)_\gamma}$ in (4.1), we have

$$X_{Rp}(q) = \frac{\Phi_{Lp}(q)}{(q;q^2)_{p+1}} - \sum_{m=0}^{p} \frac{q^{2m+1}}{(q;q^2)_{m+1}}\Phi_{Lm}(q). \tag{5.8}$$

(ii) Letting $p \to \infty$ in (i), we have

$$X_R(q) = \frac{\Phi_L(q)}{(q;q^2)_\infty} - \sum_{m=0}^{\infty} \frac{q^{2m+1}}{(q;q^2)_{m+1}}\Phi_{Lm}(q). \tag{5.9}$$

(iii) Taking $\alpha_\gamma = \frac{(-1)^\gamma q^{(\gamma+1)^2}(q;q^2)_\gamma}{(-q;q)_{2\gamma+1}}$, $\beta_\gamma = \frac{1}{(q;q^2)_\gamma}$ in (4.1), we have

$$\chi_{Rp}(q) = \frac{\Psi_{Lp}(q)}{(q;q^2)_{p+1}} - \sum_{m=0}^{p} \frac{q^{2m+1}}{(q;q^2)_{m+1}} \Psi_{Lm}(q). \tag{5.10}$$

(iv) Letting $p \to \infty$ in (iii), we have

$$\chi_R(q) = \frac{\Psi_L(q)}{(q;q^2)_\infty} - \sum_{m=0}^{\infty} \frac{q^{2m+1}}{(q;q^2)_{m+1}} \Psi_{Lm}(q). \tag{5.11}$$

(v) Taking $\alpha_\gamma = \frac{q^{\gamma(\gamma+1)/2}(-q;q)_\gamma}{(q;q^2)_{\gamma+1}}$, $\beta_\gamma = \frac{1}{(-q;q)_\gamma}$ in (4.1), we have

$$\Phi_{Rp}(q) = \frac{\rho_{Lp}(q)}{(-q;q)_{p+1}} + \sum_{m=0}^{p} \frac{q^{m+1}}{(-q;q)_{m+1}} \rho_{Lm}(q). \tag{5.12}$$

(vi) Letting $p \to \infty$ in (v), we have

$$\Phi_R(q) = \frac{\rho_L(q)}{(-q;q)_\infty} + \sum_{m=0}^{\infty} \frac{q^{m+1}}{(-q;q)_{m+1}} \rho_{Lm}(q). \tag{5.13}$$

(vii) Taking $\alpha_\gamma = \frac{q^{(\gamma+1)(\gamma+2)/2}(-q;q)_\gamma}{(q;q^2)_{\gamma+1}}$, $\beta_\gamma = \frac{1}{(-q;q)_\gamma}$ in (4.1), we have

$$\Psi_{Rp}(q) = \frac{\sigma_{Lp}(q)}{(-q;q)_{p+1}} + \sum_{m=0}^{p} \left\{q^{m+1}(-q;q)_{m+1}\right\} \sigma_{Lm}(q). \tag{5.14}$$

(viii) Letting $p \to \infty$ in (vii), we have

$$\Psi_R(q) = \frac{\sigma_L(q)}{(-q;q)_\infty} + \sum_{m=0}^{\infty} \left\{q^{m+1}(-q;q)_{m+1}\right\} \sigma_{Lm}(q). \tag{5.15}$$

5(c). Relation between Tenth Order Mock Theta Functions and Third Order Mock Theta Functions

(i) Taking $\alpha_\gamma = \frac{q^{\gamma(\gamma+1)}}{(-q;q^2)_{\gamma+1}}$, $\beta_\gamma = \frac{1}{(q;q^2)_{\gamma+1}}$ in (4.1), we have

$$\Phi_{Rp}(q^2) = \frac{\gamma_p(q)}{(q;q^2)_{p+2}} - \sum_{m=0}^{p} \frac{q^{2m+3}}{(q;q^2)_{m+2}} \gamma_m(q). \tag{5.16}$$

(ii) Letting $p \to \infty$ in (i), we have

$$\Phi_R(q^2) = \frac{\gamma(q)}{(q;q^2)_\infty} - \sum_{m=0}^{\infty} \frac{q^{2m+3}}{(q;q^2)_{m+2}} \gamma_m(q). \tag{5.17}$$

(iii) Taking $\alpha_\gamma = \frac{(-1)^\gamma q^{\gamma^2}}{(-q^2;q^2)_\gamma}$, $\beta_\gamma = \frac{1}{(-q;q^2)_\gamma}$ in (4.1), we have

(5.18) $$X_{Rp}(q) = \frac{\Phi_p(-q)}{(-q;q^2)_{p+1}} + \sum_{m=0}^{p} \frac{q^{2m+1}}{(-q;q^2)_{m+1}} \Phi_m(-q).$$

(iv) Letting $p \to \infty$ in (iii), we have

(5.19) $$X_R(q) = \frac{\Phi(-q)}{(-q;q^2)_\infty} + \sum_{m=0}^{\infty} \frac{q^{2m+1}}{(-q;q^2)_{m+1}} \Phi_m(-q).$$

(v) Taking $\alpha_\gamma = \frac{(-1)^\gamma q^{\gamma^2}}{(-q;q^2)_\gamma}$, $\beta_\gamma = \frac{1}{(-q^2;q^2)_\gamma}$ in (4.1), we have

(5.20) $$X_{Rp}(q) = \frac{1}{(-q^2;q^2)_{p+1}} \Psi_p(-q) + \sum_{m=0}^{p} \frac{q^{2m+2}}{(-q^2;q^2)_{m+1}} \Psi_m(-q).$$

(vi) Letting $p \to \infty$ in (v), we have

(5.21) $$X_R(q) = \frac{1}{(-q^2;q^2)_\infty} \Psi(-q) + \sum_{m=0}^{\infty} \frac{q^{2m+2}}{(-q^2;q^2)_{m+1}} \Psi_m(-q).$$

6. Relations between Fifth Order Mock Theta Functions and Third Order Mock Theta Functions and Their Partial Sums

(i) Taking $\alpha_\gamma = \frac{q^{\gamma^2}}{(-q;q)_\gamma}$, $\beta_\gamma = \frac{1}{(-q;q)_\gamma}$ in (4.1), we have

(6.1) $$f_p(q) = \frac{1}{(-q;q)_{p+1}} f_{0p}(q) + \sum_{m=0}^{p} \frac{q^{m+1}}{(-q;q)_{m+1}} f_{0m}(q).$$

(ii) Letting $p \to \infty$ in (i), we have

(6.2) $$f(q) = \frac{1}{(-q;q)_\infty} f_0(q) + \sum_{m=0}^{\infty} \frac{q^{m+1}}{(-q;q)_{m+1}} f_{0m}(q).$$

(iii) Taking $\alpha_\gamma = \frac{q^{\gamma^2}}{(q;q^2)_\gamma}$, $\beta_\gamma = q^\gamma$ in (4.1), we have

(6.3) $$F_{0p}(q) = q^{(p+1)^2} \Psi_p(q) + \sum_{m=0}^{p} \left\{ q^{m^2} (1 - q^{2m+1}) \right\} \Psi_m(q).$$

(iv) Letting $p \to \infty$ in (iii), we have

(6.4) $$F_0(q) = \sum_{m=0}^{\infty} q^{m^2} (1 - q^{2m+1}) \Psi_m(q).$$

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7. Expansion of Tenth Order Mock Theta Function in Terms of Partial Mock Theta Function of Tenth Order

(i) Taking $\alpha_\gamma = \frac{q^{\gamma(\gamma+1)/2}}{(q;q^2)_{\gamma+1}}$, $\beta_\gamma = q^{\gamma+1}$ in (4.1), we have

$$\Psi_{Rp}(q) = q^{p+2}\Phi_{Rp}(q) + (1-q)\sum_{m=0}^{p} q^{m+1}\Phi_{Rm}(q). \tag{7.1}$$

(ii) Letting $p \to \infty$ in (i), we have

$$\Psi_R(q) = (1-q)\sum_{m=0}^{\infty} q^{m+1}\Phi_{Rm}(q). \tag{7.2}$$

(iii) Taking $\alpha_\gamma = \frac{(-1)^\gamma q^{\gamma^2}}{(-q;q)_{2\gamma}}$, $\beta_\gamma = \frac{q^{2\gamma+1}}{1+q^{2\gamma+1}}$ in (4.1), we have

$$\begin{aligned}\chi_{Rp}(q) = {} & \frac{q^{2p+3}}{1+q^{2p+3}} X_{Rp}(q) \\ & + (1-q^2)\sum_{m=0}^{p} \frac{q^{2m+1}}{(1+q^{2m+1})(1+q^{2m+3})} X_{Rm}(q).\end{aligned} \tag{7.3}$$

(iv) Letting $p \to \infty$ in (iii), we have

$$\chi_R(q) = (1-q^2)\sum_{m=0}^{\infty} \frac{q^{2m+1}}{(1+q^{2m+1})(1+q^{2m+3})} X_{Rm}(q). \tag{7.4}$$

8. Continued Fraction Representation

We shall prove two lemmas and then with the help of these lemmas express these tenth order mock theta functions as continued fractions.

We define

$$A(\alpha;q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} \alpha^{2n}}{(-\alpha q;q)_{2n}}, \tag{8.1}$$

$$B(\alpha;q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} \alpha^n}{(q;q^2)_{n+1}}. \tag{8.2}$$

We observe

$$A(1;q) = X_R(q) \quad \text{and} \quad qA(q;q) = (1+q)\chi_R(q).$$
$$B(1;q) = \Phi_R(q) \quad \text{and} \quad qB(q;q) = \Psi_R(q).$$

Lemma 1.

$$A(\alpha;q) + \frac{\alpha^2 q}{1+\alpha q} A(\alpha q;q) - \frac{\alpha^3 q^3}{(1+\alpha q)(1+\alpha q^2)} A(\alpha q^2;q) = 1. \tag{8.3}$$

Proof.

$$\begin{aligned} A(\alpha; q) &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2} \alpha^{2n}}{(-\alpha q; q)_{2n}} \\ &= 1 - \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2} \alpha^{2n+2}}{(-\alpha q; q)_{2n+2}} \\ &= 1 - \frac{q\alpha^2}{1+\alpha q} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+2n} \alpha^{2n}}{(-\alpha q^2; q)_{2n+1}} \end{aligned}$$

So

$$\begin{aligned} \frac{1+\alpha q}{q\alpha^2}[A(\alpha q; q) - 1] + A(\alpha q; q) &= q^2 \alpha \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+4n} \alpha^{2n}}{(-\alpha q^2; q)_{2n+1}} \\ &= \frac{q^2 \alpha}{1+\alpha q^2} A(\alpha q^2; q). \end{aligned}$$

Simplifying we obtain the lemma. □

For $\alpha = 1$, we have the relation

$$A(1; q) + \frac{q}{1+q} A(q; q) - \frac{q^3}{(1+q)(1+q^2)} A(q^2; q) = 1. \tag{8.4}$$

Lemma 2.

$$B(\alpha; q) = \alpha q B(\alpha q; q) + q B(\alpha q^2; q) + 1. \tag{8.5}$$

Proof.

$$\begin{aligned} qB(\alpha q; q) &= \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q; q^2)_{n+1}} \alpha^n \\ &= \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{(q; q^2)_n} \alpha^n = \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{(q; q^2)_{n+1}} (1 - q^{2n+1}) \alpha^n \\ &= \frac{1}{\alpha} \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q; q^2)_{n+1}} \alpha^n - \frac{q}{\alpha} \sum_{n=0}^{\infty} \frac{q^{(n^2+5n)/2}}{(q; q^2)_{n+1}} \alpha^n - \frac{1}{\alpha} \\ &= \frac{1}{\alpha} B(\alpha; q) - \frac{q}{\alpha} B(\alpha q^2; q) - \frac{1}{\alpha}, \end{aligned}$$

which proves the lemma. □

For $\alpha = 1$, we have

$$B(1; q) = qB(q; q) + qB(q^2; q) + 1.$$

Continued Fractions for the Mock Theta Functions

(a) By Lemma 1

$$A(\alpha;q)+\frac{\alpha^2q}{1+\alpha q}A(\alpha q;q)-\frac{\alpha^3q^3}{(1+\alpha q)(1+\alpha q^2)}A(\alpha q^2;q)=1,$$

and we have the continued fraction

$$\begin{aligned}\frac{A(\alpha;q)}{A(\alpha q;q)}&=S(\alpha;q)+\cfrac{T(\alpha;q)}{\cfrac{A(\alpha q;q)}{A(\alpha q^2;q)}}\\&=S(\alpha;q)+\cfrac{T(\alpha;q)}{S(\alpha q;q)+\cfrac{T(\alpha q;q)}{\cfrac{A(\alpha q^2;q)}{A(\alpha q^3;q)}}}\\&=S(\alpha;q)+\cfrac{T(\alpha;q)}{S(\alpha q;q)+\cfrac{T(\alpha q;q)}{S(\alpha q^2;q)+\cfrac{T(\alpha q^2;q)}{\cfrac{A(\alpha q^3;q)}{A(\alpha q^4;q)}}}}\end{aligned}$$

where

$$\begin{aligned}S(\alpha;q)&=\left[\frac{1}{A(\alpha q;q)}-\frac{\alpha^2q}{(1+\alpha q)}\right]\\T(\alpha;q)&=\frac{\alpha^3q^3}{(1+\alpha q)(1+\alpha q^2)}\end{aligned}$$

Putting $\alpha=1$, we have

$$\frac{A(1;q)}{A(q;q)}=S(1;q)+\cfrac{T(1;q)}{S(q;q)+\cfrac{T(q;q)}{S(q^2;q)+\cfrac{T(q^2;q)}{\cfrac{A(q^3;q)}{A(q^4;q)}}}}$$

But

$$A(1;q)=X_R(q)\quad\text{and}\quad qA(q;q)=(1+q)\chi_R(q),$$

and so we have the continued fraction representation for the tenth order mock theta functions $X_R(q)$ and $\chi_R(q)$.

$$\frac{q}{1+q}\frac{\chi_R(q)}{X_R(q)} = S(1;q) + \cfrac{T(1;q)}{S(q;q) + \cfrac{T(q;q)}{S(q^2;q) + \cfrac{T(q^2;q)}{\cfrac{A(q^3;q)}{A(q^4;q)}}}} \tag{8.6}$$

(b) By Lemma 2

$$\begin{aligned}
\frac{B(\alpha;q)}{B(\alpha q;q)} &= \left(\alpha q + \frac{1}{B(\alpha q;q)}\right) + \cfrac{q}{\cfrac{B(\alpha q;q)}{B(\alpha q^2;q)}} \\
&= S(\alpha;q) + \cfrac{q}{\cfrac{B(\alpha q;q)}{B(\alpha q^2;q)}} \\
&= S(\alpha;q) + \cfrac{q}{S(\alpha q;q) + \cfrac{q}{\cfrac{B(\alpha q^2;q)}{B(\alpha q^3;q)}}}
\end{aligned}$$

$$= S(\alpha;q) + \cfrac{q}{S(\alpha q;q) + \cfrac{q}{S(\alpha q^2;q) + \cfrac{q}{\cdot}}} \tag{8.7}$$

where

$$S(\alpha;q) = \alpha q + \frac{1}{B(\alpha q;q)}.$$

Putting $\alpha = 1$, we have

$$\frac{B(1;q)}{B(q;q)} = S(1;q) + \cfrac{q}{S(q;q) + \cfrac{q}{S(q^2;q) + \cfrac{q}{\cdot}}}$$

But

$$B(1;q) = \Phi_R(q) \quad \text{and} \quad qB(q;q) = \Psi_R(q),$$

and so we have the continued fraction representation for the tenth order mock theta functions $\Phi_R(q)$ and $\Psi_R(q)$.

$$\frac{q\Phi_R(q)}{\Psi_R(q)} = S(1;q) + \cfrac{q}{S(q;q) + \cfrac{q}{S(q^2;q) + \cfrac{q}{\cdot}}} \tag{8.8}$$

Conclusion. This study of mock theta functions is interesting in the sense that a mock theta function has been expressed in a series of mock theta function of different order. Moreover we feel that a number theoretic interpretation can be given of these expansions.

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