The Steenrod Algebra and Braid Groups

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THE STEENROD ALGEBRA AND BRAID GROUPS

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1. Introduction. Let \( q \neq 0 \) be an element of \( \mathbb{Z}/(p) \) for a prime \( p \). In this paper, we define a map \( \phi_q : B_t \to \mathbb{Z}/(p) \) from the braid group \( B_t \) to \( \mathbb{Z}/(p) \) related to a cohomology of the Steenrod algebra with

\[
\phi_q(\sigma\tau) = \phi_q(\tau) + (-q)^{|\tau|}\phi_q(\sigma),
\]

where \( |\tau| = \sum_i \epsilon_i \) for \( \tau = \sigma_{t_1}^{\epsilon_1}\sigma_{t_2}^{\epsilon_2}\cdots\sigma_{t_n}^{\epsilon_n} \) (\( \sigma_i \) are generators of \( B_t \)). This map emerged from computation of the stable homotopy groups of spheres. In this paper, we argue the relation between this map and a cohomology of the Steenrod algebra. In the forthcoming paper, we shall argue the stable homotopy groups of spheres by using this map.

The braid group \( B_t \) of degree \( t \) is generated by \( \sigma_i \) (\( 1 \leq i \leq t-1 \)) and relations

\[
(R1) \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} (1 \leq i \leq t-2)
\]

\[
(R2) \quad \sigma_i\sigma_j = \sigma_j\sigma_i \quad (|i-j| \geq 2).
\]

Hence \( B_t \) has a presentation \( F_t/\langle R1, R2 \rangle \) for a free group \( F_t = \langle \sigma_1, \cdots, \sigma_{t-1} \rangle \). We notice that the symmetric group \( S_t \) has a presentation \( F_t/\langle R1, R2, \sigma_i^2 = e \rangle = B_t/\langle \sigma_i^2 = e \rangle \) (\( e \) is a unit). We denote \( \bar{\sigma} \in S_t \) for \( \sigma \in F_t \). For example, \( \bar{\sigma}_i \) is a transposition \( (i, i+1) \). Then we denote \( \sigma N = (n_{\bar{\sigma}^{-1}(1)}, \cdots, n_{\bar{\sigma}^{-1}(t)}) \).

Definition 1.1. Inductively we define \( \bar{\phi}_q : F_t \to \mathbb{Z}/(p) \) by taking

\[
\bar{\phi}_q(e) = 0, \quad \bar{\phi}_q(\sigma_i) = 1, \quad \bar{\phi}_q(\sigma_i^{-1}) = q^{-1} \quad \text{and}
\]

\[
\bar{\phi}_q(\sigma_i\sigma) = \bar{\phi}_q(\sigma) + (-q)^{|\sigma|}, \quad \bar{\phi}_q(\sigma_i^{-1}\sigma) = \bar{\phi}_q(\sigma) - (-q)^{|\sigma|-1}.
\]

Then \( \tilde{\phi}_q(\gamma\sigma) = \tilde{\phi}_q(\sigma) = \tilde{\phi}_q(\sigma\gamma) \) for \( \gamma \in \langle R1, R2 \rangle \). \( \phi_q : B_t \to \mathbb{Z}/(p) \) is induced from \( \tilde{\phi}_q \).

At Proposition 5.2, we show (1.1) and that \( \phi_q \) is well defined.
Let $A_*$ be the dual of Steenrod algebra for a prime $p$. Then
\begin{equation}
A_* = \begin{cases} 
\mathbb{Z}/(2)[\xi_1, \xi_2, \cdots] & \text{for } p = 2, \\
E(\tau_0, \tau_1, \cdots) \otimes_{\mathbb{Z}/(p)} \mathbb{Z}/(p)[\xi_1, \xi_2, \cdots] & \text{for odd prime } p.
\end{cases}
\end{equation}
where $E(\tau_0, \tau_1, \cdots)$ is the exterior algebra, $\deg \xi_i = 2^i - 1$ for $p = 2$, $= 2(p^i - 1)$ for an odd prime $p$, and $\deg \tau_i = 2p^i - 1$. We have a subalgebra
\begin{equation}
P_* = \begin{cases} 
\mathbb{Z}/(2)[\xi_1^2, \xi_2^2, \cdots] & \text{for } p = 2, \\
\mathbb{Z}/(p)[\xi_1, \xi_2, \cdots] & \text{for odd prime } p.
\end{cases}
\end{equation}
Let $c : A_* \to A_* \otimes_{\mathbb{Z}/(p)} A_*$ be a conjugation. In this paper, we use elements
\begin{equation}
\eta_i = \begin{cases} 
c(\xi_{i+1}) & \text{for } p = 2, \\
c(\tau_i) & \text{for } p > 2
\end{cases}
\quad \text{and} \quad
\gamma_i = \begin{cases} 
c(\xi_i^2) & \text{for } p = 2, \\
c(\xi_i) & \text{for } p > 2
\end{cases}
\end{equation}
For a coproduct $\Delta : A_* \to A_* \otimes_{\mathbb{Z}/(p)} A_*$,
\begin{equation}
\Delta \tau_n = \tau_n \otimes 1 + \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \tau_i \quad \text{and} \quad
\Delta \xi_n = \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \xi_i \quad (\xi_0 = 1).
\end{equation}
Hence,
\begin{equation}
\Delta \eta_n = 1 \otimes \eta_n + \sum_{i=0}^{n} \eta_i \otimes \xi_{n-i}^{p^i} \quad \text{and} \quad
\Delta \gamma_n = \sum_{i=0}^{n} \gamma_i \otimes \xi_{n-i}^{p^i} \quad (\xi_0 = 1).
\end{equation}
Let $C = \{C^t; \delta : C^t \to C^{t+1}\}$ be a cochain complex defined by
\begin{equation}
C^t = A_* \otimes_{\mathbb{Z}/(p)} P_* = A_* \otimes_{\mathbb{Z}/(p)} \cdots \otimes_{\mathbb{Z}/(p)} A_* \otimes_{\mathbb{Z}/(p)} P_* \quad \text{and} \quad
(t \text{ times})
\end{equation}
\begin{equation}
(1.\mathcal{J}) x_t \otimes \cdots \otimes x_2 \otimes x_1 \otimes m = x_t \otimes \cdots \otimes x_2 \otimes x_1 \otimes \Delta m \\
+ \sum_{i=1}^{t} (-1)^ix_t \otimes \cdots \otimes \Delta x_i \otimes \cdots \otimes x_1 \otimes m \\
(-1)^{t+1}1 \otimes x_t \otimes \cdots \otimes x_1 \otimes m,
\end{equation}
in which $x_i \in A_*$, $m \in P_*$ and $\Delta : N \to A_* \otimes N$ for $N = A_*$ or $P_*$. This cochain complex has the cohomology
\begin{equation}
H^t(C^*; \delta) = \text{Ext}^t_{A_*}(\mathbb{Z}/(p), P_*)
= \mathbb{Z}/(p)[a_0, a_1, \cdots] \quad (a_i \in \text{Ext}^1) \quad \text{(see (2.1-2)).}
\end{equation}
At Definition 2.2, we define $X(N) \in C^t$ representing $a_N = a_{n_1} \cdots a_{n_t}$ for any sequence $N = (n_1, \cdots, n_t)$ of integers $n_i \geq 0$ (see Proposition 2.3 iii)). Then, for any $\sigma \in F_t$, we define $Y_\sigma(N) \in C^{t-1}$ in Definition 3.1 such that

\begin{align*}
(1.9) \quad & \delta Y_\sigma(N) = X(N) - X(\sigma N) \quad \text{(see Proposition 3.3 i))} \\
(1.10) \quad & Y_{\sigma \tau}(N) = Y_{\tau}(N) + Y_{\sigma}(\tau N) \quad \text{(see Proposition 3.4 i))}.
\end{align*}

Moreover we define $R_\gamma(N) \in C^{t-2}$ for $\gamma \in \langle R1, R2 \rangle$ in Definition 4.1 such that

\begin{align*}
(1.11) \quad & \delta R_\gamma(N) = Y_\gamma(N) \quad \text{(see Proposition 4.2 i))}.
\end{align*}

**Definition 1.2.** (1) We define submodules of $C^t$, $C^{t-1}$ and $C^{t-2}$ by taking

\begin{align*}
C^t_X & = \langle X(N) \rangle, \quad C^{t-1}_Y(N) = \langle Y_\sigma(\tau N) | \sigma, \tau \in F_t \rangle, \\
C^{t-2}_R & = \langle R_\gamma(N) | \gamma \in \langle R1, R2 \rangle \rangle, \\
C^{t-1}_Y & = \bigcup_N C^{t-1}_Y(N) \quad \text{and} \quad C^{t-2}_R = \bigcup_N C^{t-2}_R(N)
\end{align*}

(2) We define a map $\tilde{\phi}_{q,N} : F_t \to C^{t-1}_Y(N)$ by taking

\begin{align*}
\tilde{\phi}_{q,N}(e) = 0, \quad \tilde{\phi}_{q,N}(\sigma_i) = Y_{\sigma_i}(N), \quad \tilde{\phi}_{q,N}(\sigma_i^{-1}) = q^{-1}Y_{\sigma_i}(N) \quad \text{and}
\end{align*}

\begin{align*}
\tilde{\phi}_{q,N}(\sigma_i^{\pm 1}) = \tilde{\phi}_{q,N}(\sigma) + q^{\pm 1}\tilde{\phi}_{q,N}(\sigma_i^{\pm 1}).
\end{align*}

At Proposition 5.3, we see that $\tilde{\phi}_{q,N}$ is well defined and

\begin{align*}
(1.12) \quad & \tilde{\phi}_{q,N}(\sigma \tau) = \tilde{\phi}_{q,N}(\tau) + q^{\tau}\tilde{\phi}_{q,N}(\sigma).
\end{align*}

The definitions of $\phi_q$, $\tilde{\phi}_{q,N}$ and the following theorem are the main purposes in this paper.

**Theorem 1.3.**

i) $C^t_X/\delta C^{t-1}_Y = H^t(C^*; \delta) = \text{Ext}_A^t(Z/(p), P_\ast) = Z/(p)[a_0, a_1, \cdots].$

ii) $\tilde{\phi}_{1,N}(\sigma) = Y_\sigma(N)$, and so $\tilde{\phi}_{1,N}(\langle R1, R2 \rangle) = \delta C^{t-2}_R(N).$

iii) $\tilde{\phi}_{q,N}(\langle R1, R2 \rangle)$ is a submodule of $C^{t-1}_Y(N).$
iv) We assume that \( n_i \neq n_j \) for any \( i \neq j \). If \( t \geq 5 \) and \( q^2 \neq 1 \) (\( p \)) then

\[
C_{Y}^{t-1}\left(N\right)/\phi_{q,N}\left(\langle R1, R2 \rangle\right) = \mathbb{Z}/(p) \quad \text{and}
\]

\[
\phi_{q,N} \text{ induces}
\]

\[
\phi_q : B_t = F_t/\langle R1, R2 \rangle \rightarrow C_{Y}^{t-1}\left(N\right)/\phi_{q,N}\left(\langle R1, R2 \rangle\right) = \mathbb{Z}/(p)
\]

in Definition 1.1.

The cochain complex \( C \) and its cohomology are useful in the field of the stable homotopy theory. Let \( H\mathbb{Z}/(p) \) be the Eilenberg-MacLane spectrum and \( MU_* \) a ring spectrum representing the complex cobordism theory. Then the Brown-Peterson spectrum \( BP \) is a minimal wedge summand of \( MU \) localized at a prime \( p \). Now \( H\mathbb{Z}/(p)_*\left(BP\right) = P_* \) and the Adams spectral sequence \( E_2 = \text{Ext}_{A_*}^{s,t}(\mathbb{Z}/(p), P_*) \Rightarrow \pi_*\left(BP\right) \) collapses and converges (see [9]). The Adams spectral sequence \( \{ E\left(H\mathbb{Z}/(p)\right)_*^{s,t}, d_r^{H\mathbb{Z}/(p)} \}\) with

\[
E\left(H\mathbb{Z}/(p)\right)_2^{s,t} = \text{Ext}_{A_*}^{s,t}(\mathbb{Z}/(p), \mathbb{Z}/(p))
\]

and the Novikov-Adams spectral sequence \( \{ E\left(BP\right)_*^{s,t}, d_r^{BP} \}\) with

\[
E\left(BP\right)_2^{s,t} = \text{Ext}_{BP,BP}^{s,t}(BP_*, BP_*)
\]

are used to calculate the stable homotopy groups of the spheres. For these \( E_2 \)-terms, we have the Mahowald and May spectral sequences

\[
\{ E\left(Mah\right)_{u,r}^{s,t}, d_r^{Mah} \} \quad \text{and} \quad \{ E\left(May\right)_{u,r}^{s,t}, d_r^{May} \}
\]

with

\[
E\left(Mah\right)_{u,2}^{s,t} = \text{Ext}_{P_*}^{s,u}(\mathbb{Z}/(p), \text{Ext}_{A_*}^{t,\ast}(\mathbb{Z}/(p), P_*)) \Rightarrow E\left(H\mathbb{Z}/(p)\right)_2^{s+t,u} \quad \text{and}
\]

\[
E\left(May\right)_{u,1}^{s,t} = \text{Ext}_{P_*}^{s,u}(\mathbb{Z}/(p), \text{Ext}_{A_*}^{t,\ast}(\mathbb{Z}/(p), P_*)) \Rightarrow E\left(BP\right)_2^{s,u-t}.
\]

By [4, 6], we can calculate \( d_r^{H\mathbb{Z}/(p)} \) and \( d_r^{BP} \) from \( d_r^{Mah} \) and \( d_r^{May} \), respectively. Moreover \( d_r^{Mah} \) is calculated by using \( \delta \) of (1.7), \( X\left(N\right), Y_r\left(N\right) \) and \( R_r\left(N\right) \) by [4]. We shall use the argument in this paper to calculate \( d_r^{BP} \) for \( p = 2 \) in the forthcoming paper.

This paper is organized as follows. We define \( X\left(N\right), Y_r\left(N\right) \) and \( R_r\left(N\right) \) in §2, §3 and §4, respectively. The maps \( \phi_q \) and \( \phi_{q,N} \) are argued in §5.
2. Definition of $X(N)$. In this section, we define an element $X(N) \in C^t$. Then we argue the properties of this element.

Let $C_A = \{C^t_A; \delta^A : C^t_A \to C^{t+1}_A\}$ be a cochain complex defined by

$$C^t_A = A^t_\ast \otimes_{\mathbb{Z}/(p)} A_\ast$$

and

$$\delta^A(x_t \otimes \cdots \otimes x_2 \otimes x_1 \otimes x_0) = \sum_{i=0}^{t} (-1)^i x_t \otimes \cdots \otimes \Delta x_i \otimes \cdots \otimes x_0$$

$$+ (-1)^{t+1} \otimes x_t \otimes \cdots \otimes x_0 \quad (x_i \in A_\ast).$$

Then we notice that

(2.1) \quad $C_A \supset C, \quad \delta^A | C = \delta$,
(2.2) \quad $H^t(C^*_A; \delta^A) = \text{Ext}^t_{A_\ast}(\mathbb{Z}/(p), A_\ast) = \begin{cases} 0 & \text{for } t > 0, \\ \mathbb{Z}/(p) & \text{for } t = 0 \end{cases}$ and
(2.3) \quad $\delta^A(x \otimes y) = -\delta^A x \otimes y + x \otimes \Delta y \quad \text{for } x \in A^t_\ast = C^{t-1}_A, y \in A_\ast$.

Hence we have the following lemma.

**Lemma 2.1.** For $x \in C^t$ ($t > 0$), $\delta x = 0$ if and only if there exists an element $y^A \in C^{t-1}_A$ such that $\delta^A y^A = x$.

Let $B_\ast = A_\ast / P_\ast$ be a quotient algebra, $pr : A_\ast \to B_\ast$ a projection and $\Delta_B : B_\ast \to B_\ast \otimes_{\mathbb{Z}/(p)} B_\ast$ an induced coproduct. We have a cochain complex

$$C_B = \{C^t_B = B^t_\ast; \delta^B : C^t_B \to C^{t+1}_B\}$$

defined by

$$\delta^B(x_t \otimes \cdots \otimes x_1) = x_t \otimes \cdots \otimes x_1 \otimes 1$$

$$+ \sum_{i=1}^{t} (-1)^i x_t \otimes \cdots \otimes \Delta_B x_i \otimes \cdots \otimes x_1$$

$$+ (-1)^{t+1} \otimes x_t \otimes \cdots \otimes x_1, \quad (x_i \in B_\ast).$$

Now $pr$ induces a cochain homomorphism $pr_\ast : C = A^t \otimes_{\mathbb{Z}/(p)} P_\ast \to C_B = B^t_\ast \otimes_{\mathbb{Z}/(p)} \mathbb{Z}/(p)$ and an isomorphism

(2.4) \quad $pr_\ast : \text{Ext}^t_{A_\ast}(\mathbb{Z}/(p), P_\ast) = H^t(C^*_A) \xrightarrow{\simeq} \text{Ext}^t_{B_\ast}(\mathbb{Z}/(p), \mathbb{Z}/(p)) = H^t(C^*_B)$.
by the change-of-ring theorem. Then

$$\text{Ext}_{B_*}^*(\mathbb{Z}/(p), \mathbb{Z}/(p)) = \mathbb{Z}/(p)[a_0, a_1, \cdots]$$

and $a_i \in \text{Ext}_{B_*}^1(\mathbb{Z}/(p), \mathbb{Z}/(p))$ is represented by $pr(\eta_n) \in B_* = C_B^1$, where $\eta_n \in A_*$ is the one of (1.4). Consider elements $x(n) = \eta_n \in C_A^0 = A_*$ and $X(n) = \delta^A x(n) = \sum_{i=0}^n \eta_i \otimes t_{n-i}^n \in C^1$. $X(n)$ represents $a_n$ since $pr_*(X(n)) = \eta_n \in B_* = C_B^1$. Let $N = (n_1, n_2, \cdots, n_t)$ be a sequence of integers with $n_i \geq 0$. We shall define an element $x(N) \in C_A^{t-1} = A_*^t$ so that $\delta^A x(N)$ is included in $C^t$ and represents a monomial $a_N = a_{n_1} a_{n_2} \cdots a_{n_t}$ as follows:

**Definition 2.2.** Inductively we define $x(N) \in C_A^{t-1}$ by taking $x(n_1) = \eta_{n_1}$ and

$$x(N, n_{t+1}) = x(n_1, \cdots, n_t, n_{t+1}) = \sum_{I=0}^N x(I) \otimes t_{N-I}^I \eta_{n_{t+1}} ,$$

where

$$I = (i_1, i_2, \cdots, i_t) ,$$

$$\sum_{I=0}^N = \sum_{i_1=0}^{n_1} \sum_{i_2=0}^{n_2} \cdots \sum_{i_t=0}^{n_t}$$

and

$$t_{N-I}^I = t_{n_1-i_1}^{i_1} t_{n_2-i_2}^{i_2} \cdots t_{n_t-i_t}^{i_t} .$$

Moreover we define

$$X(N) = \sum_{I=0}^N x(I) \otimes t_{N-I}^I \in C^t ,$$

and so

$$x(N, n) = X(N) \eta_n .$$
We shall prove that $\delta^A x(N) = X(N)$. For this purpose, we notice the following:

\[(a \otimes x)(b \otimes y) = (-1)^{\deg x \cdot \deg b} ab \otimes xy \quad \text{(see [11, Theorem 17.8])},\]

\[(2.6) \quad \Delta t_{N-I}^{P_I} = \sum_{J=I}^{N} t_{J-I}^{P_I} \otimes t_{N-J}^{P_J} \quad \text{and} \]

\[(2.7) \quad \Delta t_{N-I}^{P_I} \eta_n = \sum_{J=I}^{N} t_{J-I}^{P_I} \otimes t_{N-J}^{P_J} \eta_n + \sum_{J=I}^{N} \sum_{i=0}^{n} t_{J-I}^{P_I} \eta_i \otimes t_{N-J}^{P_I} t_{n-I}^{P_I} \]

by (1.6). For any expression $f(I)$,

\[(2.9) \quad \sum_{I=0}^{N} \sum_{J=I}^{N} f(I) \otimes t_{J-I}^{P_I} \otimes t_{N-J}^{P_J} = \sum_{I=0}^{N} \sum_{J=I}^{N} f(J) \otimes t_{J-I}^{P_I} \otimes t_{N-J}^{P_J} \]

\[= \sum_{I=0}^{N} \sum_{J=0}^{I} f(J) \otimes t_{J-I}^{P_I} \otimes t_{N-J}^{P_I}.\]

**Proposition 2.3.** For the above $X(N)$, we have the following.

i) $\delta^A x(N) = X(N)$.

ii) $\Delta_* X(N) = \sum_{I=0}^{N} X(I) \otimes t_{N-I}^{P_I} \in C^t \otimes P_*$, in which $\Delta_* : C^t \to C^t \otimes \mathbb{Z}/(p)$

$P_*$ is the one induced by $\Delta : P_* \to P_* \otimes \mathbb{Z}/(p) P_*$.

iii) $X(N)$ represents a monomial $a_N$.

**Proof.** i) We shall use induction on $t$. For the case of $t = 1$, (1.6) implies this part. By (2.3), (2.8), (2.9) and induction,

\[\delta^A x(N, n) = \delta^A \sum_{I=0}^{N} x(I) \otimes t_{N-I}^{P_I} \eta_n \]

\[= - \sum_{I=0}^{N} \delta^A x(I) \otimes t_{N-I}^{P_I} \eta_n + \sum_{I=0}^{N} x(I) \otimes \Delta t_{N-I}^{P_I} \eta_n \]

\[= - \sum_{I=0}^{N} X(I) \otimes t_{N-I}^{P_I} \eta_n + \sum_{I=0}^{N} \sum_{J=0}^{I} x(J) \otimes t_{J-I}^{P_I} \otimes t_{N-J}^{P_I} \eta_n.\]
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\[ + \sum_{I=0}^{N} \sum_{J=0}^{I} \sum_{i=0}^{n} x(J) \otimes t_{I-J}^{p_J} \eta_i \otimes t_{N-I}^{p_I} t_{n-i}^{p_i}. \]

By definition,

\[ X(I) = \sum_{J=0}^{I} x(J) \otimes t_{I-J}^{p_J}. \]

and

\[ x(I, i) = \sum_{J=0}^{I} x(J) \otimes t_{I-J}^{p_J} \eta_i. \]

Hence

\[ \delta^A x(N, n) = \sum_{I=0}^{N} \sum_{i=0}^{n} x(I, i) \otimes t_{N-I}^{p_I} t_{n-i}^{p_i} = X(N, n). \]

ii) By the definition, (2.7) and (2.9),

\[ \Delta_* x(N) = \sum_{I=0}^{N} x(I) \otimes \sum_{J=I}^{N} t_{J-I}^{p_J} \otimes t_{N-J}^{p_J} = \sum_{I=0}^{N} \sum_{J=0}^{I} x(J) \otimes t_{I-J}^{p_J} \otimes t_{N-I}^{p_I} = \sum_{I=0}^{N} X(I) \otimes t_{N-I}^{p_I}. \]

iii) Let \( pr_* = pr \otimes \cdots \otimes pr : C_{A^t}^{t-1} = A_*^t \to C_{B^t}^{t-1} = B_*^t \) be a homomorphism.

By induction, we see that \( pr_* x(N, n) = pr_* x(N) \otimes \eta_n = \eta_{n_1} \otimes \cdots \eta_{n_t} \otimes \eta_n. \)

This implies this proposition.

3. Definition of \( Y_{\sigma}(N) \). In this section, we define an element \( Y_{\sigma}(N) \in C_{t+1}^{t-1} \) for \( \sigma \in F_t \). Then we argue the properties of this element.

**Definition 3.1.** (1) For \( t \geq 0 \), we define \( y_{t+1}(N, n, m) \in C_{A^t}^{t-1} \) for \( N = (n_1, \cdots, n_t) \) by taking

\[ y_1(n, m) = -\eta_n \eta_m \quad \text{and} \]

\[ y_{t+1}(N, n, m) = -X(N) \eta_n \eta_m = -\sum_{I=0}^{N} x(I) \otimes t_{N-I}^{p_I} \eta_n \eta_m. \]
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Inductively we define \( y_i(N, n) \in C^{t-1}_A \) for \( 1 \leq i \leq t - 1 \) by taking

\[
y_i(N, n) = -\sum_{I=0}^{N} y_i(I) \otimes t_{N-I}^{p_i} \eta_n.
\]

(2) Next we denote \( y_e(N) = 0 \), \( y_{\sigma_i}(N) = y_i(N) \) and \( y_{\sigma_i^{-1}}(N) = y_i(N) \) for generators \( \sigma_i \in F_t \) and define \( y_{\sigma} \) inductively by \( y_{\sigma_{i\pm 1} \sigma}(N) = y_{\sigma}(N) + y_{\sigma}(\sigma N) \) for \( \sigma \in F_t \). Now we take \( Y_a(N) = \sum_{I=0}^{N} y_a(I) \otimes t_{N-I}^{p_i} \in C^{t-1} \) for \( a = i \) or \( \sigma \).

Remark 3.2. In the above definition, we notice that \( \eta_n \eta_m = -\eta_m \eta_n \), and so

\[
y_i(\sigma_i N) = -y_i(N)
\]

by induction on \( t \) for a fixed \( i \). Then

\[
y_{\sigma_i^{-1} \sigma_i \sigma}(N) = y_{\sigma}(N) + y_i(\sigma N) + y_i(\sigma_i \sigma N) = y_{\sigma}(N).
\]

Hence \( y_{\sigma}(N) \) for any \( \sigma \in F_t \) is well-defined.

By (1.6), (2.6) and (2.7), we see that

\[
\Delta \eta_n \eta_m = (1 \otimes \eta_n + \sum_{i=0}^{n} \eta_i \otimes t_{n-i}^{p_i}) (1 \otimes \eta_m + \sum_{j=0}^{m} \eta_j \otimes t_{m-j}^{p_j})
\]

\[
= 1 \otimes \eta_n \eta_m + \sum_{i=0}^{n} \eta_i \otimes t_{n-i}^{p_i} \eta_m
\]

\[
- \sum_{j=0}^{m} \eta_j \otimes \eta_n t_{m-j}^{p_j} + \sum_{i=0}^{n} \sum_{j=0}^{m} \eta_i \eta_j \otimes t_{n-i}^{p_i} t_{m-j}^{p_j},
\]

\[3.2\]
and

\[
\Delta t_{N+1}^{p^I} \eta_n \eta_m = \sum_{J=I}^{N} t_{N-J}^{p^I} \otimes t_{N-J}^{p^I} \eta_n \eta_m + \sum_{J=I}^{N} \sum_{i=0}^{n} t_{N-J}^{p^I} \eta_i \otimes t_{N-J}^{p^I} t_{n-i}^{p^I} \eta_m \\
- \sum_{J=I}^{N} \sum_{j=0}^{m} t_{N-J}^{p^I} \eta_j \otimes t_{N-J}^{p^I} t_{m-j}^{p^I} \\
+ \sum_{J=I}^{N} \sum_{i=0}^{n} \sum_{j=0}^{m} t_{N-J}^{p^I} \eta_i \eta_j \otimes t_{N-J}^{p^I} t_{n-i}^{p^I} t_{m-j}^{p^I}.
\]

**Proposition 3.3.** For the above elements, we have the following.

i) \([1.9]\) \(\delta Y_\sigma(N) = X(N) - X(\sigma N)\).

ii) \(\Delta_\sigma Y_\sigma(N) = \sum_{I=0}^{N} Y_\sigma(I) \otimes t_{N-I}^{p^I}\).

**Proof.** i) In the first place, we prove that

\[
\delta^A y_t(N) = -x(N) + x(\sigma t N) + Y_t(N).
\]

For example, by (2.3), (2.9) and (3.3),

\[
\delta^A y_{t+1}(N, n, m) = \sum_{I=0}^{N} \delta^A x(I) \otimes t_{N-I}^{p^I} \eta_n \eta_m - \sum_{I=0}^{N} x(I) \otimes \Delta t_{N-I}^{p^I} \eta_n \eta_m
\]

\[
= \sum_{I=0}^{N} X(I) \otimes t_{N-I}^{p^I} \eta_n \eta_m \\
- \sum_{I=0}^{N} \sum_{J=0}^{I} x(J) \otimes t_{N-J}^{p^I} \eta_n \eta_m \\
- \sum_{I=0}^{N} \sum_{J=0}^{I} \sum_{i=0}^{n} x(J) \otimes t_{N-J}^{p^I} \eta_i \otimes t_{n-i}^{p^I} \eta_m \\
+ \sum_{I=0}^{N} \sum_{J=0}^{I} \sum_{j=0}^{m} x(J) \otimes t_{N-J}^{p^I} \eta_j \otimes t_{n-i}^{p^I} t_{m-j}^{p^I} \\
- \sum_{I=0}^{N} \sum_{J=0}^{I} \sum_{i=0}^{n} \sum_{j=0}^{m} x(J) \otimes t_{N-J}^{p^I} \eta_i \eta_j \otimes t_{n-i}^{p^I} t_{m-j}^{p^I}.
\]
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By the definitions of $X(I)$, $X(I,i)$, $X(I,j)$ and $y_{t+1}(I,i,j)$,

$$
\delta^A y_{t+1}(N,n,m) = - \sum_{i=0}^{N} \sum_{j=0}^{m} x(I,i) \otimes t_{N-I}^{p_i} t_{n-i}^{p_j} \\
+ \sum_{i=0}^{N} \sum_{j=0}^{m} x(I,j) \otimes t_{N-I}^{p_i} t_{n-j}^{p_j} \\
+ \sum_{i=0}^{N} \sum_{j=0}^{m} \sum_{k=0}^{m} y_{t+1}(I,i,j) \otimes t_{N-I}^{p_i} t_{n-i}^{p_j} t_{n-j}^{p_k} \\
= -x(N,n,m) + x(N,m,n) + Y_{t+1}(N,n,m).
$$

Then (3.4) for any $i$ is proved by induction on $t$. In fact, by (2.3), (2.8) and (2.9),

$$
\delta^A y_i(N,n) = \sum_{I=0}^{N} \delta^A y_i(I) \otimes t_{N-I}^{p_i} - \sum_{I=0}^{N} y_i(I) \otimes \Delta t_{N-I}^{p_i} \\
= \sum_{I=0}^{N} \left\{ -x(I) + x(\sigma_i I) + Y_i(I) \right\} \otimes t_{N-I}^{p_i} \\
- \sum_{I=0}^{N} \sum_{J=0}^{I} y_i(J) \otimes t_{I-J}^{p_i} t_{N-I}^{p_i} \eta_n \\
- \sum_{I=0}^{N} \sum_{J=0}^{I} \sum_{a=0}^{n} y_i(J) \otimes t_{I-J}^{p_i} t_{n-a}^{p_i} \\
= -x(N,n) + x(\sigma_i N,n) + Y_i(N,n).
$$

Next, by (2.3), (2.9) and (3.4),

$$
\delta Y_i(N) = - \sum_{I=0}^{N} \delta^A y_i(I) \otimes t_{N-I}^{p_i} + \sum_{I=0}^{N} y_i(I) \otimes \Delta t_{N-I}^{p_i} \\
= - \sum_{I=0}^{N} \left\{ -x(I) + x(\sigma_i I) + Y_i(I) \right\} \otimes t_{N-I}^{p_i} \\
+ \sum_{I=0}^{N} \sum_{J=0}^{I} y_i(J) \otimes t^{p_i}_{I-J} t_{N-I}^{p_i}.$$
\[ = X(N) - X(\sigma_i N). \]

In the last place, we assume that i) of this proposition holds for \( \sigma \in F_i \), then
\[
\delta Y_{\sigma_i N}^+ \left( \sigma N \right) = \delta \left\{ Y_{\sigma}(N) + Y_{\sigma_i N}^+ (\sigma N) \right\} \\
= X(N) - X(\sigma N) + X(\sigma N) - X(\sigma_i N^+ \sigma N) \\
= X(N) - X(\sigma_i N). 
\]

By induction, we see i) of this proposition.

ii) This is proved by the same manner as ii) of Proposition 2.3.

Next we prepare the following proposition.

**Proposition 3.4.**

i) \( y_{\tau \sigma}(N) = y_{\tau}(N) + y_{\sigma}(\tau N), Y_{\tau \sigma}(N) = Y_{\tau}(N) + Y_{\sigma}(\tau N), \) \( y_i(\sigma_i N) = -y_i(N) \) and \( Y_i(\sigma_i N) = -Y_i(N). \)

ii) \( Y_{\sigma}(N) + Y_{\sigma^{-1}}(\sigma N) = 0 \) and \( Y_{\sigma^{-1}}(N) = -Y_{\sigma}(\sigma^{-1} N). \)

iii) If \( \tau \sigma N = \sigma N \) then \( Y_{\sigma^{-1} \tau \sigma}(N) = Y_{\tau}(\sigma N). \)

iv) \( Y_{\tau \sigma}(N) = \left( \frac{1+(-1)^n}{2} \right) Y_i(N). \)

**Proof.**

i) This is trivial by induction on a length of the word \( \sigma. \)

ii) By i), \( Y_{\sigma}(N) + Y_{\sigma^{-1}}(\sigma N) = Y_{\sigma^{-1}}(\sigma N) = Y_{\sigma}(N) = 0. \)

iii) By i), ii) and the assumption,
\[
Y_{\sigma^{-1} \tau \sigma}(N) = Y_{\tau \sigma}(N) + Y_{\sigma^{-1}}(\tau \sigma N) \\
= Y_{\sigma}(N) + Y_{\tau}(\sigma N) + Y_{\sigma^{-1}}(\sigma N) \\
= Y_{\tau}(\sigma N). 
\]

iv) By i) and induction,
\[
Y_{\sigma_i N}^{n+1} = Y_{\sigma_i}(N) + Y_{\sigma_i N}^{n} = \left( \frac{1+(-1)^n}{2} \right) Y_i(\sigma_i N) \\
= 1 - \left( \frac{1+(-1)^n}{2} \right) Y_i(N) \\
= \left( \frac{1-(-1)^{n+1}}{2} \right) Y_i(N). 
\]
Definition 3.5. We define elements $\gamma_i, \gamma_{i,j} \in F_t$ ($|i - j| \geq 2$) by

$$\gamma_i = \sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{and} \quad \gamma_{i,j} = \sigma_i^{-1} \sigma_j^{-1} \sigma_i \sigma_j,$$

and normal subgroups

$$\langle R1 \rangle = \langle \gamma_i \rangle, \quad \langle R2 \rangle = \langle \gamma_{i,j} \rangle \quad \text{and} \quad \langle R1, R2 \rangle = \langle \gamma_i, \gamma_{i,j} \rangle$$

generated by $\gamma_i, \gamma_{i,j}$ for $i + 1 < j$.

Remark 3.6. We notice that $\gamma_{i,j}^{-1} = \gamma_{j,i}$.

Consider $Y_\gamma(N)$ for these elements $\gamma \in \langle R1, R2 \rangle$. The following lemma is shown by the definitions and Proposition 3.4.

Lemma 3.7. For the above elements, we have the following.

i) $\gamma N = N$ for $\gamma \in \langle R1, R2 \rangle$, and so $Y_{\gamma' \gamma}(N) = Y_{\gamma' \gamma}(N) = Y_{\gamma}(N) + Y_{\gamma'}(N)$ for $\gamma' \in \langle R1, R2 \rangle$ and $Y_{\sigma^{-1} \gamma \sigma}(N) = Y_{\gamma}(\sigma N)$ for $\gamma \in \langle R1, R2 \rangle$ and $\sigma \in F_t$.

ii) $y_{\gamma_i}(N) = y_{i+1}(N) - y_{i+1}(\sigma_i N) + y_{i+1}(\sigma_i \sigma_{i+1} N) - y_{i}(N) + y_{i}(\sigma_{i+1} N) - y_{i}(\sigma_i \sigma_{i+1} N)$ and $Y_{\gamma_i}(N) = Y_{i+1}(N) - Y_{i+1}(\sigma_i N) + Y_{i+1}(\sigma_i \sigma_{i+1} N) - Y_{i}(N) + Y_{i}(\sigma_{i+1} N) - Y_{i}(\sigma_i \sigma_{i+1} N)$.

iii) $y_{\gamma_{i,j}}(N) = y_{j}(N) - y_{i}(N) + y_{j}(\sigma_i N) + y_{i}(\sigma_j N)$ and $Y_{\gamma_{i,j}}(N) = Y_{j}(N) - Y_{i}(N) + Y_{j}(\sigma_i N) + Y_{i}(\sigma_j N)$.

4. Definition of $R_\gamma(N)$. In this section, we define $R_\gamma(N) \in C_t^{t-2}$ for $\gamma \in \langle R1, R2 \rangle$ and prove (1.11).

Definition 4.1. For $n, m, l \geq 0$, $N = (n_1, \cdots, n_t)$ and $i$ ($1 \leq i \leq t - 1$), we define elements $r_{t+1}(N, n, m, l) \in C_A$ and $r_{i,t+1}(N, n, m) \in C_A^{t-1}$ as follows:

$$r_1(n, m, l) = \eta_n \eta_m \eta_l, \quad r_{t+1}(N, n, m, l) = X(N) \eta_n \eta_m \eta_l \quad \text{and} \quad r_{i,t+1}(N, n, m) = Y_i(N) \eta_n \eta_m.$$
Then we define $r_i(N) \in C^{t-3}_A$ for $1 \leq i \leq t - 2$ and $r_{i,j}(N) \in C^{t-3}_A$ for $1 \leq i \leq j - 2 \leq t - 3$ by taking inductively

$$r_i(N, n) = \sum_{I=0}^{N} r_i(I) \otimes t_{N-I}^{p_i} \eta_n \quad \text{for} \quad 1 \leq i \leq t - 2 \quad \text{and}$$

$$r_{i,j}(N, n) = \sum_{I=0}^{N} r_{i,j}(I) \otimes t_{N-I}^{p_j} \eta_n \quad \text{for} \quad 1 \leq i \leq j - 2 \leq t - 3.$$ 

Next we take

$$r_{\gamma_i}(N) = r_i(N), \quad r_{\gamma_i^{-1}}(N) = -r_i(N), \quad r_{\gamma_i,j}(N) = r_{i,j}(N) \quad \text{and}$$

$$r_{\gamma_i^{-1},j}(N) = -r_{i,j}(N).$$

Then we define $r_{\sigma^{-1},\gamma^\pm_1}(N) = r_{\gamma^{\pm_1}}(\sigma N)$ for $\gamma = \gamma_i, \gamma_i, \gamma_i,j$ and $\gamma_i,j$. Now any element $\rho \in \langle R1, R2 \rangle$ is given by $\rho = \rho_1 \cdots \rho_k \cdots \rho_n \in \langle R1, R2 \rangle$ where $\rho_k = \sigma^{-1}\gamma^\sigma$ for some $\sigma$ and $\gamma = \gamma_i^{\pm_1}, \gamma_i,j^{\pm_1}$. We define $r_\rho(N) = \sum_{i=1}^{n} r_{\rho_i}(N)$.

Moreover we take

$$R_{a}(N) = \sum_{I=0}^{N} r_a(I) \otimes t_{N-I}^{p_i},$$

where $r_a = r_i, r_{i,j}$ and $r_\rho$ for $\rho \in \langle R1, R2 \rangle$.

We notice that

(4.1)

$$\Delta \eta_i \eta_m \eta_l = 1 \otimes \eta_i \eta_m \eta_l + \sum_{i=0}^{n} \eta_i \otimes t_{n-i}^{p_i} \eta_m \eta_l - \sum_{j=0}^{m} \eta_j \otimes \eta_m t_{m-j}^{p_j} \eta_l$$

$$+ \sum_{k=0}^{l} \eta_k \otimes \eta_m t_{m-k}^{p_k} + \sum_{i=0}^{n} \eta_i \eta_j \otimes t_{n-i}^{p_i} t_{m-j}^{p_j} \eta_l$$

$$- \sum_{i=0}^{n} \sum_{k=0}^{l} \eta_i \eta_k \otimes t_{n-i}^{p_i} t_{m-j}^{p_j} \eta_l$$

$$+ \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{k=0}^{l} \eta_i \eta_j \eta_k \otimes t_{n-i}^{p_i} t_{m-j}^{p_j} t_{l-k}^{p_k}.$$
by (1.6), (2.6) and (3.2). Hence, by (2.6) and (2.7),

\[(4.2) \Delta t_{N-I}^{p_i} \eta_n \eta_m \eta_l = \sum_{J=I}^N \sum_{i=0}^m t_{N-I}^{p_i} \otimes t_{N-J}^{p_j} t_{N-I}^{p_i} \eta_n \eta_m \eta_l \]

\[+ \sum_{J=I}^N \sum_{i=0}^m \sum_{k=0}^l \sum_{j=0}^m \sum_{k=0}^l t_{N-I}^{p_i} \eta_i \eta_j \eta_k \otimes t_{N-J}^{p_j} t_{N-I}^{p_i} \eta_n \eta_m \eta_l \]

\[+ \sum_{J=I}^N \sum_{i=0}^m \sum_{k=0}^l \sum_{j=0}^m \sum_{k=0}^l t_{N-I}^{p_i} \eta_i \eta_j \eta_k \otimes t_{N-J}^{p_j} t_{N-I}^{p_i} \eta_n \eta_m \eta_l \]

\[+ \sum_{J=I}^N \sum_{i=0}^m \sum_{k=0}^l \sum_{j=0}^m \sum_{k=0}^l t_{N-I}^{p_i} \eta_i \eta_j \eta_k \otimes t_{N-J}^{p_j} t_{N-I}^{p_i} \eta_n \eta_m \eta_l \]

Now we have (1.11) as follows:

**Proposition 4.2.**

i) \( \delta R_\gamma(N) = Y_\gamma(N) \) for \( \gamma \in \langle R_1, R_2 \rangle \).

ii) \( \Delta \gamma R_\gamma(N) = \sum_{I=0}^N R_\gamma(I) \otimes t_{N-I}^{p_i} \) for \( \gamma \in \langle R_1, R_2 \rangle \).

iii) \( R_{\sigma^{-1} \gamma}(N) = R_\gamma(\sigma N) \)

\( R_{\gamma'}(N) = R_{\gamma \gamma'}(N) = R_\gamma(N) + R_{\gamma'}(N) \) for \( \gamma, \gamma' \in \langle R_1, R_2 \rangle \) and \( \sigma \in F_\ell \).
Proof. i) By using Proposition 2.3 i), Lemma 3.7 ii), (2.3), (2.9) and (4.2), we see that

\[
\delta^A r_{t+1}(N, n, m, l) = - \sum_{l=0}^{N} \delta^A x(I) \otimes t_{N-l}^{p_i} \eta_n \eta_m \eta_l + \sum_{l=0}^{N} x(I) \otimes \Delta t_{N-l}^{p_i} \eta_n \eta_m \eta_l
\]

\[
= \sum_{l=0}^{N} \sum_{i=0}^{n} x(I, i) \otimes t_{N-l}^{p_i} \eta_n \eta_m \eta_l
\]

\[- \sum_{l=0}^{N} \sum_{j=0}^{m} x(I, j) \otimes t_{N-l}^{p_i} \eta_n \eta_m \eta_l
\]

\[+ \sum_{l=0}^{N} \sum_{i=0}^{n} \sum_{j=0}^{m} y_{t+1}(I, i, j) \otimes t_{N-l}^{p_i} t_{m-j}^{p_j} \eta_l
\]

\[- \sum_{l=0}^{N} \sum_{i=0}^{n} \sum_{k=0}^{l} y_{t+1}(I, i, k) \otimes t_{N-l}^{p_i} t_{m-i}^{p_k} \eta_l
\]

\[+ \sum_{l=0}^{N} \sum_{j=0}^{m} \sum_{k=0}^{l} y_{t+1}(I, j, k) \otimes t_{N-l}^{p_j} t_{n-j}^{p_k} \eta_l
\]

\[+ \sum_{l=0}^{N} \sum_{j=0}^{m} \sum_{k=0}^{l} r_{t+1}(I, i, j, k) \otimes t_{N-l}^{p_i} t_{m-j}^{p_j} t_{l-k}^{p_k}
\]

\[= -y_{t+2}(N, n, m, l) + y_{t+2}(N, m, n, l)
\]

\[- y_{t+2}(N, l, n, m) + y_{t+1}(N, n, m, l)
\]

\[- y_{t+1}(N, n, l, m) + y_{t+1}(N, m, l, n)
\]

\[+ R_{t+1}(N, n, m, l)
\]

\[= -y_{t+1}(N, n, m, l) + R_{t+1}(N, n, m, l).
\]

Then, by induction, we can prove that

(4.3) \[\delta^A r_i(N) = -y_{\gamma_i}(N) + R_i(N).\]

In fact, by (2.3), (2.8) and (2.9),

\[\delta^A r_i(N, n) = - \sum_{l=0}^{N} \delta^A r_i(I) \otimes t_{N-l}^{p_i} \eta_n + \sum_{l=0}^{N} r_i(I) \otimes \Delta t_{N-l}^{p_i} \eta_n
\]
\[
\begin{align*}
&= -\sum_{I=0}^{N} \{-y_{\gamma_{i}}(I) + R_{i}(I)\} \otimes t_{N-I}^{p_{i}} \eta_{n} \\
&\quad + \sum_{I=0}^{N} R_{i}(I) \otimes t_{N-I}^{p_{i}} \eta_{n} \\
&\quad + \sum_{I=0}^{N} \sum_{a=0}^{n} r_{i}(I, a) \otimes t_{N-I}^{p_{i}} t_{n-a}^{p_{a}} \\
&= -y_{\gamma_{i}}(N, n) + R_{i}(N, n).
\end{align*}
\]

Now,
\[
\delta R_{i}(N) = -\sum_{I=0}^{N} \delta^{A} r_{i}(I) \otimes t_{N-I}^{p_{i}} + \sum_{I=0}^{N} r_{i}(I) \otimes \Delta t_{N-I}^{p_{i}}
\]
\[
= -\sum_{I=0}^{N} \{-y_{\gamma_{i}}(I) + R_{i}(I)\} \otimes t_{N-I}^{p_{i}}
\]
\[
+ \sum_{I=0}^{N} \sum_{J=0}^{I} r_{i}(J) \otimes t_{I-J}^{p_{i}} \otimes t_{N-I}^{p_{i}}
\]
\[
= Y_{\gamma_{i}}(N).
\]

Next, by Lemma 3.7, (2.3), (2.9), (3.3) and (3.4),
\[
\delta^{A} r_{i, t+1}(N, n, m) = -\sum_{I=0}^{N} \delta^{A} y_{i}(I) \otimes t_{N-I}^{p_{i}} \eta_{n} \eta_{m} + \sum_{I=0}^{N} y_{i}(I) \otimes \Delta t_{N-I}^{p_{i}} \eta_{n} \eta_{m}
\]
\[
= -\sum_{I=0}^{N} \{-x(I) + x(\sigma_{i} I) + Y_{i}(I)\} \otimes t_{N-I}^{p_{i}} \eta_{n} \eta_{m}
\]
\[
+ \sum_{I=0}^{N} Y_{i}(I) \otimes t_{N-I}^{p_{i}} \eta_{n} \eta_{m}
\]
\[
+ \sum_{I=0}^{N} \sum_{a=0}^{n} y_{i}(I, a) \otimes t_{N-I}^{p_{i}} t_{n-a}^{p_{a}} \eta_{m}
\]
\[
- \sum_{I=0}^{N} \sum_{b=0}^{m} y_{i}(I, b) \otimes t_{N-I}^{p_{i}} t_{m-b}^{p_{b}}
\]
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\[ + \sum_{I=0}^{N} \sum_{a=0}^{n} \sum_{b=0}^{m} r_{i,t+1}(I, a, b) \otimes t_{N-I}^{p^a} t_{n-a}^{p^b} \]

\[ = -y_{t+1}(N, n, m) + y_{t+1}(\sigma I N, n, m) \]

\[ + y_{t}(N, n, m) - y_{t}(N, m, n) + R_{i,t+1}(N, n, m) \]

\[ = -y_{\gamma, t+1}(N, n, m) + R_{i,t+1}(N, n, m). \]

Then, by induction, we can prove that

\[(4.4) \quad \delta^A r_{i,j}(N) = -y_{\gamma, i,j}(N) + R_{i,j}(N). \]

In fact, by (2.3), (2.8) and (2.9),

\[ \delta^A r_{i,j}(N, n) = -\sum_{I=0}^{N} \delta^A r_{i,j}(I) \otimes t_{N-I}^{p^j} \eta_n + \sum_{I=0}^{N} r_{i,j}(I) \otimes \Delta t_{N-I}^{p^j} \eta_n \]

\[ = -\sum_{I=0}^{N} \{ -y_{\gamma, i,j}(I) + R_{i,j}(I) \} \otimes t_{N-I}^{p^j} \eta_n \]

\[ + \sum_{I=0}^{N} R_{i,j}(I) \otimes t_{N-I}^{p^j} \eta_n \]

\[ + \sum_{I=0}^{N} \sum_{a=0}^{n} r_{i,j}(I, a) \otimes t_{N-I}^{p^j} t_{n-a}^{p^a} \]

\[ = -y_{\gamma, i,j}(N, n) + R_{i,j}(N, n). \]

Now we see that

\[ \delta R_{i,j}(N) = -\sum_{I=0}^{N} \delta^A r_{i,j}(I) \otimes t_{N-I}^{p^j} \]

\[ + \sum_{I=0}^{N} r_{i,j}(I) \otimes \Delta t_{N-I}^{p^j} \]

\[ = -\sum_{I=0}^{N} \{ -y_{\gamma, i,j}(I) + R_{i,j}(I) \} \otimes t_{N-I}^{p^j} \]

\[ + \sum_{I=0}^{N} \sum_{J=0}^{I} r_{i,j}(J) \otimes t_{I-J}^{p^j} \otimes t_{N-I}^{p^j} \]

\[ = Y_{\gamma, i,j}(N). \]

By the definition of $R_{\gamma}(N)$ and the above results, Lemma 3.7 implies this proposition i).

ii) This is proved by the same manner as ii) of Proposition 2.3.

iii) By the definition, this is trivial.
5. $\phi_q$ and $\tilde{\phi}_{q,N}$. In this section, we argue the properties of $\phi_q$ and $\tilde{\phi}_{q,N}$ and prove Theorem 1.3.

In the first place, we prepare the following lemma.

Lemma 5.1.

i) $\tilde{\phi}_q(\sigma_i^{-1} \sigma_i \sigma) = \phi_q(\sigma) = \tilde{\phi}_q(\sigma_i \sigma_i^{-1} \sigma).

ii) $\tilde{\phi}_q(\sigma \tau) = \phi_q(\tau) + (-q)^{||\tau||} \tilde{\phi}_q(\sigma).

iii) $\tilde{\phi}_q((\sigma \tau) \delta) = \phi_q(\sigma (\tau \delta)) = \phi_q(\delta) + (-q)^{||\delta||} \phi_q(\tau) + (-q)^{|\tau \delta|} \tilde{\phi}_q(\sigma).

iv) $\tilde{\phi}_q(\sigma^{-1}) = -(-q)^{-|\sigma|} \tilde{\phi}_q(\sigma).

v) $\tilde{\phi}_q(\sigma^{-1} \gamma \sigma) = (-q)^{|\alpha|} \phi_q(\gamma) + \tilde{\phi}_q(\sigma) - (-q)^{|\gamma|} \tilde{\phi}_q(\sigma).

vi) $\tilde{\phi}_q(\gamma_i) = 0$ and $\tilde{\phi}_q(\gamma_{i,j}) = 0.

vii) For $\gamma \in \langle R1, R2 \rangle$, $\tilde{\phi}_q(\gamma) = 0$ and $\tilde{\phi}_q(\gamma \sigma) = \phi_q(\sigma) = \tilde{\phi}_q(\sigma \gamma)$.

Proof. i) By definition,

$\tilde{\phi}_q(\sigma_i^{-1} \sigma_i \sigma) = \phi_q(\sigma_i \sigma) + (-q)^{|\sigma|+1} q^{-1}$

$= \phi_q(\sigma) + (-q)^{|\sigma|} + (-q)^{|\sigma|+1} q^{-1} = \tilde{\phi}_q(\sigma).

As the same way, we see $\tilde{\phi}_q(\sigma_i \sigma_i^{-1} \sigma) = \tilde{\phi}_q(\sigma).

ii) We assume $\tilde{\phi}_q(\sigma \tau) = \phi_q(\tau) + (-q)^{|\tau|} \tilde{\phi}_q(\sigma)$. Then, by definition,

$\tilde{\phi}_q(\sigma_i \sigma \tau) = \phi_q(\sigma \tau) + (-q)^{|\sigma \tau|}$

$= \phi_q(\tau) + (-q)^{|\tau|} \phi_q(\sigma) + (-q)^{|\tau|+|\sigma|} = \tilde{\phi}_q(\tau) + (-q)^{|\tau|} \tilde{\phi}_q(\sigma_i \sigma).

As the same way, $\tilde{\phi}_q(\sigma_i^{-1} \sigma \tau) = \phi_q(\tau) + (-q)^{|\tau|} \tilde{\phi}_q(\sigma_i^{-1} \sigma)$. By induction on the word $\sigma$, we see ii).

iii) By ii), this is trivial.

iv) By ii), $\tilde{\phi}_q(\sigma^{-1}) + (-q)^{-|\sigma|} \tilde{\phi}_q(\sigma) = \phi_q(\sigma^{-1}) = \tilde{\phi}_q(e) = 0.

v) This is easy by iii) and iv).
vi) By ii),
\[
\tilde{\phi}_q(\gamma_i) = \tilde{\phi}_q(\sigma_{i+1}) - q\tilde{\phi}_q(\sigma_i) + q^2\tilde{\phi}_q(\sigma_{i+1}) - q^3\tilde{\phi}_q(\sigma_i^{-1}) + q^2\tilde{\phi}_q(\sigma_i^{-1}) - q\tilde{\phi}_q(\sigma_i) = 1 - q + q^2 - q^2 + q - 1 = 0
\]

By the same way, we see the another equation.

vii) is seen by ii), v), vi) and \(|\gamma| = 0\).

This lemma i) implies that \(\bar{\phi}_q\) is well defined. By vii), \(\bar{\phi}_q\) induces
\[
\phi_q : B_t = F_t/\langle R1, R2 \rangle \to \mathbb{Z}/(p).
\]

Hence we have the following.

**Proposition 5.2.**

i) \(\phi_q\) is well defined.

ii) \(\phi_q(\sigma \tau) = \phi_q(\tau) + (-q)^{\mid\tau\mid}\phi_q(\sigma)\).

iii) \(\phi_q(\sigma^{-1}) = -(-q)^{-\mid\sigma\mid}\phi_q(\sigma)\).

iv) \(\phi_q(\sigma^{-1} \gamma \sigma) = (-q)^{\mid\sigma\mid}\phi_q(\gamma) + \phi_q(\sigma) - (-q)^{\mid\gamma\mid}\phi_q(\sigma)\).

Next consider the map \(\tilde{\phi}_{q,N} : F_t \to C^{-1}_y(N)\) of Definition 1.2 (2).

**Proposition 5.3.**

i) \(\tilde{\phi}_{q,N}(\sigma_i^{-1} \sigma_i \sigma) = \tilde{\phi}_{q,N}(\sigma) = \tilde{\phi}_{q,N}(\sigma_i^{-1} \sigma)\). Hence \(\tilde{\phi}_{q,N}\) is well defined.

ii) \(\tilde{\phi}_{q,N}(\sigma \tau) = \tilde{\phi}_{q,N}(\tau) + q^{\mid\tau\mid}\tilde{\phi}_{q,N}(\sigma)\).

iii) \(\tilde{\phi}_{q,N}(\sigma \tau \delta) = \tilde{\phi}_{q,N}(\sigma(\tau \delta)) = \tilde{\phi}_{q,N}(\sigma) + q^{\mid\delta\mid}\phi_{q,N}(\sigma) + q^{\mid\tau\delta\mid}\phi_{q,N}(\sigma)\).

iv) \(\tilde{\phi}_{q,N}(\sigma^{-1}) = -(-q)^{\mid\sigma\mid}\phi_{q,N}(\sigma)\).

v) \(\tilde{\phi}_{q,N}(\sigma^{-1} \gamma \sigma) = q^{\mid\sigma\mid}\phi_{q,N}(\gamma) + \phi_{q,N}(\sigma) - q^{\mid\gamma\mid}\phi_{q,N}(\sigma)\).
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Proof. i) By definition and Proposition 3.4 i),
\[ \tilde{\phi}_{q,N}(\sigma_i^{-1}\sigma_i\sigma) = \tilde{\phi}_{q,N}(\sigma) + q^{l(\tau)}\tilde{\phi}_{q,N}(\sigma_i) + q^{l(\tau)+1}\tilde{\phi}_{q,N}(\sigma_i^{-1}) \]
\[ = \tilde{\phi}_{q,N}(\sigma) + q^{l(\tau)}Y_i(\sigma\sigma) + q^{l(\tau)+1}Y_i(\sigma_i\sigma) \]
\[ = \tilde{\phi}_{q,N}(\sigma) \]

The another equation is seen by the same way.

ii) By induction on word \( \sigma \) and
\[ \tilde{\phi}_{q,N}(\sigma_i^{\pm 1}\sigma_i) = \{ \tilde{\phi}_{q,N}(\tau) + q^{l(\tau)}\tilde{\phi}_{q,N}(\sigma) \} + q^{l(\tau)+1}\tilde{\phi}_{q,N}(\sigma_i^{\pm 1}) \]
\[ = \tilde{\phi}_{q,N}(\tau) + q^{l(\tau)}\tilde{\phi}_{q,N}(\sigma_i^{\pm 1}) \]

we see ii).

iii) is given by ii).

iv) is given by ii) and \( \tilde{\phi}_{q,N}(\sigma_i\sigma_i^{-1}) = \tilde{\phi}_{q,N}(e) = 0 \).

v) is given by iii) and iv).

Now we argue \( \tilde{\phi}_{q,N}(\langle R_1, R_2 \rangle) \). The following lemma is proved by Proposition 5.3.

Lemma 5.4. For \( \gamma, \gamma' \in \langle R_1, R_2 \rangle \) and \( \sigma \in F_t \), we have the following.

i) \( \tilde{\phi}_{q,N}(\sigma^{-1}\gamma\sigma) = q^{l(\tau)}\tilde{\phi}_{q,N}(\gamma) \).

ii) \( \tilde{\phi}_{q,N}(\gamma\gamma') = \tilde{\phi}_{q,N}(\gamma) + \tilde{\phi}_{q,N}(\gamma') \). Hence \( \tilde{\phi}_{q,N}(\langle R_1, R_2 \rangle) \) is a submodule of \( C_{Y^{-1}}(N) \).

iii) \( \tilde{\phi}_{q,N}(\sigma_\gamma) = \tilde{\phi}_{q,N}(\sigma) + \tilde{\phi}_{q,N}(\gamma) \) and
\( \tilde{\phi}_{q,N}(\gamma\sigma) = \tilde{\phi}_{q,N}(\sigma^{-1}\gamma\sigma) = \tilde{\phi}_{q,N}(\sigma) + \tilde{\phi}_{q,N}(\sigma^{-1}\gamma\sigma) \).

Hence \( \tilde{\phi}_{q,N} \) induces a map
\( \phi_{q,N} : B_t = F_t/\langle R_1, R_2 \rangle \rightarrow C_{Y^{-1}}(N)/\tilde{\phi}_{q,N}(\langle R_1, R_2 \rangle) \).

iv)
\[ \tilde{\phi}_{q,N}(\gamma_i) = Y_{i+1}(N) - Y_i(N) - q\{Y_{i+1}(\sigma_i N) - Y_i(\sigma_{i+1}N)\} \]
\[ + q^2\{Y_{i+1}(\sigma_i\sigma_{i+1}N) - Y_i(\sigma_{i+1}\sigma_iN)\} \]
\[ \tilde{\phi}_{q,N}(\gamma_{i,j}) = Y_j(N) - Y_i(N) - q\{Y_j(\sigma_i N) - Y_i(\sigma_j N)\}. \]
Next we define elements \( Z_{i,\sigma}(N) = Y_1(N) - (-1)^{\sigma|\sigma} Y_i(\sigma N) \) and a module \( C^t_{Z^{-1}}(N) = (Z_{i,\sigma}(N)). \)

**Lemma 5.5.**

i) If \( p = 2 \) then \( C^t_{Y^{-1}}(N)/C^t_{Z^{-1}}(N) = \mathbb{Z}/(2) \). If \( p > 2 \) and \( n_i \neq n_j \) for any \( i \neq j \) then \( C^t_{Y^{-1}}(N)/C^t_{Z^{-1}}(N) = \mathbb{Z}/(p) \). If \( p > 2 \) and \( n_i = n_j \) for some \( i \neq j \) then \( C^t_{Y^{-1}}(N)/C^t_{Z^{-1}}(N) = 0 \).

ii) \( \tilde{\phi}_{q,N}(\langle R1, R2 \rangle) \subset C^t_{Z^{-1}}(N) \).

iii) If \( q^2 \neq 1 \) (p) and \( t \geq 5 \) then \( \tilde{\phi}_{q,N}(\langle R1, R2 \rangle) = C^t_{Z^{-1}}(N) \).

**Proof.**

i) \( \{ Y_i(\sigma N) \mid \sigma \in F_i \} \) is a basis of \( C^t_{Y^{-1}}(N) \), and so is \( \{ Y_1(N) \} \cup \{ Z_{i,\sigma}(N) \mid \sigma \in F_i \} \). Hence \( C^t_{Y^{-1}}(N)/C^t_{Z^{-1}}(N) \) has a generator \( Y_1(N) \). If \( p = 2 \) then \( Y_i(\sigma N) \neq 0 \). Therefore \( C^t_{Y^{-1}}(N)/C^t_{Z^{-1}}(N) = \mathbb{Z}/(2) \).

We assume \( p > 3 \). If \( n_i \neq n_j \) for any \( i \neq j \) then \( Y_i(\sigma N) \neq 0 \) for any \( i \) and \( \sigma \in F_i \), and so \( C^t_{Y^{-1}}(N)/C^t_{Z^{-1}}(N) = \mathbb{Z}/(p) \). If \( n_i = n_j \) for some \( i \neq j \) then there exists an element \( \sigma \in F_i \) so that \( n_{\sigma^{-1}(1)} = n_{\sigma^{-1}(2)} \), i.e., \( \sigma_i \sigma N = \sigma N \), and so \( Y_1(\sigma N) = Y_1(\sigma_1 \sigma N) = -Y_1(\sigma N) \). Hence \( Y_1(\sigma N) = 0 \) by \( p > 2 \) and \( Y_1(N) = Z_{i,\sigma}(N) \subset C^t_{Z^{-1}}(N) \). Thus \( C^t_{Y^{-1}}(N)/C^t_{Z^{-1}}(N) = 0 \).

ii) By definition, if \( |\sigma| = |\tau| \) then \( Y_i(\sigma N) - Y_j(\tau N) = (-1)^{|\sigma|} \{ Z_{i,\tau}(N) - Z_{i,\sigma}(N) \} \). We see ii) by Lemma 5.4.

iii) By Lemma 5.4,

\[
\tilde{\phi}_{q,N}(\sigma^{-1} \gamma_{i,j} \sigma) = q^{|\sigma|} \{ Y_j(\sigma N) - Y_i(\sigma N) \} \]

\[
- q^{|\sigma|+1} \{ Y_j(\sigma \sigma N) - Y_i(\sigma j \sigma N) \}
\]

and

\[
\tilde{\phi}_{q,N}((\sigma \sigma j \sigma)^{-1} \gamma_{i,j} \sigma_1 \sigma j \sigma) = q^{|\sigma|+2} \{ Y_j(\sigma \sigma j \sigma N) - Y_i(\sigma \sigma j \sigma N) \} \]

\[
- q^{|\sigma|+3} \{ Y_j(\sigma \sigma j \sigma \sigma N) - Y_i(\sigma j \sigma j \sigma N) \} = q^{|\sigma|+3} \{ Y_j(\sigma N) - Y_i(\sigma N) \} \]

\[
- q^{|\sigma|+2} \{ Y_j(\sigma j \sigma N) - Y_i(\sigma j \sigma N) \}.
\]

Therefore

\[
\tilde{\phi}_{q,N}(\sigma^{-1} \gamma_{i,j} \sigma) - q^{-1} \tilde{\phi}_{q,N}((\sigma \sigma j \sigma)^{-1} \gamma_{i,j} \sigma_1 \sigma j \sigma) \]

\[
= (1 - q^2)q^{|\sigma|} \{ Y_j(\sigma N) - Y_i(\sigma N) \}.
\]
and so $Y_j(\sigma N) \equiv Y_i(\sigma N) \mod \tilde{\phi}_{q,N}((R_1,R_2))$ for $|i-j| \geq 2$ if $q^2 \neq 1$ (p).
Hence $Y_i(\sigma N) \equiv Y_1(\sigma N)$ for $i \geq 3$. By $t \geq 5$, $Y_2(\sigma N) \equiv Y_4(\sigma N) \equiv Y_1(\sigma N)$.
Thus

(5.1) \quad Y_i(\sigma N) \equiv Y_1(\sigma N) \mod \tilde{\phi}_{q,N}((R_1,R_2)) \quad \text{for any } i.

By this equation,

\[ Y_j(\sigma_{j+1}^\pm \sigma N) \equiv Y_1(\sigma_{j+1}^\pm \sigma N) \equiv Y_j(\sigma_{j+1}^\pm \sigma N) = -Y_j(\sigma N) \equiv -Y_1(\sigma N). \]

By induction, $Y_i(\sigma N) \equiv (-1)^{|\sigma|}Y_1(N) \mod \tilde{\phi}_{q,N}((R_1,R_2))$. Thus

\[ Z_{i,\sigma}(N) \in \tilde{\phi}_{q,N}((R_1,R_2)). \]

Now we can prove Theorem 1.3.

Proof of Theorem 1.3. i-ii) are trivial by the definitions, (1.8) and Proposition 3.3.
iii) is Lemma 5.4 ii).
iv) We define a map $\psi : C_t^{i-1}(N) \to \mathbb{Z}/(p)$ by taking $\psi(Y_i(\sigma N)) = (-1)^{|\sigma|}$.
If $n_i \neq n_j$ for any $i \neq j$ then $\psi$ is well defined. By Definition 1.1 and 1.2, $\tilde{\phi}_{q} = \psi \tilde{\phi}_{q,N}$. Now Ker $\psi = C_t^{i-1}(N) = \tilde{\phi}_{q,N}((R_1,R_2))$ by Lemma 5.5, and so $\tilde{\phi}_{q,N}$ induces $\phi_q$.

References

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