The Comparison Theorem of
Hilbert-space-valued Tangent Sequences

Yu He*        Peide Liu†

*University Of Science And Technology Of China
†Wuhan University

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THE COMPARISON THEOREM OF
HILBERT-SPACE-VALUED TANGENT SEQUENCES

Yu HE and Peide LIU

Let \((\Omega, \mathcal{F}, \mu)\) be a complete probability space, \(X\) be a Hilbert space. When \(X\) has a Schauder basis \((e_i)_{i \geq 1}\), we consider \(\varphi: X \to \mathbb{R}^\infty = \{(a_i)_{i \geq 1} | a_i \in \mathbb{R}\}, \varphi(\sum_{i \geq 1} a_i e_i) = (a_i)_{i \geq 1}\); Let \(f\) be an \(X\)-valued random variable, then \(\varphi(f)\) is a series of random functions, there exists RCPD (regular conditional probability distribution) \(P_{\varphi(f)}\) of \(\varphi(f)\) w.r.t. \(B\), where \(B\) is a subalgebra of \(\mathcal{F}\). Let \(B^\infty\) be the Borel algebra of \(\mathbb{R}^\infty\), \(B_X\) be the Borel algebra of \(X\), \(\varphi(B_X) = \{\varphi(B) | B \in B_X\}\). Let \(\chi_A\) be the characteristic function of \(A \in \mathcal{F}\).

In this article, integrability means Bochner integrability.

**Lemma 1.** Let \(X\) be a Banach space, \(f\) be an \(X\)-valued random variable with almost separable values, \(\mathcal{B}\) be a subalgebra of \(\mathcal{F}\), then there exists regular conditional probability distribution of \(f\) w.r.t. \(B\) denoted by \(P_f\).

*Proof.* see [1].

**Theorem 2.** Let \(X\) be a Hilbert space, \(f\) be an \(X\)-valued integrable random variable, then \(E(f|\mathcal{B})(t) = \int_X xP_f(t, dx)\). a.e.

*Proof.* Since an \(X\)-valued integrable random variable is strong measurable, it is almost separably-valued by the Pettis theorem. We need only consider the case where \(X\) is a separable Hilbert space. Let \((e_n)_{n \geq 1}\) be an orthonormal basis of \(X\), \(p_n\) respectively \(q_n\) be the projections of \(X\) respectively \(\mathbb{R}^\infty\) to the \(n\)'th coordinate, then for all \(x \in X\), \(p_n(x) = q_n(\varphi(x))\);
\[ E(f|B)(t) = E\left( \sum_{n=1}^{\infty} (p_n f) e_n \bigg| B \right)(t) = \sum_{n=1}^{\infty} E(p_n f|B)(t)e_n \]

(Since \( \left\| \sum_{n=1}^{k} p_n f(t)e_n \right\| \leq \|f(t)\| \))

\[ = \sum_{n=1}^{\infty} E\left( q_n(\varphi(f))|B \right)(t)e_n = \sum_{n=1}^{\infty} \int_{\mathbb{R}^\infty} q_n(y) P_{\varphi f}(t,dy)e_n \]

\[ = \sum_{n=1}^{\infty} \int_X q_n(\varphi(x)) P_f(t,dx)e_n = \sum_{n=1}^{\infty} \int_X p_n(x)e_n P_f(t,dx) \]

\[ = \int_X \sum_{n=1}^{\infty} p_n(x)e_n P_f(t,dx) = \int_X x P_f(t,dx). \]

**Theorem 3.** Let \( X, Y \) be Hilbert spaces, \( f \) be an \( X \)-valued random variable with almost separable values, \( h: X \to Y \) be Borel measurable, \( h \circ f \) be integrable, Then \( E(h \circ f|B)(t) = \int_X h(x) P_f(t, dx). \) a.e.

**Proof.** Because \( h: X \to Y \) is measurable, we can define

\[ P_{h \circ f}(t, B) = P_f(t, h^{-1}(B)), \quad \forall t \in \Omega, \ B \in B_Y \]

then \( \forall t \in \Omega, P_{h \circ f}(t, \ast) \) is a probability measure on \( B_Y. \ \forall B \in B_Y, \)

\[ P_{h \circ f}(t, B) = P_f(t, h^{-1}(B)) = E\left( f^{-1}(h^{-1}(B)) | B \right)(t) = E((h \circ f)^{-1}(B)|B)(t) \text{ a.e.} \]

So \( P_{h \circ f} \) is a regular distribution of \( h \circ f \) w.r.t. \( B. \) Choosing regular distribution pair such as these and using Theorem 2, we have

\[ E(h \circ f|B)(t) = \int_Y y P_{h \circ f}(t, dy) \]

\[ = \int_X h(x) P_f(t, dx). \]

**Definition 4.** Let \( (\mathcal{F}_n)_{n \geq 0} \) be an increasing sub-\( \sigma \)-algebra sequence of \( \mathcal{F}, (d_n)_{n \geq 1}, (e_n)_{n \geq 1} \) are \( X \)-valued random variables w.r.t \( (\mathcal{F}_n)_{n \geq 1}. \) We call \( (d_n)_{n \geq 1} \) and \( (e_n)_{n \geq 1} \) tangent, if \( \forall A \in B_X, \forall n \geq 1, P(d_n^{-1}(A)|\mathcal{F}_{n-1}) = \)
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\( P(e_n^{-1}(A) \mid \mathcal{F}_{n-1}) \) a.e. We call \((d_n)_{n \geq 1}\) conditionally symmetric, if \((-d_n)_{n \geq 1}\) and \((d_n)_{n \geq 1}\) are tangent.

Let \( \Phi: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) be an increasing function satisfying the condition \( \Delta_2 \), it means: \( \exists C > 0 \) such that \( \forall x \geq 0, \Phi(2x) \leq C \Phi(x) \), and \( \Phi(0) = 0 \). Easily, we have: \( \forall x, y \geq 0, \Phi(x + y) \leq C \Phi(x) + C \Phi(y) \). Let \((d_n)_{n \geq 1}\) be a random variable sequence, we define

\[
    d_n^* = 0, \quad d_n^* = \sup_{1 \leq k \leq n} \|d_k\|, \quad d_n^* = \sup_{n \geq 1} \|d_n\|.
\]

**Lemma 5.** Let \( X \) be a Hilbert space with orthonormal basis \((e_i)_{i \geq 1}\), \( \varphi(\sum_{i \geq 1} a_i e_i) = (a_i)_{i \geq 1} \), then

\[
    \varphi(B_X) = B^\infty \cap \varphi(X) = \{ A \cap \varphi(X) \mid A \in B^\infty \}.
\]

**Proof.** Let \( p_n \) be the projection of \( \mathbb{R}^\infty \) to the first \( n \) coordinates, \( B^n \) be the Borel algebra of \( \mathbb{R}^n \). Then \( B^\infty = \sigma(T), T = \bigcup_{n \geq 1} p_n^{-1}(B^n) \), where \( \sigma(T) \) is the \( \sigma \) algebra generated by \( T \). So \( \sigma(T \cap \varphi(X)) = B^\infty \cap \varphi(X) \subseteq \varphi(B_X) \), \( B^\infty \cap \varphi(X) = \varphi(B_X) \).

**Lemma 6.** Let Hilbert spaces \( X, Y \) have orthonormal bases \((e_{2k-1})_{k \geq 1}, (e_{2k})_{k \geq 1} \) respectively, we take product topology on \( X \times Y \), then \( B_{X \times Y} = B_X \times B_Y \).

**Proof.** We define \( \varphi: Z = X \times Y \rightarrow \mathbb{R}^\infty, \varphi(\sum_{i \geq 1} a_i e_i) = (a_i)_{i \geq 1} \),

\[
    \mathbb{R}_1^\infty = \{(a_n)_{n \geq 1} \mid a_{2k} = 0, \forall k \geq 1\}
\]

\[
    \mathbb{R}_2^\infty = \{(a_n)_{n \geq 1} \mid a_{2k-1} = 0, \forall k \geq 1\}
\]

\( B_i^\infty \) is the Borel algebra of \( \mathbb{R}_i^\infty (i = 1, 2) \). Then

\[
    \varphi(B_Z) = B^\infty \cap \varphi(Z) = (B_1^\infty \times B_2^\infty) \cap (\varphi(X) \times \varphi(Y))
\]

\[
    = (B_1^\infty \cap \varphi(X)) \times (B_2^\infty \cap \varphi(Y))
\]

\[
    = \varphi(B_X) \times \varphi(B_Y) = \varphi(B_X \times B_Y).
\]

So \( B_Z = B_X \times B_Y \).
Lemma 7. Let $X$ be a Hilbert space with basis, $(d_n)_{n \geq 1}$ be an $X$-valued conditionally symmetric sequence. We define

$$\lambda_n = (d_1, \ldots, d_n): \Omega \to X^n = Y,$$

$$\xi_n = (d_1, \ldots, d_{n-1}, -d_n): \Omega \to Y,$$

then both $\lambda_n$, $\xi_n$ are measurable w.r.t. $B_Y$, we can take their RCPDs $P_{\lambda_n}$, $P_{\xi_n}$ w.r.t. $\mathcal{F}_{n-1}$ and $E \in \mathcal{F}_{n-1}$ such that $\mu(E) = 0$, and $\forall t \in \Omega \setminus E$, $\forall A \in B_Y$, $P_{\lambda_n}(t, A) = P_{\xi_n}(t, A)$.

Proof. By Lemma 6, $B_Y = B^n_X$, so $\forall A_i \in \mathcal{F}$, $\lambda_n$, $\xi_n$ are measurable w.r.t. $B_Y$. Since $X$ is separable, $X$ is secondly denumbrable, we can take a countable set $\mathcal{A} = \{A_i \mid i \in \mathbb{N}\}$ consisted of open sets of $X$ such that $\mathcal{A}$ generates the topology $\mathcal{T}_X$ of $X$. The $\sigma$ algebra generated by $\mathcal{T}_X$ is $\sigma(\mathcal{T}_X) = B_X$, so $\sigma(\mathcal{A}) = B_X$, $\sigma(\mathcal{A}^n) = B^n_X = B_Y$.

$$\forall B_k \in B_X, \quad P_{\lambda_n}(t, \bigcap_{k=1}^n B_k) = P(\lambda_{n-1}^{-1}(\bigcap_{k=1}^n B_k) \mid \mathcal{F}_{n-1})(t)$$

$$= P\left(\bigcap_{k=1}^n d_k^{-1}(B_k) \mid \mathcal{F}_{n-1}\right)(t)$$

$$= \prod_{k=1}^{n-1} \chi_{D_k} E(\chi_{D_k} \mid \mathcal{F}_{n-1})(t)$$

(where $D_k = d_k^{-1}(B_k)$)

$$= \prod_{k=1}^{n-1} \chi_{D_k} E(\chi_{E_n} \mid \mathcal{F}_{n-1})(t)$$

(where $E_n = (-d_n)^{-1}(B_n)$)

$$= E\left(\prod_{k=1}^{n-1} \chi_{D_k} \circ \chi_{E_n} \mid \mathcal{F}_{n-1}\right)(t)$$

$$= P(\xi_{n-1}^{-1}(\bigcap_{k=1}^n B_k) \mid \mathcal{F}_{n-1})(t)$$

$$= P_{\xi_n}(t, \bigcap_{k=1}^n B_k). \text{ a.e.}$$

For $k_i \in \mathbb{N}$, we take $E(k_1, \ldots, k_n) \in \mathcal{F}_{n-1}$, such that $\mu(E(k_1, \ldots, k_n)) = 0$, $\forall t \in \Omega \setminus E(k_1, \ldots, k_n)$, $P_{\lambda_n}(t, \prod_{i=1}^n A_{k_i}) = P_{\xi_n}(t, \prod_{i=1}^n A_{k_i})$. (1)
Let $E = \bigcup \{ E(k_1, \ldots, k_n) \mid k_i \in \mathbb{N}, 1 \leq i \leq n \}$, then $\mu(E) = 0$, $\forall t \in \Omega \setminus E$, $\forall k_i \in \mathbb{N}$, (1) holds. Since $P_{\lambda_n}(t, *)$ and $P_{\xi_n}(t, *)$ are probability measures on $\mathcal{B}_Y$, by (1), they are equal on $\mathcal{A}$ that generates $\mathcal{B}_Y$, so $\forall t \in \Omega \setminus E$, $\forall A \in \mathcal{B}_Y$, $P_{\lambda_n}(t, A) = P_{\xi_n}(t, A)$. For $t \in E$, we take

$$P_{\lambda_n}(t, A) = \mu(\lambda_n^{-1}(A)), \quad P_{\xi_n}(t, A) = \mu(\xi_n^{-1}(A)).$$

**Lemma 8.** Let $X$ be a Hilbert space, $(d_n)_{n \geq 1}$ be a conditionally symmetric sequence, $d_n \in L_1(\mu, X)$. We denote $f_n = \sum_{k=1}^{n} d_k$, then $f = (f_n)_{n \geq 1}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n \geq 1}$, and we have decomposition $f_n = g_n + h_n$, such that $g = (g_n)_{n \geq 1}$ and $h = (h_n)_{n \geq 1}$ are martingales w.r.t. $(\mathcal{F}_n)_{n \geq 1}$, where

$$g_n = \sum_{k=1}^{n} a_k \chi_{A_k}, \quad h_n = \sum_{k=1}^{n} b_k \chi_{B_k},$$

$$A_k = \{ \|d_k\| \leq 2d^*_{k-1} \}, \quad B_k = \{ \|d_k\| > 2d^*_{k-1} \}$$

**Proof.** Similar to the proof of Theorem 2, we need only consider the case where $X$ is a separable Hilbert space. We define $B: X^n \to X$ by

$$\beta(x_1, \ldots, x_n) = x_n \chi(\|x_n\| \leq 2y_{n-1})$$

$$y_{n-1} = \max\{\|x_1\|, \ldots, \|x_{n-1}\|\}, \quad y_0 = 0$$

then

$$E(a_n|\mathcal{F}_{n-1})(t) = E(\beta(d_1, \ldots, d_n)|\mathcal{F}_{n-1})(t) = E(\beta \circ \lambda_n|\mathcal{F}_{n-1})(t)$$

$$= \int_{X^n} \beta(x) P_{\lambda_n}(t, dx) = \int_{X^n} \beta(x) P_{\xi_n}(t, dx)$$

$$= E(\beta \circ \xi_n|\mathcal{F}_{n-1})(t) = E(-a_n|\mathcal{F}_{n-1})(t) \text{ a.e.}$$

So $E(a_n|\mathcal{F}_{n-1}) = 0$ a.e., $g$ is a martingale. Similarly, $h$ is a martingale. We denote the RCPD of $(d_n)_{n \geq 1}$ and $(-d_n)_{n \geq 1}$ w.r.t. $\mathcal{F}_{n-1}$ by $P_+, P_-$ respectively, then $P_+ = P_-$ a.e., similarly to Lemma 7, using separability of $X$. By this result

$$E(d_n|\mathcal{F}_{n-1})(t) = \int_X x P_+(t, dx) = \int_X x P_-(t, dx)$$

$$= E(-d_n|\mathcal{F}_{n-1})(t) \Rightarrow E(d_n|\mathcal{F}_{n-1}) = 0 \text{ a.e.},$$

so $f$ is a martingale.
Lemma 9. Let $X$ be a Hilbert space, then there exists a constant $C > 0$ dependent only on $\Phi$, such that for all $L^1(\mu, X)$ bounded martingale $f = (f_n)_{n \geq 1}$ satisfying $\|d_n\| \leq w_{n-1}$, where $d_n = f_n - f_{n-1}$, $w_n$ is $F_n$ measurable, we have

\begin{align*}
(1) \quad E\Phi(f^*) &\leq CE\Phi(S(f)) + CE\Phi(w^*) \\
(2) \quad E\Phi(S(f)) &\leq CE\Phi(f^*) + CE\Phi(w^*)
\end{align*}

Proof.

\[
S_n(f) = \left( \sum_{k=1}^{n-1} \|d_k\|^2 + \|d_n\|^2 \right)^{1/2} \leq \left( S_{n-1}(f)^2 + w_{n-1}^2 \right)^{1/2} \\
\leq S_{n-1}(f) + w_{n-1} = \varrho_{n-1}
\]

For $\beta > 0$, $\lambda > 0$, we define a stopping time

\[ S = \inf\{n \mid \varrho_n > \beta \lambda \}. \]

We consider martingale $f^{(S)} = (f_{n \wedge S})_{n \geq 0}$ and define a stopping time

\[ T = \inf\{n \mid \|f_{n}^{(S)}\| > \lambda \}. \]

Then $\forall \alpha > 1$, denoting $|A|$ the measure of $A$, we have

\begin{align*}
(3) \quad |\{f^* > \alpha \lambda\}| &\leq |\{f^{(S)} > \alpha \lambda, S = \infty\}| + |\{S < \infty\}| \\
&\leq |\{f^{(S)} > \alpha\}| + |\{S < \infty\}| \\
&\leq |\{f^{(S)} - f_{T-1}^{(S)} > (\alpha - 1) \lambda\}| + |\{S < \infty\}|
\end{align*}

Now we consider a new $\sigma$ algebra sequence $(\mathcal{F}'_n)_{n \geq 0}$, where $\mathcal{F}'_n = \mathcal{F}_{n+T}$.

We define $g' = (g'_n)_{n \geq 0}$, $g'_n = f_{n+T}^{(S)} - f_{T-1}^{(S)}$, then $g'$ is a martingale, because

\[
E(g'_{n+1}|\mathcal{F}'_n) = E(f_{n+T}^{(S)} - f_{T-1}^{(S)}|\mathcal{F}_{n+T}) \\
= E(f_{n+T+1}^{(S)}\wedge S - f_{T-1}^{(S)}\wedge S|\mathcal{F}_{n+T}) \\
= E(f_{n+T+1}^{(S)}\wedge S|\mathcal{F}_{n+T}) - f_{T-1}^{(S)}\wedge S
\]

and

\[
E(f_{n+T+1}^{(S)}\wedge S|\mathcal{F}_{n+T}) \\
= E(f_{S\wedge (S\leq n+T)}\chi_{S\leq n+T}\wedge S|\mathcal{F}_{n+T}) + E(f_{n+T+1}\chi_{S\geq n+T+1}|\mathcal{F}_{n+T}) \\
= f_{S\wedge (S\leq n+T)} + f_{n+T}\chi_{S\geq n+T+1} + E(f_{n+T+1}|\mathcal{F}_{n+T}) \\
= f_{S\wedge (S\leq n+T)} + f_{n+T}\chi_{S\geq n+T-1} = f_{(n+T)\wedge S}
\]
So \( E(g'_{n+1}|F'_n) = g_n \).

Because \( f^{(S)}_{T-1} = \sup_{m \geq 0} \| f^{(S)}_{m+T-1} \| \), if \( \exists m \leq T - 1 \), such that \( f^{(S)*} = \| f^{(S)}_m \| \), then \( f^{(S)*} = \| f^{(S)}_m \| \). If \( m > T - 1 \), \( f^{(S)*} > f^{(S)}_m \), we have

\[
\begin{align*}
    f^{(S)*} - f^{(S)}_{T-1} &\leq \sup_{m \geq T} \| f^{(S)}_m \| - \| f^{(S)}_{T-1} \| \leq \sup_{m \geq T} \| f^{(S)}_m \| - \| f^{(S)}_{T-1} \|.
    \end{align*}
\]

\[
S(g') = \left( \sum_{n \geq 0} \| f^{(S)}_{n+T+1} - f^{(S)}_{n+T} \|^2 \right)^{1/2}
\]

\[
= \left( \sum_{n \geq 0} \| f^{(S)}_{n+T+1} - f^{(S)}_{n+T} \|^2 \right)^{1/2} \chi_{\{T < \infty\}}
\]

\[
\leq S(f^{(S)}) \chi_{\{T < \infty\}} \leq c_{T-1} \chi_{\{T < \infty\}} \leq \beta \lambda \chi_{\{S < \infty\}}.
\]

|\{f^{(S)*} > \alpha \lambda\}| \leq |\{(g')^* > (\alpha - 1) \lambda\}|

\[
\leq E(g')^*/(\alpha - 1) \lambda \leq CES(g')/(\alpha - 1) \lambda
\]

(\text{using [5, p.414, Theorem 7])}

\[
\leq C \beta |\{T < \infty\}|/(\alpha - 1) \leq C \beta |\{f^{(S)*} > \lambda\}|/(\alpha - 1)
\]

\[
\leq C \beta |\{f^* > \lambda\}|/(\alpha - 1).
\]

So \( |\{f^* > \alpha \lambda\}| \leq \left( C \beta |\{f^* > \lambda\}|/(\alpha - 1) \right) + |\{S(f) + w^* > \beta \lambda\}| \) it means that \((f^*, S(f) + w^*)\) satisfies “good \( \lambda \) inequality”, so we have (1).

The proof of (2) is similar. With \( \| f_n \| \leq f^*_n + w_{n-1} = \varrho_{n-1} \), we define a stopping time

\[
S = \inf \{ n \mid \varrho_n > \beta \lambda \}, \quad \forall \beta > 0, \lambda > 0
\]

We consider martingale \( f^{(S)} = (f_{n+T})_{n \geq 0} \), and define a stopping time

\[
T = \inf \{ n \mid S_n(f^{(S)}) > \lambda \}
\]

then for \( \alpha > 1 \), we have

\[
\begin{align*}
|\{S(f) > \alpha \lambda\}| &\leq |\{S(f^{(S)}) > \alpha \lambda\}| + |\{S < \infty\}|
\leq |\{S(f^{(S)}) - S_{T-1}(f^{(S)}) > (\alpha - 1) \lambda\}| + |\{S < \infty\}|
\end{align*}
\]
Because
\[
S(f^{(S)}) - S_{T-1}(f^{(S)}) \leq \left( \sum_{n \geq T} \| f_n^{(S)} - f_{n-1}^{(S)} \|^2 \right)^{1/2} \\
= (S(f^{(S)})^2 - S_{T-1}(f^{(S)})^2)^{1/2}, \\
\sup_{m \geq T} \| f_m^{(S)} - f_{T-1}^{(S)} \| \leq 2 f^{(S)}_* \chi_{\{T < \infty\}} \\
\leq 2\beta\lambda \chi_{\{T < \infty\}}.
\]
So
\[
\left| \{ S(f^{(S)}) > \alpha \lambda \} \right| \leq \left| \{ S(f^{(S)}) - S_{T-1}(f^{(S)}) > (\alpha - 1)\lambda \} \right| \\
\leq \left| \left( (S(f^{(S)})^2 - S_{T-1}(f^{(S)})^2)^{1/2} > (\alpha - 1)\lambda \right) \right| \\
\leq E (S(f^{(S)})^2 - S_{T-1}(f^{(S)})^2)^{1/2}/(\alpha - 1)\lambda \\
= E \left( \sum_{n \geq T} \| f_n^{(S)} - f_{n-1}^{(S)} \|^2 \right)^{1/2} / (\alpha - 1)\lambda \\
\leq CE \left( \sup_{m \geq T} \| f_m^{(S)} - f_{T-1}^{(S)} \| \right) / (\alpha - 1)\lambda
\]
(using [5, p.411, Theorem 4])
\[
\leq 2\beta C \{ T < \infty \} / (\alpha - 1) \\
= 2\beta C \left| \{ S(f^{(S)}) > \lambda \} \right| / (\alpha - 1) \\
\leq 2\beta C \left| \{ S(f) > \lambda \} \right| / (\alpha - 1).
\]
So
\[
\left| \{ S(f) > \alpha \lambda \} \right| \leq 2\beta C \left| \{ S(f) > \lambda \} \right| / (\alpha - 1) + \left| \{ f^* + w^* > \beta \lambda \} \right|,
\]
it means \((S(f), f^* + w^*)\) satisfies "good \(\lambda\) inequality", (2) holds.

**Theorem 10.** Let \(X\) be a Hilbert space, then there exists a constant \(C\) dependent only on \(\Phi\), such that for all integrable conditionally symmetric \(X\)-valued sequence \((d_n)_{n \geq 1}\) with respect to \((\mathcal{F}_n)_{n \geq 0}\), denoting \(f_n = \sum_{1 \leq k \leq n} d_k\), we have
\[
C^{-1} E\Phi(S(f)) \leq E\Phi(f^*) \leq CE\Phi(S(f)).
\]
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Proof. Let

\[ g_n = \sum_{1 \leq k \leq n} a_k = \sum_{1 \leq k \leq n} d_k \chi_{A_k}, \quad A_k = \{ \|d_k\| \leq 2d_{k-1}^* \} \]

\[ h_n = \sum_{1 \leq k \leq n} b_k = \sum_{1 \leq k \leq n} d_k \chi_{B_k}, \quad B_k = \{ \|d_k\| > 2d_{k-1}^* \} \]

then

\[ f^* \leq g^* + h^* \leq g^* + \sum_{n \geq 1} \|b_n\| \]

\[ S(g) \leq S(f) + S(h) \leq S(f) + \sum_{n \geq 1} \|b_n\| \]

\[ E\Phi(f^*) \leq CE\Phi(g^*) + CE\Phi\left(\sum_{n \geq 1} \|b_n\|\right) \]

By Lemma 9, we have

\[ E\Phi(g^*) \leq CE\Phi S(g) + CE\Phi(2d^*) \]

\[ \leq CE\Phi (S(f)) + CE\Phi\left(\sum_{n \geq 1} \|b_n\|\right) + CE\Phi(d^*). \]

\[ d^* \leq S(f) \to \sum_{n \geq 1} \|b_n\| \leq 2d^* \leq 2S(f) \]

So \( E\Phi(f^*) \leq CE\Phi(S(f)) \). The proof of the other inequality is similar, using \( S(f) \leq S(g) + \sum_{n \geq 1} \|b_n\|, \ g^* \leq f^* + \sum_{n \geq 1} \|b_n\| \).

Remark 11. Let \( \Phi(x) = x^p, \ 0 < p < +\infty \), we can obtain (11.2) in [2] and (1.6), (1.7) of Theorem 1 in [3] with different constants by inequality \( \|f\|_p \leq \|f^*\|_p \), but the constants in [3] are the best possible.

Lemma 12. There exists a constant \( C \) depending only on \( \Phi \) such that for all nonnegative \( \mathbb{R} \)-valued tangent sequences \((d_n)_{n \geq 1}, (e_n)_{n \geq 1}\), we have

\[ E\Phi\left(\sum_{n \geq 1} d_n\right) \leq CE\Phi\left(\sum_{n \geq 1} e_n\right) \]

Proof. see [4, Theorem 2].
Corollary 13. Let $X$ be a Hilbert space, then there exists a constant $C$ depending only on $\Phi$ such that for all $X$-valued conditionally symmetric sequences $(d_n)_{n \geq 1}, (e_n)_{n \geq 1}, d_n, e_n \in L_1(\mu, X)$, when $(\|d_n\|)_{n \geq 1}$ and $(\|e_n\|)_{n \geq 1}$ are tangent, we have

$$E\Phi(f^*) \leq C E\Phi(g^*)$$

where $f_n = \sum_{1 \leq k \leq n} d_k, g_n = \sum_{1 \leq k \leq n} e_k$.

Proof. From Lemma 12,

$$E\Phi(f^*) \leq C E\Phi(S(f)),$$
$$E\Phi(S(g)) \leq C E\Phi(g^*)$$

and we take $\Phi(t^{1/2})$ in stead of $\Phi(t)$ and take tangent sequences $(\|d_n\|^2)_{n \geq 1}$, $(\|e_n\|^2)_{n \geq 1}$, and get

$$E\Phi(S(f)) \leq E\Phi(S(g))$$

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Yu He
Department of Mathematics
University of Science and Technology of China
HILBERT-SPACE-VALUED TANGENT SEQUENCES

PeiDe Liu
WuHan University
WuHan, HuBei 430072, P.R.China

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Current Address:
Yu He
Department of Mathematics, ZhongShan University,
GuangZhou, GuangDong 510275, P.R.China