Note on Schreier Semigroup Rings

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Let $D$ be an integral domain with the quotient field $q(D)$. Let $c$ be an element of $D$. Assume that, if $c$ is a divisor of $a_1a_2$ (for $a_1, a_2 \in D$), then $c$ is a product of a divisor of $a_1$ and a divisor of $a_2$. Then $c$ is called a primal element of $D$. If each divisor of $c$ is a primal element of $D$, then $c$ is called a completely primal element of $D$. $D$ is called a Schreier ring if $D$ is an integrally closed ring in which every element is primal ([2]). Let $S$ be a semigroup $\cong 0$ of a torsion-free abelian (additive) group. Then $S$ is called a grading monoid (or a $g$-monoid) ([6]). Let $D[X; S] = \{ \sum_{finite} a_s X^s \mid a_s \in D, s \in S \}$ be the semigroup ring of $S$ over $D$ ($X$ is a symbol). [3] is a general reference on $D[X; S]$. For various ring-theoretic properties $\Pi$, conditions for $D[X; S]$ to have the property $\Pi$ have been obtained (cf. [1, 3, 5]). The aim of this note is to obtain conditions for $D[X; S]$ to be a Schreier ring.

**Lemma 1** ([2]). Let $D$ be an integrally closed domain, and let $T$ be a multiplicative system of $D$ generated by completely primal elements. If the quotient ring $DT$ is a Schreier ring, then $D$ is a Schreier ring.

For elements $s, t$ of a $g$-monoid $S$, if $t = s + s'$ for some $s' \in S$, then $s$ is called a divisor of $t$. For elements $s, t_1, \cdots, t_n$ of $S$, if $s$ is a divisor of $t_1, \cdots, t_n$, then $s$ is called a common divisor of $t_1, \cdots, t_n$. The group $\{ s_1 - s_2 \mid s_1, s_2 \in S \}$ is called the quotient group of $S$, and is denoted by $q(S)$. We note that $q(S)$ is a totally ordered abelian group ([3, COROLLARY 3.4]). An element $x$ of $q(S)$ is called integral over $S$, if $nx \in S$ for some $n \in \mathbb{N}$. If every integral element of $q(S)$ is contained in $S$, then $S$ is called an integrally closed semigroup. Let $G$ be a torsion-free abelian (additive) group, and $\Gamma$ a totally ordered abelian group. A homomorphism $v$ of $G$ to $\Gamma$ is called a valuation on $G$. The semigroup $\{ x \in G \mid v(s) \geq 0 \}$ is called the valuation semigroup of $G$ associated with $v$. A valuation semigroup of $q(S)$ which contains $S$ is called a valuation oversemigroup of $S$. Let $c$ be an element of $S$. Assume that, if $c$ is a divisor of $a_1 + a_2$ (for $a_1, a_2 \in S$), then $c$ is a sum of a divisor of $a_1$ and a divisor of $a_2$. Then $c$ is called a primal element of $S$. $S$ is called a Schreier semigroup if $S$ is an integrally closed semigroup in which every element is primal. We
consider the following condition:

(*) For every finite subsets \(\{s_1, \ldots, s_n\}, \{t_1, \ldots, t_m\}\) of \(S\) and an element \(s\) of \(S\), if \(s\) is a common divisor of \(s_1 + t_1, s_1 + t_2, \ldots, s_i + t_j, \ldots, s_n + t_m\), then \(s\) is a sum of a common divisor of \(s_1, \ldots, s_n\), and a common divisor of \(t_1, \ldots, t_m\).

If \(S\) is integrally closed and satisfies the condition \((*)\), then \(S\) is a Schreier semigroup.

**Lemma 2** ([3, THEOREM 12.8]). \(S\) is integrally closed if and only if \(S\) is the intersection of all the valuation oversemigroups of \(S\).

Let \(v\) be a valuation on \(q(S)\). Let \(f = \sum a_i X^{s_i}\) be an element of \(D[X; S]\), where each \(a_i \neq 0\) and \(s_i \neq s_j\) for \(i \neq j\). We set \(v^*(f) = \inf v(s_i)\).

**Lemma 3.** (1) ([3, THEOREM 15.7]) \(v^*\) naturally induces a valuation on \(q(D[X; S])\).

(2) ([3, COROLLARY 12.11]) \(D[X; S]\) is integrally closed if and only if \(D\) is integrally closed and \(S\) is integrally closed.

**Lemma 4.** (1) ([4, (4.5)PROPOSITION]) Let \(G\) be a torsion-free abelian group. Then \(D[X; G]\) is a Schreier ring if and only if \(D\) is a Schreier ring.

(2) ([4, (4.6)PROPOSITION]) \(D[X; S]\) is a Schreier ring if and only if \(D\) and \(K[X; S]\) are Schreier rings and \(S\) is a Schreier semigroup, where \(K = q(D)\).

**Lemma 5.** Let \(k\) be a field. If \(k[X; S]\) is a Schreier ring, then \(S\) satisfies the condition \((*)\).

**Proof.** Let \(s, s_1, \ldots, s_n, t_1, \ldots, t_m\) be a finite number of elements of \(S\) such that \(s\) is a common divisor of \(s_1 + t_1, s_1 + t_2, \ldots, s_i + t_j, \ldots, s_n + t_m\). Set \(f = X^{s_1} + \ldots + X^{s_n}\) and \(g = X^{t_1} + \ldots + X^{t_m}\). Then \(X^s\) is a divisor of \(fg\) in \(k[X; S]\). Hence there exist a divisor \(f_1\) of \(f\) and a divisor \(g_1\) of \(g\) such that \(X^{s'} = f_1 g_1\). Noting that \(S\) is a subsemigroup of a totally ordered abelian group \(q(S)\), we may assume that \(f_1 = X^a\) and \(g_1 = X^b\) for \(a, b \in S\). It follows that \(a\) is a common divisor of \(s_1, \ldots, s_n\), and \(b\) is a common divisor of \(t_1, \ldots, t_m\), and \(a + b = s\). Therefore \(S\) satisfies the condition \((*)\).
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Lemma 6. Let $k$ be a field, and let $S$ be an integrally closed semigroup which satisfies the condition $(\ast)$. Then $k[X; S]$ is a Schreier ring.

Proof. By Lemma 3(2), $k[X; S]$ is integrally closed. Let $s \in S$. We will show that $X^s$ is a primal element of $k[X; S]$. Thus let $f, g$ be non-zero elements of $k[X; S]$ such that $fg = X^s h$ for some $h \in k[X; S]$. Set $f = \sum_1^n a_i X^{s_i}, g = \sum_1^m b_i X^{t_i}$, where each $a_i$ and $b_j$ are non-zero elements of $k$, $s_i \neq s_j$ for $i \neq j$, and $t_k \neq t_l$ for $k \neq l$. Let $1 \leq k \leq n$, and $1 \leq l \leq m$. Let $V$ be a valuation oversemigroup of $S$, and let $v$ be the valuation on $q(S)$ associated with $V$. Then we have

$$v(s_k) + v(t_l) \geq v^*(f) + v^*(g) = v^*(X^s h) = v(s) + v^*(h).$$

It follows that $v(s_k + t_l - s) \geq 0$, and hence $s_k + t_l - s \in V$. Since $V$ is arbitrary, $s_k + t_l - s \in S$ by Lemma 2. Hence $s$ is a divisor of $s_k + t_l$. Since $k$ and $l$ are arbitrary, $s$ is a common divisor of $s_1 + t_1, s_1 + t_2, \cdots, s_n + t_m$. Hence there exist a common divisor $a$ of $s_1, \cdots, s_n$, and a common divisor $b$ of $t_1, \cdots, t_m$ such that $s = a + b$. Then $X^a$ is a divisor of $f$, $X^b$ is a divisor of $g$, and $X^a X^b = X^s$. Therefore $X^s$ is a primal element of $k[X; S]$. Since $s$ is arbitrary, we see that, for every element $s$ of $S$, $X^s$ is a completely primal element of $k[X; S]$. $T = \{X^s \mid s \in S\}$ is a multiplicative system of $k[X; S]$ generated by completely primal elements, and we have $k[X; S]_T = k[X; G]$, where $G = q(S)$. By Lemma 4(1), $k[X; G]$ is a Schreier ring. By Lemma 1, $k[X; S]$ is a Schreier ring.

Lemmas 5 and 6 imply the following,

Proposition 7. Let $k$ be a field. Then $k[X; S]$ is a Schreier ring if and only if $S$ is an integrally closed semigroup which satisfies the condition $(\ast)$.

Lemma 4 (2) and Proposition 7 imply the following,

Theorem 8. $D[X; S]$ is a Schreier ring if and only if $D$ is a Schreier ring, $S$ is an integrally closed semigroup which satisfies the condition $(\ast)$.

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