On Rings of Certain Type Associated With Simple Ring-extensions

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ON RINGS OF CERTAIN TYPE ASSOCIATED WITH SIMPLE RING-EXTENSIONS

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Let \( R \) be a Noetherian domain (which is commutative and has a unit), let \( R[X] \) be a polynomial ring, let \( \alpha \) be an element of an algebraic extension field of the quotient field \( K \) of \( R \) and let \( \pi: R[X] \to R[\alpha] \) be the \( R \)-algebra homomorphism sending \( X \) to \( \alpha \). Let \( \varphi_\alpha(X) \) be the monic minimal polynomial of \( \alpha \) over \( K \) with \( \deg \varphi_\alpha(X) = d \) and write

\[
\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d.
\]

Then \( \eta_i \in K \) \((1 \leq i \leq d)\) are uniquely determined by \( \alpha \). Put \( d = [K(\alpha): K] \), \( I_{\eta_i} := R : R \eta_i \) and \( I_{[\alpha]} := \bigcap_{i=1}^d I_{\eta_i} \). If \( \text{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X] \), we say that \( \alpha \) is anti-integral over \( R \) (cf. [3]). Put \( J_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_d) \). Then \( J_{[\alpha]} = c(I_{[\alpha]}\varphi_\alpha(X)) \), where \( c(\ ) \) denotes the ideal generated by the coefficients of the polynomials in \( (\ ) \), that is, the content ideal of \( (\ ) \).

If \( J_{[\alpha]} \not\subseteq p \) for all \( p \in \text{Dp}_1(R) := \{ p \in \text{Spec}(R) \mid \text{depth} R_p = 1 \} \), the element \( \alpha \) is called a super-primitive element over \( R \). A super-primitive element over \( R \) is anti-integral over \( R \) (cf. [3, Theorem 1.12]). It is also known that any algebraic element over a Krull domain \( R \) is super-primitive over \( R \) (cf. [3, Theorem 1.13]), and hence \( \alpha \) is anti-integral over \( R \). We also note here that \( I_{[\alpha]} = R \iff R[\alpha] \) is integral over \( R \) and that \( J_{[\alpha]} = R \iff R[\alpha] \) is flat over \( R \), provided that \( \alpha \) is anti-integral over \( R \).

Assume that \( \alpha \) is an anti-integral element of degree \( d \geq 2 \) over \( R \). We have seen in [1] that \( R[\alpha] \cap R[\alpha^{-1}] = R \oplus I_{[\alpha]} \zeta_1 \oplus \cdots \oplus I_{[\alpha]} \zeta_{d-1} \) as an \( R \)-module, where \( \zeta_i := \alpha^i + \eta_1 \alpha^{i-1} + \cdots + \eta_d (1 \leq i \leq d) \). Note that \( \zeta_d = \varphi_\alpha(\alpha) = 0 \). Put \( \zeta_0 := 1 \) and \( \eta_0 := 1 \) for convenience. It is easy to see that \( \alpha \zeta_i = \zeta_{i+1} - \eta_{i+1} (0 \leq i \leq d-1) \) by definition. Note that \( R[\alpha] \cap R[\alpha^{-1}] \) and \( R[\alpha] \) are birational and their quotient fields are equal to \( K(\alpha) \).

Let \( H \) be an ideal of \( R \) and put \( C_H := R + H \zeta_1 + \cdots + H \zeta_{d-1} \), which is an \( R \)-submodule in \( K(\alpha) = K \oplus K \zeta_1 \oplus \cdots \oplus K \zeta_{d-1} \). Our objective of this paper is to investigate when \( C_H \) forms a subring of \( K(\alpha) \) \((R\text{-algebra)}\).

Throughout this paper, we use the above notation and conventions unless otherwise specified. Our general reference for unexplained terminology is [2].

We start with the following lemma, which is obviously seen.

**Lemma 1.** Assume that \( \alpha \) is an anti-integral element of degree \( d \)

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over $R$. Let $H$ denote an ideal of $R$. Then $C_H$ forms a subring of $K(\alpha)$ if and only if $H^2\zeta_i\zeta_j \subseteq C_H$ for all $i, j \in \{1, 2, \ldots, d - 1\}$ with $j \leq i$.

**Proposition 2.** Let $H$ denote an ideal of $R$. Assume that $\alpha$ is an anti-integral element over $R$ of degree $d = 2$. Then $C_H$ forms a subring of $K(\alpha)$ if and only if $H^2\eta_1 \subseteq H$ and $H^2\eta_2 \subseteq R$.

**Proof.** Note $\zeta_1 = \alpha + \eta_1$ and hence $\zeta_1^2 = \zeta_1(\alpha + \eta_1) = \alpha\zeta_1 + \eta_1\zeta_1 = \zeta_2 - \eta_2 + \eta_1\zeta_1 = \eta_1\zeta_1 - \eta_2$ (here $\zeta_2 = 0$). Assume that $C_H$ is a subring of $K(\alpha)$. Then $H^2\zeta_1^2 \in C_H$ yields that for any $h \in H$, $h^2\eta_1\zeta_1 - h^2\eta_2 \in R + H\zeta_1 \subseteq K \oplus K\zeta_1$. Consequently, $H^2\eta_1 \subseteq H$ and $H^2\eta_2 \subseteq R$. Conversely, take $a + h_1\zeta_1, b + h_2\zeta_1 \in C_H$ with $h_1, h_2 \in H$ and $a, b \in R$. Then $(a + h_1\zeta_1)(b + h_2\zeta_1) = ab + (ah_2 + bh_1)\zeta_1 + h_1h_2\zeta_1^2 \in R + H\zeta_1$ because $h_1h_2\zeta_1^2 \in H^2\zeta_1^2 = H^2(\eta_1\zeta_1 - \eta_2) \subseteq R + H\zeta_1 = C_H$.

We consider the case $d = 3$, and compute $\zeta_i\zeta_j$ by a definite calculation.

**Example 3.** Assume that $\alpha$ is an anti-integral element of degree $d = 3$ over $R$. Note $\zeta_1 = \alpha + \eta_1$, $\zeta_2 = \alpha\zeta_1 + \eta_2$ and $\zeta_3 = \alpha\zeta_2 + \eta_3$. Thus

\[
\zeta_1^2 = \zeta_1(\alpha + \eta_1) = \alpha\zeta_1 + \eta_1\zeta_1 = \zeta_2 - \eta_2 + \eta_1\zeta_1,
\]

\[
\zeta_2\zeta_1 = \zeta_2(\alpha + \eta_1) = \alpha\zeta_2 + \zeta_2\eta_1 = \zeta_3 - \eta_3 + \eta_1\zeta_2 = \zeta_3 + \eta_1\zeta_2
\]

and

\[
\zeta_2^2 = \zeta_2(\alpha\zeta_1 + \eta_2) = (\alpha\zeta_2)\zeta_1 + \zeta_2\eta_2 = (\zeta_3 - \eta_3)\zeta_1 + \zeta_2\eta_2 = \zeta_3\zeta_1 + \eta_2\zeta_2
\]

because $\zeta_3 = 0$.

Assume that $C_H$ is a subring of $K(\alpha)$. Then we have $H^2\eta_1 \subseteq H$, $H^2\eta_2 \subseteq H$ and $H^2\eta_3 \subseteq H$ since $\zeta_1^2, \zeta_2\zeta_1$ and $\zeta_2^2$ are in $C_H$.

Conversely, if $H^2\eta_1 \subseteq H$, $H^2\eta_2 \subseteq H$ and $H^2\eta_3 \subseteq H$, then $\zeta_1^2, \zeta_2\zeta_1$ and $\zeta_2^2$ are in $C_H$. So we conclude that $C_H$ is a subring of $k(\alpha)$ by Lemma 1.

**Lemma 4.** Assume that $\alpha$ is an anti-integral element of degree $d(\geq 3)$ over $R$ and that $1 \leq j \leq i < d$.

(i) If $i + j < d$, then

\[
(\ast) \quad \zeta_i\zeta_j = \zeta_{i+j} - \sum_{t=1}^{j} \eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{j-1} \eta_{j-s}\zeta_{i+s}.\]

(ii) If $i + j \geq d$, then

\[
(\ast\ast) \quad \zeta_i\zeta_j = -\sum_{t=1}^{d-i} \eta_{i+t}\zeta_{j-t} + \sum_{s=0}^{d-i-1} \eta_{j-s}\zeta_{i+s}.\]

(Note that $d - i > 0$ and $d - i - 1 \geq 0$ because $1 \leq j \leq i < d$.)
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Proof. Note first that \( \eta_0 := 1, \zeta_0 := 1, \zeta_d = 0 \) and \( \zeta_{i+1} = \alpha \zeta_i + \eta_{i+1} \). Now we compute \( \zeta_i \zeta_j \) as follows:

\[
\begin{align*}
\zeta_i \zeta_j &= \zeta_i (\alpha \zeta_{j-1} + \eta_j) \\
&= \alpha \zeta_i \zeta_{j-1} + \eta_j \zeta_i \\
&= (\zeta_{i+1} - \eta_{i+1}) \zeta_{j-1} + \eta_j \zeta_i \\
&= \zeta_{i+1} \zeta_{j-1} - \eta_{i+1} \zeta_{j-1} + \eta_j \zeta_i \\
&= \zeta_{i+2} \zeta_{j-2} - \eta_{i+2} \zeta_{j-2} + \eta_{j-1} \zeta_{i+1} - \eta_{i+1} \zeta_{j-1} + \eta_j \zeta_i \\
&= \zeta_{i+2} \zeta_{j-2} - (\eta_{i+2} \zeta_{j-2} + \eta_{i+1} \zeta_{j-1}) + (\eta_j \zeta_i + \eta_{j-1} \zeta_{i+1})
\end{align*}
\]

\[\cdots\]

\[\cdots\]

(i) Repeat the above process, we have

\[
\begin{align*}
\zeta_i \zeta_j &= \zeta_{i+j} \zeta_0 - \sum_{t=1}^{j} \eta_{i+t} \zeta_{j-t} + \sum_{s=0}^{j-1} \eta_{j-s} \zeta_{i+s},
\end{align*}
\]

that is,

\[
\zeta_i \zeta_j = \zeta_{i+j} - \sum_{t=1}^{j} \eta_{i+t} \zeta_{j-t} + \sum_{s=0}^{j-1} \eta_{j-s} \zeta_{i+s}.
\]

(ii) Put \( \ell := i + j - d \). Then \( j \leq i < d \) yields \( \ell < d - 1 \) and \( j - \ell \geq 1 \). Thus continuing the above process yields

\[
\begin{align*}
\zeta_i \zeta_j &= \zeta_{d} \zeta_{\ell} - \sum_{t=1}^{j-\ell} \eta_{i+t} \zeta_{j-t} + \sum_{s=0}^{j-\ell-1} \eta_{j-s} \zeta_{i+s},
\end{align*}
\]

that is,

\[
\begin{align*}
(\ast\ast) \quad \zeta_i \zeta_j &= - \sum_{t=1}^{d-i} \eta_{i+t} \zeta_{j-t} + \sum_{s=0}^{d-i-1} \eta_{j-s} \zeta_{i+s}.
\end{align*}
\]

Remark 5. Assume that \( \alpha \) is an anti-integral element of degree \( d(\geq 3) \) over \( R \) and that \( 1 \leq j \leq i < d \). For \((i,j)\) with \( i + j < d \), let \( \Delta_1^{(j)} := \{ j - t \mid t = 1, \ldots, j \} \) and \( \Delta_1^{(i,j)} := \{ i + s \mid s = 0, \ldots, j - 1 \} \). For \((i,j)\) with \( i + j \geq d \), let \( \Delta_2^{(j,i)} := \{ j - t \mid t = 1, \ldots, d - i \} \) and \( \Delta_2^{(i)} := \{ i + s \mid s = 0, \ldots, d - i - 1 \} \).
(i) If \( i + j < d \), then \( \Delta^{(i,j)}_1 \cap \Delta^{(j)}_1 = \emptyset \) because \( j - t < i + s \), and
\[
\bigcup_{i+j<d}(\Delta^{(i,j)}_1 \cup \Delta^{(j)}_1) = \{0, 1, 2, \ldots, d-1\}.
\]
(ii) Assume \( i + j \geq d \). Then \( \Delta^{(i)}_2 \cap \Delta^{(j,i)}_2 = \emptyset \) because \( j - t < i + s \).
Put \( e_k := -1 \) if \( k \in \Delta^{(j,i)}_2 \) and \( e_k := 1 \) if \( k \in \Delta^{(i)}_2 \). Then
\[
(**) \quad \zeta_i \zeta_j = \sum_{k \in \Delta^{(j,i)}_2 \cup \Delta^{(i)}_2} e_k \eta_{i+j-k} \zeta_k.
\]
Consider \( \zeta^2_{d-1} \). Then we have
\[
\zeta^2_{d-1} = \zeta_{d-1}(\alpha \zeta_{d-2} + \eta_{d-1})
= \alpha \zeta_{d-1} \zeta_{d-2} + \eta_{d-1} \zeta_{d-1}
= (\zeta_d - \eta_d) \zeta_{d-2} + \eta_{d-1} \zeta_{d-1}
= -\eta_d \zeta_{d-2} + \eta_{d-1} \zeta_{d-1}.
\]
Hence if \( \zeta^2_{d-1} \in \mathcal{C}_H \), then \( H^2 \eta_d \in H \).

**Theorem 6.** Assume that \( \alpha \) is an anti-integral element of degree \( d(\geq 3) \) over \( R \). Let \( H \) be an ideal of \( R \). Then \( \mathcal{C}_H := R + H \zeta_1 + \cdots + H \zeta_{d-1} \) is a subring of \( K(\alpha) \) if and only if \( H^2 \eta_i \subseteq H \) for all \( i \) (\( 1 \leq i \leq d \)).

**Proof.** Our conclusion follows Lemmas 1 and 4 and Remark 5.

**Corollary 7.** Assume that \( \alpha \) is an anti-integral element of degree \( d(\geq 3) \) over \( R \). Let \( H \) be an ideal of \( R \). If \( \mathcal{C}_H \) is a subring of \( K(\alpha) \), then every element in \( H \eta_i \) is integral over \( R \) for all \( i \) (\( 1 \leq i \leq d \)).

**Proof.** We have \( H^2 \eta_i \subseteq H \) for all \( i \) (\( 1 \leq i \leq d \)) by Theorem 6. So \( H \eta_i \subseteq H :_KH \). Since \( H \) is a finitely generated ideal, any element in \( H :_KH \) is integral over \( R \). Thus every element of \( H \eta_i \) is integral over \( R \) for all \( i \) (\( 1 \leq i \leq d \)).

**Corollary 8.** Assume that \( R \) is normal and that \( \alpha \) is of degree \( d(\geq 3) \) over \( R \). Let \( H \) be an ideal of \( R \). If \( \mathcal{C}_H \) is a subring of \( K(\alpha) \), then \( H \subseteq I_{[\alpha]} \) and hence \( \mathcal{C}_H \subseteq R[\alpha] \cap R[\alpha^{-1}] \).

**Proof.** Since \( R \) is normal, the element \( \alpha \) is anti-integral over \( R \) by [3]. We have \( H \eta_i \subseteq H :_KH = R \) for all \( i \) (\( 1 \leq i \leq d \)), that is, \( H \subseteq I_{[\alpha]} \).
Theorem 9. Let $H$ be an ideal of $R$. Assume that $\alpha$ is an anti-integral element of degree $d(\geq 2)$ over $R$.

1. If $C_H$ is a subring of $K(\alpha)$, then $H^2 \subseteq I_{[\alpha]}$.
2. Assume that $\operatorname{grade}(H) > 1$. Then $C_H$ is a subring of $K(\alpha)$ if and only if $\eta_i \in R$ for all $i$ if and only if $\alpha$ is integral over $R$.
3. If $I_{[\alpha]} \neq R$ and if $C_H$ is a subring of $K(\alpha)$, then $\operatorname{grade}(H) \leq 1$.
4. When $H$ is an invertible ideal and $d \geq 3$, $H \subseteq I_{[\alpha]}$ if and only if $C_H$ is a subring of $K(\alpha)$.

Proof. (1) Assume that $d \geq 3$. Since $C_H$ is a ring, we have $H^2 \eta_i \subseteq H$ for all $(1 \leq i \leq d)$ by Theorem 6. Hence $H^2 \eta_i \subseteq H \subseteq R$, which means that $H^2 \subseteq I_{\eta_i}$ for all $(1 \leq i \leq d)$. Thus $H^2 \subseteq I_{[\alpha]}$. Next assume that $d = 2$. Then Proposition 2 yields our conclusion.

(2) Assume that $C_H$ is a subring of $K(\alpha)$. Then $H^2 \subseteq I_{[\alpha]}$ by (1). Hence $1 < \operatorname{grade}(H) \leq \operatorname{grade}(I_{[\alpha]})$, which means that $I_{[\alpha]} = R$ because $I_{[\alpha]}$ is a divisorial ideal of $R$. So we have $I_{\eta_i} = R$ for all $i$ $(1 \leq i \leq d)$, that is, $\eta_i \in R$ for all $i$ $(1 \leq i \leq d)$. Furthermore $\alpha$ is integral over $R$ by [3]. Conversely, assume that $\alpha$ is integral over $R$. Then $I_{[\alpha]} = R$ (cf. [3]), and hence $\eta_i \in R$ for all $(1 \leq i \leq d)$. So if $d \geq 3$, $H^2 \eta_i \subseteq H$ yields that $C_H$ is a ring by Theorem 6. If $d = 2$, $C_H$ is a ring by Proposition 2.

(3) Our conclusion follows the result (2).

(4) Since $d \geq 3$, $C_H$ is a subring if and only if $H^2 \eta_i \subseteq H$ for all $i$ $(1 \leq i \leq d)$. Since $H$ is invertible, $H^2 \eta_i \subseteq H$ for all $i$ $(1 \leq i \leq d)$ if and only if $H \eta_i \subseteq R$ for all $i$ $(1 \leq i \leq d)$ if and only if $H \subseteq I_{[\alpha]}$.

Theorem 10. Assume that $\alpha$ is an anti-integral element of degree $d(\geq 2)$ over $R$. Let $H$ be an ideal of $R$. If $H \subseteq I_{[\alpha]}$, then $C_H$ is a subring of $K(\alpha)$.

Proof. The inclusion $H \subseteq I_{[\alpha]}$ yields that $H^2 \eta_i \subseteq H(\subseteq R)$ for all $i$ $(1 \leq i \leq d)$. Hence $C_H$ is a ring by Proposition 2 and Theorem 6.

Corollary 11. Assume that $R$ is a locally factorial domain and that $\alpha$ is an element of degree $d(\geq 2)$. The collection $\Delta := \{C_H \mid H$ is an ideal of $R$ and $C_H$ is a subring of $K(\alpha)\}$ has the maximum member $C_{I_{[\alpha]}}(= R[\alpha] \cap R[\alpha^{-1}])$. 

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Proof. Note first that $\alpha$ is anti-integral over $R$ because $R$ is a Krull domain (cf. [3]). Take $C_H \in \Delta$. Then $H^2 \subseteq I_{[\alpha]}$ by Theorem 9(1). Since $I_{[\alpha]}$ is a divisorial ideal, we have grade($H$) = grade($H^2$) = 1. Since $R$ is locally factorial, every non-zero ideal of grade one is invertible. So $H$ is invertible. Hence $H \subseteq I_{[\alpha]}$ by Theorem 9(4).

Finally, we close this paper by the following result concerned with the ring $R[\alpha] \cap R[\alpha^{-1}]$.

**Proposition 12.** Assume that $\alpha$ is an anti-integral element of degree $d$. If an element $a$ in $R$ is a non-zero-divisor on $R/I_{[\alpha]}$, then $R[\alpha a] \cap R[\alpha^{-1}] = R + I_{[\alpha]}(a \zeta_1) + \cdots + I_{[\alpha]}(a^{d-1} \zeta_{d-1})$.

Proof. We have only to show the inclusion ($\subseteq$) because $a^d I_{[\alpha]} \zeta_i \subseteq R[\alpha a] \cap R[\alpha^{-1}]$. Take an element $\beta \in R[\alpha a] \cap R[\alpha^{-1}]$ and write $\beta = x_n(a\alpha)^n + \cdots + x_1(a\alpha) + x_0 = y_0 + y_1(\alpha^{-1}) + \cdots + y_m(\alpha^{-1})m$ with $x_i, y_j \in R$. Then we have $a^m \beta = x_n a^n \alpha^{n+m} + \cdots + x_0 \alpha^m = y_0 \alpha^m + y_1 \alpha^{m-1} + \cdots + y_m$. Put $f(X) := x_n a^n X^{n+m} + \cdots + x_0 X^m - (y_0 X^m + y_1 X^{m-1} + \cdots + y_m) \in \text{Ker}(\pi)$, where $\pi: R[X] \to R[\alpha]$ denotes the canonical $R$-homomorphism. Since $\alpha$ is an anti-integral element over $R$, $\text{Ker}(\pi) = I_{[\alpha]} \varphi_{\alpha}(X) R[X]$. Hence $x_n a^n \in I_{[\alpha]}$. Since $\alpha$ is a non-zero-divisor of $R/I_{[\alpha]}$, we have $x_n \in I_{[\alpha]}$. Put $\beta_\alpha(X) := x_n a^n \alpha X^n + \cdots + x_1 a \alpha + x_0$. Since $x_n \varphi_{\alpha}(X) \in R[X]$, considering $\text{deg}(\beta_\alpha(X) - x_n \varphi_{\alpha}(X) X^{n-d}) < n$ if $n \geq d$, we can use the induction on $n$ and assume that $n < d$. So considering $\beta - x_n a^n \zeta_n$ instead of $\beta$, we can conclude that $\beta \in R + I_{[\alpha]}(a \zeta_1) + \cdots + I_{[\alpha]}(a^{d-1} \zeta_{d-1})$ by induction on $n$.

**Proposition 13.** Assume that $\alpha$ is an anti-integral element of degree $d$ over $R$. If an element $a$ in $R$ is a non-zero-divisor on $R/I_{[\alpha^{-1}]}$, then $R[\alpha^{-1} a] \cap R[\alpha^{-1}] = R[\alpha] \cap R[\alpha^{-1}]$.

Proof. We have only to show the inclusion ($\subseteq$). Take an element $\beta \in R[\alpha^{-1}] \cap R[\alpha^{-1}]$ and write $\beta = x_n(a^{-1}\alpha)^n + \cdots + x_1(a^{-1}\alpha) + x_0 = y_0 + y_1(\alpha^{-1}) + \cdots + y_m(\alpha^{-1})m$ with $x_i, y_j \in R$. Then $a^n \beta = x_n a^n \alpha + \cdots + x_0 a^n = a^n y_0 + a^n y_1(\alpha^{-1}) + \cdots + a^n y_m(\alpha^{-1})m$. Since $\alpha$ is anti-integral over $R$, so is $\alpha^{-1}$ (cf. [1]). Putting $f(X) := a^m y_m X^{m+n} + \cdots + a^n x_0 X^n + \cdots + x_n$, we have $f(\alpha^{-1}) = 0$ and hence $f(X) \in I_{[\alpha^{-1}] \varphi_{\alpha^{-1}}(X)}$ (Thus we have $a^n \eta_m \varphi_{\alpha^{-1}}(X) \in I_{[\alpha^{-1}] \varphi_{\alpha^{-1}}(X)}$. Hence $y_m \varphi_{\alpha^{-1}}(X) \in R[X]$. Put $\beta_\alpha(X) := y_0 + y_1 X + \cdots + y_m X^m$. Considering $\text{deg}(\beta_\alpha(X) - y_m \varphi_{\alpha^{-1}}(X)) < m$, we can assume that $m < d$ by induction. Let $\varphi_{\alpha^{-1}}(X) := X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ be the monic minimal
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polynomial of $\alpha^{-1}$ over $K$ and let $\zeta'_i := (\alpha^{-1})^i + \eta'_i (\alpha^{-1})^{i-1} + \cdots + \eta'_i \eta_i$ $(1 \leq i \leq d)$. Considering $\beta - y_m \zeta_m$ instead of $\beta$, we obtain that $\beta - y_m \zeta_m \in R + I_{a^{-1}} \zeta'_1 + \cdots + I_{a^{-1}} \zeta'_d - 1 = R[\alpha^{-1}] \cap R[\alpha]$ (cf. [1]) by induction on $m$. Thus $\beta \in R[\alpha^{-1}] \cap R[\alpha]$. Therefore we have $R[\alpha^{-1}] \cap R[\alpha] \subseteq R[\alpha] \cap R[\alpha^{-1}]$.

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