Associated Riemannian manifolds and motions

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ASSOCIATED RIEMANNIAN MANIFOLDS AND MOTIONS

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In this paper, we shall investigate the associated Riemannian manifold $V_N (N = n(n + 1)/2)$ with a Riemannian manifold $V_n$ that a Riemannian metric which is naturally derived from the one of $V_n$ is given on the bundle space of the associated principal fibre bundle of $V_n$ and the motions on $V_n$ in connection with $V_N$.

In §§1—3, we shall calculate the parameters of the connection and the curvature forms of $V_N$ with respect to the canonical frames. In §4, we shall investigate the conditions in order that $V_N$ becomes an Einstein space. In §§5, 6, we shall give the equations of geodesics in $V_N$ and an elementary exposition of the relations between the Levi-Civita connection of $V_n$ and the associated Riemannian manifold $V_N$ with $V_n$. In §§7, 8, we shall show that a mapping derived from the differential mapping of a motion of $V_n$ becomes a motion of $V_N$ and investigate the properties of motions of $V_n$ by means of such motions of $V_N$. In §9, we shall investigate sequences of motions of $V_n$ and prove that under a suitable condition, we can derive a differentiable tangent vector field of motion from a sequence of motions which is a geometrical treatment of such sequences of motions of $V_n$ investigated in [6, 7]. Lastly in §10, we shall give an elementary exposition of holonomy groups of $V_n$.

§ 1. Definitions

Let $V_n$ be an n-dimensional Riemannian manifold and let $\mathfrak{B} = \{B, \varphi, V_n, O_n, O_s\}$ be the associated principal fibre bundle of the tangent fibre bundle of $V_n$ as a differentiable manifold with a Riemannian metric, that is

i) $p : B \rightarrow V_n$ be the projection.

ii) for any point $x \in V_n$, the fibre $p^{-1}(x) = O_n(x)$ is homeomorphic to the n-dimensional orthogonal group $O_n$, and

iii) $O_n(x) \ni b = \{x, e_1, \ldots, e_n\}$ is an orthonormal frame at $x$ such that $e_i, i = 1, 2, \ldots, n$, are unit tangent vectors to $V_n$ at $x$ and mutually orthogonal.

Let be given the line element of $V_n$ by


(1, 1) \[ ds^2 = \sum_1^n \omega_i \omega_i \]

and the equations of structure of \( V_v \) with respect to \( b \) are

(1, 2) \[
\begin{align*}
\delta \omega_{ij} &= \sum \omega_k \wedge \omega_{ij}^k, \\
\omega_{ij} &= -\omega_{ji}, \\
\Omega_{ij} &= \frac{1}{2} \sum R_{ijkl} \omega_l \wedge \omega_k.
\end{align*}
\]

where \( R_{ijkl} \) are the components of curvature tensor of \( V_v \).

Now, define a Riemannian manifold \( V_r \) of dimension \( N = \frac{n(n + 1)}{2} \)
whose underlying manifold is \( B \) and whose line element is given by

(1, 4) \[
ds^2 = \sum \omega_i \omega_i + \rho \sum_{i < j} \omega_i \omega_j,
\]

\( \rho = \text{constant} \neq 0. \)

We shall represent (1, 4) by local coordinates in \( V_v \). Let \( x^1, x^2, \ldots, x^n \)
be local coordinates of a neighborhood in \( V_v \) on which the line element of
\( V_v \) is written as

(1, 1') \[ ds^2 = \sum g_{ij}(x) dx^i dx^j. \]

Let \( X_i, i = 1, 2, \ldots, n, \) be tangent vectors such as

\[ X_i = \partial / \partial x^i = y^l e_i, \]

\[ g_{ij} = X_i \cdot X_j = \sum y^l y^j. \]

In matrix notations, putting

(1, 5) \[ G = (g_{ij}), \quad Y = (y^i), \]

then we have

\[ G = YY', \]

where \( Y' \) denotes the transposed matrix of \( Y \). If we put

\[ \pi = (\pi^i), \quad \pi^i = \sum_{k} l_k \cdot dx^k, \]

where \( l_k \)'s are the Christoffel symbols made by \( g_{ij} \), then we have, as is well known,

\[ dG = \pi G + G \pi', \]

and

\[ \omega = Y^{-1} \pi Y - Y^{-1} dY, \quad \omega = (\omega_{ij}). \]

Hence we have
\[ \sum_{i<j} \omega_{ij} = - \frac{1}{2} \text{Trace} \omega \omega \]
\[ = - \frac{1}{2} \text{Trace} (Y^{-1} \pi Y - Y^{-1} \pi dY - Y^{-1} dYY^{-1} \pi Y + Y^{-1} dYY^{-1} dY) \]
\[ = - \frac{1}{2} \text{Trace} (\pi \pi - 2dYY^{-1} \pi + dYY^{-1} dYY^{-1}). \]

Accordingly, (4) can be written as
\[ (1, 4') \quad ds_N^2 = \sum_{i} g_{ij}(x) dx^i dx^j \]
\[ - \frac{1}{2} \rho \text{Trace} (\pi \pi - 2dYY^{-1} \pi + dYY^{-1} dYY^{-1}), \]
where \[ Y = (y^i), \quad \pi = (\sum_i h_k^i dx^k), \quad G = (g_{ij}), \quad G = YY^\prime. \]

§ 2. Parameters of the connection of Levi-Civita of \( V_N \). In this section, we shall determine the parameters of the connection of Levi-Civita of \( V_N \) with respect to orthonormal frames of \( V_N \). According to the ordinary method, let us put
\[
\begin{cases}
\frac{1}{\rho} \sum \omega_{ik} \wedge \theta_{ik} + \frac{1}{\rho} \sum k \otimes k \wedge \theta_{ik} = 0, \\
\frac{1}{\rho} \sum \omega_{ij} = - \frac{1}{\rho} \sum \omega_{ij} + \frac{1}{\rho} \sum \omega_{ij} = 0, \\
\end{cases}
\]
(2, 1)

\[
\begin{cases}
\omega_k \wedge (\theta_{ij} - \omega_{ij}) + \omega_{ij} \wedge 0 = 0, \\
\omega_k \wedge (\frac{1}{\rho} \theta_{ij} - \frac{1}{\rho} R_{jkh} \omega_k) \wedge \omega_k = 0, \\
\end{cases}
\]
(2, 2)

As is well known, we can solve (2, 1), (2, 2) with respect to \( \theta_{ij}, \theta_{ij} \), \( \theta_{ij} ; k \), \( \theta_{ij} ; k \) and shall obtain an unique solution. We get from (2, 1), (2, 2) the equations
\[
\begin{align*}
\theta_{ij} + A_{ijh} \omega_h &= 0, \\
\frac{1}{\rho} \theta_{ij} \wedge \omega_k &= 0, \\
\end{align*}
\]
(2, 3)

From the first of the above equations, we may put
\[
\begin{align*}
\theta_{ij} &= \frac{1}{\rho} \theta_{ij} + A_{ijh} \omega_h, \\
\frac{1}{\rho} \theta_{ij} \wedge \omega_k &= B_{ijh} \omega_k, \\
\end{align*}
\]
(2, 4)

1) In the following we shall use the summation convention.
where $A$'s, $B$'s and $C$'s satisfy the following relations:

$$
\begin{align*}
A_{kh} &= A_{hk} = - A_{kh}, \\
B_{khj} &= - B_{ikhj} = - B_{kijh}, \\
C_{khjl} &= - C_{khlj} = - C_{khjl} = C_{jkh},
\end{align*}
$$

(2, 5)

The first of (2, 5) yields

(2, 6)

$$A_{kh} = 0.$$  

Substituting (2, 4) into the second of (2, 3), we get

$$
- \omega_k \wedge \left\{ \frac{1}{\rho} B_{khj'\omega_h} + \frac{1}{2\rho^2} C_{ijkl\omega_{lm}} + \frac{1}{2} R_{ijkl\omega_h} \right\}
+ \frac{1}{4} \omega_{kh} \wedge (2)_{kh'}_{ij} + \delta_{ikh} \omega_{kj} - \delta_{ikk} \omega_{kj} - \delta_{jkh} \omega_{lk} + \delta_{jkh} \omega_{kl}) = 0,
$$

from which we may put

(2, 7)

$$
\frac{1}{\rho} B_{khj'\omega_h} + \frac{1}{2\rho^2} C_{ijkl\omega_{lm}} + \frac{1}{2} R_{ijkl\omega_h} = D_{ijkl\omega_h} + \frac{1}{2} E_{ijkl\omega_h},
$$

$$
\theta_{kh'}_{ij} + \frac{1}{2} (\delta_{ikh} \omega_{kj} - \delta_{ikk} \omega_{kj} - \delta_{ikk} \omega_{lk} + \delta_{jkh} \omega_{kl}) = - E_{ijkl\omega_h} - \frac{1}{2} F_{klhijklm},
$$

(2, 8)

where $D$'s, $E$'s and $F$'s satisfy the following relations:

$$
\begin{align*}
D_{ijkl} &= D_{ijlk}, \\
E_{ijhkl} &= - E_{ijhkl} = - E_{ijkl}, \\
F_{klhijklm} &= - F_{klhijklm} = - F_{klhijklm} = - F_{klhijklm} = F_{ijkl},
\end{align*}
$$

(2, 9)

Since $\omega_i$, $\omega_{jk}$ are linearly independent, we get from (2, 7) the relations

(2, 10)

$$D_{ijh} = \frac{1}{\rho} B_{hij} + \frac{1}{2} R_{ijh},$$

(2, 11)

$$E_{ijkl} = \frac{1}{\rho} C_{ijkl}.$$  

By (2, 5) and (2, 9), we get easily

(2, 12)

$$D_{ijh} = 0,$$

accordingly

(2, 13)

$$B_{hij} = - \frac{1}{2} \rho^2 R_{ijh} = \frac{1}{2} \rho^2 R_{hij},$$
On the other hand, from (2, 2) we get

\[(2, 14) \quad E_{ijkh} = - E_{khlj}, \]
\[(2, 15) \quad F_{khlj} = - F_{ijkh}. \]

By (2, 11) and (2, 5) we get

\[E_{ijkh} = E_{khlj}. \]

This equation and (2, 14) follow the relation

\[(2, 16) \quad E_{ijkh} = C_{ijkh} = 0. \]

Analogously, we get easily from (2, 9), (2, 15)

\[(2, 17) \quad F_{ijkh} = 0. \]

Thus, we obtain the solution of (2, 1) under the condition (2, 2) as follows:

\[
\begin{aligned}
\theta_{ij} &= \omega_{ij} + \frac{1}{4} \rho^2 R_{ijkh} \omega_{kh}, \\
\theta_{ij,k} &= - \frac{1}{2} \rho R_{ijkh} \omega_{kh} \\
\theta_{ijkh} &= \frac{1}{2} (\partial_{i} \omega_{j} + \partial_{j} \omega_{i} - \partial_{k} \omega_{j} - \partial_{k} \omega_{i}).
\end{aligned}
\]

\[\quad (2, 18) \]

§ 3. Curvature forms in \( V_\mathcal{N} \). Making use of (2, 1), (2, 18), we shall calculate the curvature forms of \( V_\mathcal{N} \).

We have

\[
\begin{aligned}
\Omega_{ij} &= d\theta_{ij} - \theta_{ik} \wedge \theta_{kj} - \frac{1}{2} \theta_{i:j:k} \wedge \theta_{k:h:j} \\
&= d\omega_{ij} + \frac{1}{4} \rho^2 d(R_{ijkh} \omega_{kh}) \\
&\quad - (\omega_{ik} + \frac{1}{4} \rho^2 R_{ijkm} \omega_{im}) \wedge (\omega_{kj} + \frac{1}{4} \rho^2 R_{kjm} \omega_{mn}) \\
&\quad + \frac{1}{8} \rho^2 R_{khlj} \omega_{il} \wedge R_{iklj} \omega_{hm} \\
&\quad = \Omega_{ij} + \frac{1}{4} \rho^2 (R_{ijkh} \omega_{ih} + R_{ijkm} \omega_{il} + R_{iklj} \omega_{jm}) \\
&\quad + R_{ijhm} \omega_{ik} + R_{ijkm} \omega_{ih} \wedge \omega_{kh} \\
&\quad + \frac{1}{4} \rho^2 R_{ijkh} (\Omega_{kh} + \omega_{ih} \wedge \omega_{ih}) \\
&\quad - \frac{1}{4} \rho^2 (R_{iklm} \omega_{im} \wedge \omega_{kj} + R_{kjm} \omega_{ik} \wedge \omega_{ih}) \\
&\quad - \frac{1}{16} \rho^2 R_{khlm} R_{kjlm} \omega_{im} \wedge \omega_{et} \\
&\quad + \frac{1}{8} \rho^2 R_{km} R_{kml} \omega_{il} \wedge \omega_{im},
\end{aligned}
\]
that is

\[
II_{ij} = \Omega_{ij} + \frac{1}{4} \rho \bar{R}_{ijkl} \Omega_{kh} + \frac{1}{8} \rho \bar{R}_{khlm} R_{klmn} \omega_l \wedge \omega_m \\
+ \frac{1}{4} \rho \bar{R}_{ijkh} \omega_l \wedge \omega_k \\
- \frac{1}{4} \rho \bar{R}_{njkh} \omega_l \wedge \omega_k
\]

(3, 1)

where a comma denotes the covariant differentiation of \( V_n \). We have

\[
II_{ik;jk} = \frac{1}{2} \rho d(R_{jikh} \omega_h) \\
- (\omega_{ik} + \frac{1}{4} \rho \bar{R}_{ihlm} \omega_{lm}) \wedge \left( \frac{1}{2} \rho R_{jkh} \omega_k \right) \\
- \frac{1}{8} \rho R_{hkl} \omega_s \wedge (\delta_{ik} \omega_{lj} + \delta_{ik} \omega_{lj} - \delta_{ik} \omega_{lj} - \delta_{ik} \omega_{lk})
\]

\[
= \frac{1}{2} \rho (R_{jkh} \omega_l + R_{ihlm} \omega_{lm} + R_{ihkm} \omega_{lm} + R_{iklm} \omega_{lm}) \wedge \omega_h \\
+ \frac{1}{2} \rho R_{jik} \omega_l \wedge \omega_h + \frac{1}{2} \rho R_{jkh} \omega_s \wedge \omega_h + \frac{1}{8} \rho R_{hkm} R_{jkh} \omega_s \wedge \omega_{lm}
- \frac{1}{8} \rho (R_{jhs} \omega_s \wedge \omega_l + R_{hkm} \omega_s \wedge \omega_h - R_{hkl} \omega_s \wedge \omega_j \\
- R_{hls} \omega_s \wedge \omega_h)
\]

\[
= \frac{1}{2} \rho R_{jkh} \omega_l \wedge \omega_h \\
+ \frac{1}{4} \rho (R_{jkh} \omega_l \wedge \omega_h + R_{hkm} \omega_s \wedge \omega_l) \\
+ \frac{1}{8} \rho \bar{R}_{hlm} R_{jkh} \omega_s \wedge \omega_{lm}.
\]

On the other hand, we have

\[
R_{jkh} \omega_l \wedge \omega_h = -(R_{jkh} \omega_l \wedge \omega_h + R_{jkh} \omega_l \wedge \omega_h) = R_{jkh} \omega_h \wedge \omega_l + R_{jkh} \omega_l \wedge \omega_h,
\]

that is,

\[
2R_{jkh} \omega_l \wedge \omega_h = R_{jkh} \omega_l \wedge \omega_h.
\]

Thus we obtain

\[
II_{ik;jk} = \frac{1}{4} \rho R_{jkh} \omega_l \wedge \omega_h \\
+ \frac{1}{4} \rho (R_{ihr} \omega_h \wedge \omega_i - R_{ihkl} \omega_h \wedge \omega_j) \\
+ \frac{1}{8} \rho \bar{R}_{hlm} R_{jkh} \omega_s \wedge \omega_{lm}.
\]

(3, 2)
Finally, we have

\[ II_{1;kh} = d\theta_{1;j,kh} - \theta_{1;j,1} \wedge \theta_{1;k,h} - \frac{1}{2} \theta_{1;j,1m} \wedge \theta_{1;m,1kh} \]

\[ = \frac{1}{2} (\hat{\alpha}_{ik}(\Omega_{jk} + \omega_{jk} \wedge \omega_{ik}) + \hat{\alpha}_{jk}(\Omega_{ik} + \omega_{ik} \wedge \omega_{jk}) - \hat{\alpha}_{ik}(\Omega_{jk} \wedge \omega_{ij} \wedge \omega_{ik}) - \hat{\alpha}_{jk}(\Omega_{ik} \wedge \omega_{ij} \wedge \omega_{jk}) + \frac{1}{4} \rho^2 R_{ijkl} R_{kmn} \omega_m \wedge \omega_n - \frac{1}{8} (\hat{\alpha}_{ij} \omega_{jk} + \hat{\alpha}_{jk} \omega_{ij} - \hat{\alpha}_{ik} \omega_{jk} - \hat{\alpha}_{jk} \omega_{ik}) \times (\hat{\alpha}_{ik} \omega_{jk} \wedge \omega_{ik} + \hat{\alpha}_{jk} \omega_{ij} \wedge \omega_{jk} - \hat{\alpha}_{ik} \omega_{jk} \wedge \omega_{ik}) \]

that is

\[ II_{1;kh} = \frac{1}{2} (\hat{\alpha}_{ik} \Omega_{jk} + \hat{\alpha}_{jk} \Omega_{ik} - \hat{\alpha}_{ij} \Omega_{jk} - \hat{\alpha}_{jk} \Omega_{ik}) + \frac{1}{4} \rho^2 R_{ijkl} R_{kmn} \omega_m \wedge \omega_n \]

(3, 3)

\[ + \frac{1}{4} (\hat{\alpha}_{ik} \omega_{jk} \wedge \omega_{ik} + \hat{\alpha}_{jk} \omega_{ij} \wedge \omega_{jk} - \hat{\alpha}_{ik} \omega_{jk} \wedge \omega_{ik}) \]

Now, let us put

(3, 4)

\[ II_{AB} = \frac{1}{2} P_{ABC} \hat{\omega}_C \wedge \hat{\omega}_D, \]

where

\[ A, B, \cdots = i, j, \{ij\}, \cdots, \]

\[ \hat{\omega}_i = \omega_i, \hat{\omega}_{ij} = \rho \omega_{ij}, \]

then \( P_{ABC} \) are components of the Riemann-Cristoffel tensor of \( V_\kappa \) with respect to the orthonormal frames which are derived from the ones of \( V_\omega \).

In the first place, we get from (3, 1) the equations as follow.

(3, 5)

\[ P_{1,jkh} = R_{1,jkh} + \frac{1}{4} \rho^2 R_{1,jlm} R_{lmk} \]

\[ + \frac{1}{8} \rho^2 (R_{lmk} R_{lmj} - R_{lml} R_{mj,k}), \]

(3, 6)

\[ P_{1,jkh} = \frac{1}{2} \rho R_{1,jkh}, \]

Since

\[ - \frac{1}{4} \rho^2 R_{1,jkh} \omega_k \wedge \omega_{ij} - \frac{1}{16} \rho^4 R_{1,lm} R_{1,jkm} \omega_m \wedge \omega_{ij} \]
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\[ = \frac{1}{8} \rho^2 \left( R_{1,jm} \delta_{kh} - R_{1,jh} \delta_{mk} \right) \omega_{kl} \wedge \omega_{mh} + \frac{1}{32} \rho^3 \left( R_{1,ih} R_{j,mh} - R_{1,im} R_{j,h} \right) \omega_{kl} \wedge \omega_{mh}, \]

we have

\[ (3, 7) \quad P_{[jk][im]} = \frac{1}{2} \left( R_{1,jkm} \delta_{ih} - R_{1,jhm} \delta_{ik} - R_{1,jkm} \delta_{im} + R_{1,jkh} \delta_{jm} \right) + \frac{1}{4} \rho^2 \left( R_{1,ih} R_{j,mh} - R_{1,im} R_{j,h} \right). \]

Since we have

\[ \frac{1}{4} \rho \left( R_{1,ih} \omega_{ij} \wedge \omega_{ik} - R_{1,ih} \omega_{ij} \wedge \omega_{il} \right) + \frac{1}{8} \rho^2 R_{1,ih} R_{j,kh} \omega_{il} \wedge \omega_{im} = \frac{1}{4} \rho \left( R_{1,ih} \delta_{ik} - R_{1,ih} \delta_{jm} \right) \omega_{ih} \wedge \omega_{im} + \frac{1}{8} \rho^2 R_{1,ih} R_{j,km} \omega_{il} \wedge \omega_{im}, \]

we get from (3, 1)

\[ (3, 8) \quad P_{[jk][im]} = \frac{1}{4} \left( R_{1,ih} \delta_{km} - R_{1,jkm} \delta_{ij} - R_{1,jhm} \delta_{ik} + R_{1,jkh} \delta_{jm} \right) \]

\[ - \frac{1}{4} \rho^2 R_{1,ih} R_{j,kh}, \]

and

\[ (3, 9) \quad P_{[jk][lm]} = 0. \]

Lastly, since

\[ \frac{1}{4} \delta_{il} \omega_{ij} \wedge \omega_{ih} + \delta_{j} \omega_{ij} \wedge \omega_{ik} - \delta_{ik} \omega_{ij} \wedge \omega_{il} - \delta_{jk} \omega_{ij} \wedge \omega_{ik} \]

\[ = \frac{1}{16} \left( \delta_{il} \delta_{ij} \omega_{il} \wedge \omega_{ih} - \delta_{j} \delta_{il} \omega_{ij} \wedge \omega_{ik} - \delta_{ik} \delta_{jl} \omega_{ij} \wedge \omega_{il} - \delta_{jk} \delta_{ih} \omega_{ij} \wedge \omega_{ik} \right) \]

\[ = \frac{1}{16} \left( \delta_{il} \delta_{ij} \omega_{il} \wedge \omega_{ih} + \delta_{j} \delta_{il} \omega_{ij} \wedge \omega_{ik} - \delta_{ik} \delta_{jl} \omega_{ij} \wedge \omega_{il} + \delta_{jk} \delta_{ih} \omega_{ij} \wedge \omega_{ik} \right) \]

\[ = \frac{1}{32} \left( \delta_{il} \delta_{ij} \omega_{il} \wedge \omega_{ih} + \delta_{j} \delta_{il} \omega_{ij} \wedge \omega_{ik} - \delta_{ik} \delta_{jl} \omega_{ij} \wedge \omega_{il} + \delta_{jk} \delta_{ih} \omega_{ij} \wedge \omega_{ik} \right) \]

\[ = \frac{1}{32} \left( \delta_{il} \delta_{ij} \omega_{il} \wedge \omega_{ih} + \delta_{j} \delta_{il} \omega_{ij} \wedge \omega_{ik} - \delta_{ik} \delta_{jl} \omega_{ij} \wedge \omega_{il} + \delta_{jk} \delta_{ih} \omega_{ij} \wedge \omega_{ik} \right) \]

we have

\[ 1) \text{ Where } \delta_{ij}^{kh} \text{ denote the generalized Kronecker's } \delta. \]
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\[ P_{[ij][kh][lm][ab]} \]

\[ = \frac{1}{4} \left\{ \dot{\gamma}^{j} \dot{\gamma}^{k} \dot{\gamma}^{l} \dot{\gamma}^{m} - \dot{\gamma}^{j} \dot{\gamma}^{k} \dot{\gamma}^{l} \dot{\gamma}^{m} - \dot{\gamma}^{j} \dot{\gamma}^{k} \dot{\gamma}^{l} \dot{\gamma}^{m} + \dot{\gamma}^{j} \dot{\gamma}^{k} \dot{\gamma}^{l} \dot{\gamma}^{m} - \dot{\gamma}^{j} \dot{\gamma}^{k} \dot{\gamma}^{l} \dot{\gamma}^{m} \right\} \]

(3, 10) shows that \( V_{N} \) can not always become flat.

Now, we shall calculate components of the Ricci curvature tensor with respect to the canonical orthonormal frames derived from those of \( V_{N} \) from (3, 5)—(3, 10).

Let us put

\[ (3, 11) \quad P_{AB} = \sum_{C} P_{ABC}. \]

We have

\[ P_{ij} = \sum_{k} P_{ik} \dot{k} + \frac{1}{2} \sum_{h,k} P_{i(hk)j(hk)} \]

\[ = R_{ik} \dot{j} + \frac{1}{4} \rho \dot{2} R_{ik} \dot{m} R_{lm} \dot{j} \]

\[ + \frac{1}{8} \rho \dot{2} (R_{lm} \dot{j} R_{mn} \dot{k} - R_{lm} \dot{k} R_{mn} \dot{j}) \]

\[ + \frac{1}{8} \left( R_{i(jh)k} \delta_{j} \dot{k} - R_{i(jh)k} \delta_{j} \dot{k} - R_{i(jk)h} \delta_{j} \dot{k} - R_{i(jk)h} \delta_{j} \dot{k} \right) \]

\[ - \frac{1}{8} \rho \dot{2} R_{i(kh)j} \dot{k}, \]

that is

\[ (3, 12) \quad P_{ij} = R_{ij} + \frac{1}{4} \rho \dot{2} R_{ik} \dot{m} R_{lm} \dot{j}. \]

Nextly, we have

\[ P_{i[jk]} = P_{i[hk]j} \dot{h} + \frac{1}{2} P_{i[lm][jkl]} \dot{m} \]

\[ = P_{i[hj]k} \dot{h} = - \frac{1}{2} \rho R_{i[hk],j} \dot{h} \dot{k} \]

\[ = \frac{1}{2} \rho (R_{ikh,j} + R_{ihk,j}), \]

that is

\[ (3, 13) \quad P_{i[jk]} = - \frac{1}{2} \rho (R_{i,j,k} - R_{i,k,j}). \]

Then, we have
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\[ P_{(ij\{kh\})} = P_{(ij\{kh\})} + \frac{1}{2} P_{(ij\{kh\})\{\{kh\}\{\}} \]

\[ = - \frac{1}{4} \rho^2 R_{ijlm} R_{lmkh} \]

\[ + \frac{1}{8} \{ \delta_{i}^{\nu} \rho_{\mu l}^{\nu} - \delta_{i}^{\nu} \rho_{\mu \nu}^{l} - \delta_{i}^{\nu} \rho_{\mu \nu}^{l} + \delta_{i}^{\nu} \rho_{\mu \nu}^{l} \}

\[ - \delta_{i}^{\nu} \rho_{\mu l}^{\nu} + \delta_{i}^{\nu} \rho_{\mu \nu}^{l} + \delta_{i}^{\nu} \rho_{\mu \nu}^{l} - \delta_{i}^{\nu} \rho_{\mu \nu}^{l} \} \]

\[ = - \frac{1}{4} \rho^2 R_{ijlm} R_{lmkh} - \frac{1}{2} \rho (n-2) \rho_i^{kh}, \]

hence

(3.14) \[ P_{(ij\{kh\})} = - \frac{1}{4} \rho^2 R_{ijlm} R_{lmkh} - \frac{1}{2} (n-2) \delta_{ij}^{kh}. \]

Lastly we get from (3.12), (3.14) the scalar curvature of \( V_N \) as follows.

\[ P = P_{\mu} + \frac{1}{2} P_{(ij\{kl\})} \]

\[ = R_{\mu} + \frac{1}{4} \rho^2 R_{klmn} R_{iklm} - \frac{1}{8} \rho^2 R_{ijlm} R_{ijlm} - \frac{n(n-1)(n-2)}{4}, \]

that is

(3.15) \[ P = - \frac{1}{4} n(n-1)(n-2) + R + \frac{1}{8} \rho^2 R_{ijlm} R_{ijlm}. \]

§ 4. Some special cases. In this section, we shall consider the spaces \( V_n \) whose associated Riemannian spaces \( V_N \) are Einstein spaces, that is

(4.1) \[ P_{\mu \nu} = \frac{P}{N} \delta_{\mu \nu}. \]

These equations are written by (3.12)–(3.14) as

(4.2) \[ \mathit{R}_{ij,k} - \mathit{R}_{ik,j} = 0, \]

(4.3) \[ \mathit{R}_{ij} + \frac{1}{4} \rho^2 \mathit{R}_{klmn} \mathit{R}_{iklm} = \frac{P}{N} \delta_{ij}, \]

(4.4) \[ - \frac{(n-2)}{2} \mathit{\delta}_{ij}^{kh} = \frac{1}{4} \rho^2 \mathit{R}_{ijlm} R_{lmkh} = \frac{P}{N} \mathit{\delta}_{ij}^{kh}. \]

By contraction, we get from (4.3), (4.4), (3.15)
\[ R + \frac{1}{4} \rho \dddot{R}_{ijklmn} R_{ijklmn} = \frac{P}{N} n \]

\[ = \frac{2}{n+1} \left\{ - \frac{n(n-1)(n-2)}{4} + R + \frac{1}{8} \rho \dddot{R}_{ijklmn} R_{ijklmn} \right\} \]

and

\[ - \frac{1}{2} n(n-1)(n-2) - \frac{1}{4} \rho \dddot{R}_{ijklmn} R_{ijklmn} = \frac{P}{N} n(n-1) \]

\[ = \frac{2(n-1)}{n+1} \left\{ - \frac{n(n-1)(n-2)}{4} + R + \frac{1}{8} \rho \dddot{R}_{ijklmn} R_{ijklmn} \right\}, \]

that is

\[ (n-1)R + \frac{1}{4} n \rho \dddot{R}_{ijklmn} R_{ijklmn} = -\frac{1}{2} n(n-1)(n-2), \]

which shows that (4, 3) and (4, 4) are linearly dependent.

Now, we get by (3, 15) and (4, 5) the equation

\[ (4, 6) \quad P = \frac{(n+1)}{2n} R - \frac{(n+1)(n-1)(n-2)}{4} \]

Substituting (4, 6) into (4, 3) and (4, 4), we have

\[ (4, 3') \quad R_{ij} + \frac{1}{4} \rho \dddot{R}_{iklm} R_{ijklmn} = \left( \frac{1}{n^2} R - \frac{(n-1)(n-2)}{2n} \right) \delta_{ij}, \]

(4, 7)

\[- \frac{1}{4} \rho \dddot{R}_{ijklmn} R_{kijlm} = \left( \frac{1}{n^2} R + \frac{n-2}{2n} \right) \delta^{k}_{ij}. \]

By contraction, we get from (4, 7)

\[ (4, 8) \quad - \frac{1}{4} \rho \dddot{R}_{iklm} R_{ijklmn} = \left( \frac{n-1}{n^2} R + \frac{(n-1)(n-2)}{2n} \right) \delta_{ij}, \]

hence this and (4, 3') follow the equation

\[ (4, 9) \quad R_{ij} = \frac{1}{n} R \delta_{ij}. \]

(4, 9) shows that if \( V_{n} \) is an Einstein space, then \( V_{n} \) is also an Einstein space.

If \( n > 2 \) and \( V_{n} \) is an Einstein space, then, as is well known, (4, 2) is automatically satisfied. If \( n = 2 \), since \( R_{ij} = \frac{1}{2} R \delta_{ij} \), (4, 2) becomes

\[ R_{11} = R_{22} = 0, \]

that is \( R = \) constant. But, if \( V_{n} \) for \( V_{2} \) is an Einstein space, then, by
means of (4, 6), \( R \) is constant because \( N > 2 \). Then we obtain the theorem.

**Theorem 1.** Let \( V_n \) be an \( n \)-dimensional Riemannian space and \( V^*_n \) be the Riemannian space of dimension \( N = \frac{1}{2}n(n + 1) \) associated with \( V_n \). In order that \( V^*_n \) be an Einstein space, it is necessary and sufficient that \( V_n \) be an Einstein space and satisfy the equation

\[
- \frac{1}{4} \rho R_{ij}^{lm} R_{im}^{kh} = \frac{1}{n} \left( \frac{R}{n} + \frac{n-2}{2} \right) \delta_{ij}.
\]

**Proof.** The necessity of the conditions is evident by the arguments above. We shall prove the sufficiency.

Since \( V_n \) is an Einstein space, (4, 2) is clearly satisfied for \( n > 2 \). When \( n = 2 \), (4, 2) is equivalent to \( R = \text{constant} \) but it can be derived from (4, 7).

We get from (3, 15), (4, 7)

\[
\frac{P}{N} = \frac{2}{n(n+1)} \left\{ - \frac{n(n-1)(n-2)}{4} + R - \frac{1}{2} \left( \frac{n-1}{n} R \right) \right. \\
\left. + \frac{(n-1)(n-2)}{2} \right\}
\]

\[
= - \frac{(n-1)(n-2)}{2n} + \frac{R}{n^2}.
\]

On the other hand, we get from (4, 7)

\[
- \frac{n-2}{2} \delta_{ij} - \frac{1}{4} \rho R_{ij}^{km} R_{im}^{kh} \\
= - \frac{n-2}{2} \delta_{ij} + \left( \frac{1}{n^2} R + \frac{n-2}{2n} \right) \delta_{ij} \\
= \left( - \frac{(n-1)(n-2)}{2n} + \frac{R}{n^2} \right) \delta_{ij} = \frac{P}{N} \delta_{ij}.
\]

Analogously we have

\[
R_{ij} + \frac{1}{4} \rho R_{kim} R_{jim} \\
= \frac{R}{n} \delta_{ij} - \left( \frac{n-1}{n^2} R + \frac{(n-1)(n-2)}{2n} \right) \delta_{ij} = \frac{P}{N} \delta_{ij}.
\]

Thus we see that the system of equations (4, 2), (4, 8) and (4, 4) is equivalent to the one of (4, 7) and (4, 9).

Now, let \( V_n \) be a space of constant curvature, that is whose curvature tensor satisfies the equations.
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\[ R_{ijkl} = - K(\partial_i \partial_j \partial_k - \partial_i \partial_k \partial_j), \]
\[ K = \text{constant} \]

with respect to orthonormal frames. Since

\[ R_{ij} = -(n-1)K_{ij}, \quad R = -n(n-1)K, \]
\[ \frac{1}{n^2} R + \frac{n-2}{2n} = -\frac{n-1}{n} K + \frac{n-2}{2n} \]

and

\[ -\frac{1}{4} \rho^a R_{ij} \rho^b R_{ij} = -\frac{1}{2} \rho^a K^a_{ij} \]

it follows that if \( \rho \) is a constant such that

\[ (4, 10) \]
\[ \rho^a K^a = \frac{2(n-1)}{n} K = \frac{n-2}{n} \]

then \( V \) becomes an Einstein space. Thus we have a corollary.

**Corollary.** An \( n \)-dimensional Riemann space of constant curvature \( K \) has an Einstein space as its associated Riemann space if and only if \( K > \frac{n-2}{2(n-1)} \) (or \( -R > \frac{n(n-2)}{2} \)).

Lastly, we shall consider the special case \( n = 3 \). Putting

\[ (4, 11) \]
\[ R_{312} = - K_{11}, \quad R_{314} = - K_{22}, \quad R_{331} = - K_{33}, \quad R_{123} = - K_{12}, \quad R_{231} = - K_{23}, \quad R_{321} = - K_{32} \]

we have

\[ (4, 12) \]
\[ \begin{cases} R_{11} = - K_{11} - K_{12}, & R_{22} = K_{22}, \\ R_{22} = - K_{21} - K_{11}, & R_{33} = K_{33}, \\ R_{33} = - K_{33} - K_{22}, & R_{12} = K_{12} \end{cases} \]

and

\[ (4, 13) \]
\[ R = -2 \sum_{i=1}^{3} K_{ii}. \]

Then, \( (4, 9) \) becomes

\[ (4, 14) \]
\[ K_{11} = K_{12} = K_{22} = k, \]
\[ K_{23} = K_{32} = K_{33} = 0. \]

Since
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\[ R_{\Sigma i m} R_{i m \Sigma j} = 2 \sum K_{i j} K_{i j} = \frac{2}{s}, \]
\[ R_{\Sigma i m} R_{i m \Sigma j} = 2 \sum K_{i j} K_{i j} = O, \]

etc.,

(4, 7) becomes

(4, 15)

\[ \frac{1}{2} \rho^2 \kappa^2 = \frac{2}{3} \kappa - \frac{1}{6} \]

Since \( R = -6r \), (4, 15) yields the following corollary.

**Corollary.** A 3-dimensional Einstein space has an Einstein space as its associated Riemann space if and only if \( -R > \frac{3}{2} \).

**Remark.** (4, 7) follows that \( R + \frac{1}{2} n(n - 2) < 0 \).

§ 5. Geodesics in \( V_n \). The differential equations of geodesics in \( V_n \) are

\[
\frac{d \omega_i + \theta_{ki} \omega_k + \frac{1}{2} \rho \gamma_{kh;ij} \omega_k}{\omega_i} = \rho d \omega_{ij} + \theta_{k;ij} \omega_k + \frac{1}{2} \rho \gamma_{kh;ij} \omega_k
\]

Since we have by means of (2, 18)

\[
d \omega_i + \theta_{ki} \omega_k + \frac{1}{2} \rho \gamma_{kh;ij} \omega_k
\]

\[
= d \omega_i + (\omega_{ki} + \frac{1}{4} \rho R_{iki} \omega_k) \omega_k - \frac{1}{4} \rho^2 R_{ki} \omega_{ij} \omega_k
\]

\[
= d \omega_i + \omega_{ki} \omega_k - \frac{1}{2} \rho^2 R_{ijkl} \omega_{ij} \omega_k
\]

and

\[
\rho d \omega_{ij} + \theta_{k;ij} \omega_k + \frac{1}{2} \rho \gamma_{kh;ij} \omega_k
\]

\[
= \rho d \omega_{ij} + \frac{1}{2} \rho R_{ijkl} \omega_k \omega_k + \frac{1}{4} (\delta_{kl} \omega_{ij} + \delta_{kl} \omega_{ij} - \delta_{ij} \omega_{kl} - \delta_{kl} \omega_{ij}) \omega_k
\]

\[
= \rho d \omega_{ij}
\]

the equations of geodesics in \( V_n \) become
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\[
(5,1) \quad \frac{d\omega_i + \omega_i \omega_j}{\omega_i} - \frac{1}{2} \rho^2 R_{ijkl} \omega_j \omega_k \omega_{kl} = \frac{d\omega_j}{\omega_j}
\]

Let \( \bar{C} \) be a geodesic in \( V_n \) and \( C \) be its image in \( V_s \) by the projection \( p : V_n \rightarrow V_s \). Let \( \tau, s \) be the arclengths of \( \bar{C}, C \) respectively. Then, (5, 1) is written as

\[
\begin{align*}
\frac{d\omega_i}{d\tau} + \omega_k \frac{d\omega_i}{d\tau} - \frac{1}{2} \rho^2 R_{ijkl} \omega_j \omega_k \omega_{kl} &= 0, \\
\frac{d\omega_j}{d\tau} &= 0.
\end{align*}
\]

Hence, we have

\[
(5,2) \quad \frac{d\omega_i}{d\tau} + \omega_k \frac{d\omega_i}{d\tau} = \frac{1}{2} \rho^2 R_{ijkl} \omega_j \omega_k \omega_{kl} = c_{ij} d\tau,
\]

where \( c_{ij} = c_{ji} \) are constants. In a local coordinate neighborhood \( (x^\alpha) \), if we put

\[
(5,3) \quad dx^\alpha = y_i^\alpha d\omega_i, \quad g^\mu^\nu = y_i^\mu y_i^\nu,
\]

then since we have

\[
\omega_{ik} y_k^\alpha = y_i^\mu \cdot \frac{\partial y_k^\alpha}{\partial x^\mu} + dy_i^\alpha,
\]

where \( \frac{\partial}{\partial x^\mu} \) is the Christoffel symbols made by \( g_{\mu^\nu} \). (5, 2) is written as

\[
(5,2') \quad \begin{align*}
\frac{D}{d\tau} \frac{dx^\alpha}{d\tau} &= \frac{1}{2} \rho^2 R_{\mu^\alpha \nu^\rho} \frac{dx^\alpha}{d\tau} y_i^\nu y_j^\rho c_{ij}, \\
\frac{D}{d\tau} y_i^\alpha &= c_{ij} y_j^\alpha,
\end{align*}
\]

where \( \frac{\partial}{\partial x^\mu} \) is the Christoffel symbols made by \( g_{\mu^\nu} \) and \( D \) denotes the covariant differential in \( V_s \). From (5, 2), we see that

\[
\frac{ds}{d\tau} = k = \text{constant} \quad 0 < k < 1.
\]

This equation shows that a geodesic in \( V_n \) has a constant angle with the field \( t^t \) of \( n \)-dimensional horizontal tangent subspaces \( t_b \subset T_b(B), b \in B \), which will be defined in §6. By means of (1, 4), we must have
(5, 3) \[ 1 = -k^2 + \frac{1}{2} \rho^2 c_{ij} e_{ij}. \]

In the case \( k = 0 \), \( C \) is clearly a point curve, hence \( \mathcal{C} \) is the image of a one parameter subgroup of \( O_\nu \) by an admissible mapping of the fibre bundle \( \mathcal{B} \). In the case \( k = 1 \), we have \( c_{ij} = 0 \) by the above equation, hence \( C \) is a geodesic in \( V_\nu \) and the points of \( \mathcal{C} \) are the parallel displaced orthonormal frames of \( V_\nu \) along \( C \). For an example, if \( V \) is a 2-sphere, then \( C \) is a circle on the sphere.

§ 6. The Levi-Civita's connection of \( V_\nu \) and its explanation in \( V_\nu \).

According to §1, let \( \mathcal{B} = \{ B, p, V_\nu, O_\nu, O_\nu \} \) be the associated principal fibre bundle of \( V_\nu \). Any point \( b \in B \) is represented as

\[ b = (x(b), e_i(b)) \]

where \( x(b) = p(b) \) and \( e_i(b), \ i = 1, 2, \ldots, n \), are unit tangent vectors to \( V_\nu \) at \( x(b) \) and orthogonal each others.

Let \( v_i(b), v_{ij}(b), \ i < j, \ i, j = 1, 2, \ldots, n \), be tangent vectors to \( B \) dual to \( \omega_i(b), \omega_{ij}(b), \ i < j, \ i, j = 1, 2, \ldots, n \).

In the following, for a differentiable mapping \( f \) of a differentiable manifold \( X \) into another differentiable manifold \( Y \), we shall denote the differential mapping of \( f \) by \( f_* : T(X) \rightarrow T(Y) \) and the dual mapping of \( f_* \) by \( f^* : T^*(Y) \rightarrow T^*(X) \), where \( T(X), T(Y) (T^*(X), T^*(Y)) \) are the spaces of tangent (cotangent) vectors to \( X, Y \) respectively.

Since \( \omega_i(b) = p^* \omega_i(b) \), where \( \omega_i(b) \) in the right-hand side is regarded as a cotangent vector to \( V_\nu \) at \( x(b) \) such that \( \tilde{\omega}_{ij} = \langle \omega_i(b), e_j(b) \rangle \), we have \( \langle \omega_i(b), p^* v_j(b) \rangle = \langle \omega_i(b), v_j(b) \rangle = \tilde{\omega}_{ij} \), hence

\[ p^* v_i(b) = e_i(b). \]

Analogously we have

\[ p^* v_{ij}(b) = 0. \]

For any \( \alpha = ((a^I_\alpha)) \in O_\nu \), we denoted the right translation corresponding to \( \alpha \) by \( r(\alpha) \) which is defined by

\[ r(\alpha)(b) = (x(b), \ a_I^I(\alpha) e_j(b)), \]

where \( ((a^I_\alpha)) \) is an \( n \)-dimensional orthogonal matrix. Since we have

\[ a^I_\alpha(x_\alpha) = a^I_\alpha(x) a^I_\alpha(x_\alpha), \ \alpha_1, \alpha_2 \in O_\nu, \]
it follows that
\[ r(a_1) \circ r(a_2) = r(a_2 \circ a_1). \]

Now, we shall consider \((r(a))^*\). Let \(b = f'(x) = (x, e_t(x))\) be a differentiable local cross-section of \(\mathcal{B}\) defined on a neighborhood \(U\) in \(V_x\). Let us put
\[ e_i(b) = y^i_t(b) e_t(x), \quad x = p(b), \]
then we can consider \(x(b), y^i_t(b)\) as local coordinates of the point \(b\). Let us put
\[ \theta_2(x) = f^* \omega_2(b), \quad \theta_{2h}(x) = f^* \omega_{2h}(b), \]
then we have the equations
\[ \begin{cases} 
\omega_i(b) = z^i_t(b) \theta_i(x), \\
\omega_{ij}(b) = y^i_t(b) z^j_t(b) \theta_{ij}(x) + z^i_t(b) d y^j_t(b)
\end{cases} \]
in the coordinates \(x(b), y^i_t(b)\), where \(y^i_t(b) z^i_t(b) = \delta^i_t\).
Since we have from (6, 4) (6, 6)
\[ r(a)(b) = (x, a^i_t(a) y^j_t(b) e_t(x)), \quad x = p(b), \]
we get
\[ r(a) \circ r(a)(b) = a^i_j(a^{-1}) \omega_j(b) = a^i_j(a) \omega_j(b), \]
\[ r(a) \circ r(a)(b) = a^i_j(a) a^j_k(a) \omega_{ik}(b). \]
Accordingly, we get
\[ \begin{cases} 
r(a)^* \omega_i(b) = a^i_j(a) \omega_i(r(a)(b)), \\
r(a)^* \omega_{ij}(b) = a^i_j(a) a^k_l(a) \omega_{ij}(r(a)(b)).
\end{cases} \]

Now, let be \(I_b\) the tangent subspace to \(B\) at \(b\) spaned by \(v_i(b), \ldots, v_n(b)\) which define a differentiable field \(I\) on \(V_x\). By (6, 2), (6, 10) it follows that
\[ \begin{cases} 
r(a)^* I_b = I_{r(a)(b)}, \quad a \in O_x, \\
p^* I_b = T_{p(b)}(V_x).
\end{cases} \]
Let \(\mu_b : T_b(B) \to T_b(O_x(x)), \quad p(b) = x\), be the projection defined by
(6, 12) \[ \mu_b(\sum v_i b_i(b) + \sum_{i<j} v_i b_{ij}(b)) = \sum_{i<j} v_i b_{ij}(b) \]

and denote also by \( b \) the admissible mapping \( O_\alpha \rightarrow O_\alpha(x) \) defined by \( b(\alpha) = r(\alpha)(b) \), \( \alpha \in O_\alpha \). Putting \( \pi_\alpha = (b_*)^{-1}(b_\alpha) : T_b(B) \rightarrow T_{\alpha}(O_\alpha) \), \( e \) the identity of \( O_\alpha \), we obtain a \( L(O_\alpha) \)-valued differential from \( \pi \) defined on \( B \) by \( \pi | T_b(B) = \pi_\alpha \). Then it follows from (6, 11) that \( r(\alpha)^* \pi = ad(\alpha^{-1})^* \pi \).

Let \( \iota_x : O_\alpha(x) \rightarrow B \) be the imbedding mapping then we get from (6, 8)

(6, 13) \[ b_\alpha^* \omega_{ij}(b(\alpha)) = a_\alpha^i(\alpha^{-1})da_{ij}^i(\alpha) = a_\alpha^j(\alpha)da_{ij}^j(\alpha) = \omega_{ij}(\alpha), \]

which are left invariant differential forms on \( O_{\alpha} \).

Let \( \tilde{\omega}_{ij}(\alpha) \) be the tangent vector fields on \( O_{\alpha} \) dual to \( (\omega_{ij}(\alpha)) \). Since

\[
\langle \omega_{ij}(\alpha), (b_\alpha)_* \tilde{\omega}_{ab}(e) \rangle = \langle \omega_{ij}(e), \tilde{\omega}_{ab}(e) \rangle, \\
\langle \omega_{ij}(b_\alpha), (\iota_x)_* \tilde{\omega}_{ab}(e) \rangle = \langle \iota_x^* \omega_{ij}(b), b_\alpha \tilde{\omega}_{ab}(e) \rangle \\
= \langle 0, b_\alpha \tilde{\omega}_{ab}(e) \rangle = 0,
\]

we have

(6, 14) \[ (\iota_x)_* \omega_{ij}(\alpha) = \omega_{ij}(b). \]

Let \( : p^{-1}(U) \rightarrow B \) be the imbedding mapping and define a mappings \( \rho : U \rightarrow U \times O_\alpha, \phi : U \times O_\alpha \rightarrow B \) by

\[
\rho(x) = x \times e, \\
\phi(x, \alpha) = (x, \alpha f(e_\iota(f(x))).
\]

Then we have a \( L(O_\alpha) \)-valued differential from \( \theta = (\iota_x \phi)^* \) on \( U \). Since \( (\iota_x \phi)(x) = (x, e_\iota(f(x))) \), we get from (6, 8)

\[
(\iota_x \phi)^* \omega_{ij}(f(x)) = \theta_{ij}(x), \\
(\iota_x \phi)^* \omega_{ij}(f(x)) = \theta_{ij}(x) = I_\alpha^{\iota}(x) e_\iota(f(x))
\]

hence

\[
(\iota_x \phi)^* e_{ik}(f(x)) = \omega_{ik}(f(x)) + \frac{1}{2} I_{\alpha}^{\iota}(x) \omega_{ij}(f(x)).
\]

Accordingly, we have by (6, 14)

\[
\langle \tilde{\omega}, e_k(x) \rangle = \langle \pi, (\iota_x \phi)^* e_k(x) \rangle \\
= \langle \pi, \omega_k(f(x)) + \frac{1}{2} I_{\alpha}^{\iota}(x) \omega_{ij}(f(x)) \rangle \\
= \frac{1}{2} \langle \pi, I_{\alpha}^{\iota}(x) \omega_{ij}(f(x)) \rangle = \frac{1}{2} I_{\alpha}^{\iota}(x) \omega_{ij}(e).
\]
On the other hand, we can define canonically a $L(O_1)$-valued differential form $\theta$ on $U$ from $\theta_{ij}$ by

$$\theta(v) = \langle \delta_i(x), v \rangle \delta_{ij}(e), \; v \in T_\ast(V, \nu)$$

this shows that

$$\theta = \tilde{\theta}.$$  

That is, the parameters $\hat{\theta}$ on $U \subset V_\nu$ derived from the connection in the sense of C. Ehresmann [3] defined by the field of tangent subspaces $\nu$, by the local cross section $f : U \rightarrow B$ are the parameters $\theta$ on $U$ of the Levi-Civita connection of $V_\nu$ with respect to the field of orthonormal frames defined by $f$.

§ 7. Motions of $V_\nu$ derived from motions of $V_\nu$. Let $f$ be a motion of $V_\nu$, that is a homeomorphism onto itself such that $(f(x_1), f(x_2)) = (x_1, x_2), x_1, x_2 \in V_\nu$, where $(x_1, x_2)$ denotes the Riemannian distance in $V_\nu$ between $x_1$ and $x_2$. As is well known, $f$ is differentiable. Furthermore we have

$$(f \ast X_1) \cdot (f \ast X_2) = X_1 \cdot X_2, \; X_1, X_2 \in T_\ast(V, \nu),$$

where $X_1 \cdot X_2$ denotes the inner product of $X_1$ and $X_2$. Accordingly, we can define a differentiable homeomorphism $f = \chi(f)$ on $B$ by

$$(7, 2) \quad f(b) = (f(x(b)), f \ast e_i(b)), \; b \in B.$$  

Since $p_\ast f = f \ast p$, we have $p_\ast f \ast \nu_i(b) = f_\ast p_\ast \nu_i(b) = f_\ast e_i(b) = e_i(f(b))$ by $(6, 2), \; (7, 2)$, hence $f_\ast \nu_i(b) = \nu_i(f(b)) + \text{a linear combinations of } \nu_i(f(b))$ and $f_\ast \nu_i(f(b)) = \nu_i(B) + \text{a linear combination of } \nu_i(f(b))$. On the other hand, since we can consider $\omega_i(b)$ as differential forms in $V_\nu$, we obtain

$$(7, 3) \quad f_\ast \omega_i(f(b)) = \omega_i(b).$$

Furthermore, we get from $(1, 2), \; (7, 3)$

$$d \omega_i = \sum_k \omega_k \wedge f_\ast \omega_k,$$

$$f_\ast \omega_i = - f_\ast \omega_i,$$

hence we have

$$(7, 4) \quad f_\ast \omega_i(f(b)) = \omega_i(b).$$

Thus we obtain the following theorem.
Theorem 2. If \( f \) is a motion of \( V_n \), then the transformation 
\( \bar{f} = \chi(f) \) derived from \( f \) by (7, 2) is also a motion of \( V_n \) and
\[
(7, 5) \quad \chi(f_1 \circ f_2) = \chi(f_1) \circ \chi(f_2).
\]

Now, denoting the group of motions of \( V_n \) by \( M(V_n) \), we get easily from (6, 9) the following theorem.

Theorem 3. Any right translation of \( B \) is a motion of \( V_n \) and commutes with \( \chi(f) \), \( f \in M(V_n) \).

It is sufficient to prove the second part of the theorem.

For \( \alpha \in O_n \), \( f \in M(V_n) \), \( b \in B \), we have
\[
(\alpha)(\chi(f)(b)) = r(\alpha)((f(x(b)), f_*e_j(b)))
\]
\[
= (f(x(b)), a(\alpha) f_*e_j(b))
\]
\[
= (f(x(b)), f_*a(\alpha)e_j(b))
\]
\[
= \chi(f)((x)(b), a(\alpha)e_j(b))
\]
\[
= \chi(f)(r(\alpha)(b)).
\]

Hence we have the relation
\[
(7, 6) \quad r(\alpha) \circ \chi(f) = \chi(f) \circ r(\alpha).
\]

We see also easily that
\[
(7, 7) \quad r(O_n) \cap \chi(M(V_n)) = 1,
\]
where 1 denotes the identity transformation.

§ 8. Some mappings on \( V_n \). Now, let \( S^{n-1} \) be the \((n-1)\)-dimensional unit sphere: \( \sum w^i w^i = 1 \) in an \( n \)-dimensional Euclidean space \( R^n \).

For any complete Riemannian manifold \( V_n \), we define a mapping \( \Phi : \) \( B \times S^{n-1} \times R \rightarrow B \) as follows:

For \( b \in B \), \( w = (w^1, \ldots, w^n) \in S^{n-1}, s \in R \), let \( \gamma(b, w, s) \) be the geodesic arc in \( V_n \), starting at \( p(b) = x(b) \) whose tangent unit vector at \( x(b) \) is \( w^i e_i(b) \) and whose length is \( s \). Let \( \tilde{F}(b, w, s) \) be the end point of \( \gamma(b, w, s) \).

By parallel displacing \( e_i(b) \) along this geodesic, we get a curve \( \overline{\gamma}(b, w, s) \) in \( B \) whose points are these frames, hence \( p(\overline{\gamma}(b, w, s)) = \gamma(b, w, s) \). Let \( \Phi(b, w, s) \) be the end point of \( \overline{\gamma}(b, w, s) \).

The mapping \( \Phi \) is clearly differentiable and have the following properties:
\[
(8, 1) \quad p(\Phi(b, w, s)) = F(b, w, s).
\]
\[ (8, 2) \quad r(\alpha) \circ \Phi(b, w, s) = \Phi(r(\alpha)b, \alpha^{-1}w, s), \quad \alpha \in O_n, \]
\[ (8, 3) \quad F(b, w, s) = F(r(\alpha)b, \alpha^{-1}w, s), \]
\[ (8, 4) \quad \Phi(b, w, 0) = b. \]

Furthermore, since any motion of \( V_n \) preserves geodesics and parallel displaced vector fields, it follows that
\[ (8, 5) \quad \tilde{f}(\Phi(b, w, s)) = \Phi(\tilde{f}(b), w, s), \]
\[ \tilde{f} = \chi(f), \quad f \in \mathcal{M}(V_n), \]
and by \( p \circ \tilde{f} = f \circ p \) and \((8, 2)\),
\[ (8, 6) \quad f \circ F(b, w, s) = F(\tilde{f}(b), w, s). \]

Let \( \{f_m\}, m = 1, 2, \ldots \), be a sequence of motions on \( V_n \). For a fixed point \( b_0 \in B \), we suppose that \( \tilde{f}_m(b_0) \), \( \tilde{f}_m = \chi(f_m) \), converge to a point \( b'_0 \). For any point \( b \in B \), we can take an \( \alpha \in O_n \), a \( w \in S^{n-1} \) and a real number \( s \) such that \( b = r(\alpha)((\Phi(b_0, w, s)) \). Hence, by \((7, 5)\), \((8, 5)\), we get
\[
\lim f_m(b) = \lim \tilde{f}_m(r(\alpha)((\Phi(b_0, w, s)))
\]
\[ = \lim r(\alpha)((f_m(\Phi(b_0, w, s)))
\]
\[ = r(\alpha)(\lim \Phi(\tilde{f}_m(b_0), w, s)), \]
that is
\[ (8, 7) \quad \lim \tilde{f}_m(b) = r(\alpha)((\Phi(b'_0, w, s)). \]

Thus we can define a limiting map \( \hat{f} : B \to B \) by
\[ (8, 8) \quad \hat{f}(b) = \lim \chi(f_m)(b), \quad b \in B, \]
which is clearly a motion of \( V_n \). By the above equation, we get easily
\[ (8, 9) \quad \hat{f} \circ r(\alpha) = r(\alpha) \circ \hat{f}, \quad \alpha \in O_n. \]

Furthermore, since we have \( p(\hat{f}(b)) = \lim f_m(p(b)) \), we get a limiting map \( f : V_n \to V_n \) by
\[ f(x) = \lim f_m(x), \quad x \in V_n, \]
such that \( f \in \mathcal{M}(V_n) \) and
\[ f \circ p = p \circ \hat{f}. \]

Now, we have by \((8, 1)\), \((8, 7)\) the relation
\[ f(x) = F(b'_0, w, s), \]
\[ x = \hat{f}(b), \quad b = r(\alpha)(\Phi(b_0, w, s)). \]
On the other hand, we get from (8, 5) the equation
\[ p(\tilde{f}(\psi(b_0, w, s))) = f(p(\psi(b_0, w, s))) = \tilde{f}(x) \]
\[ = p(\psi(\tilde{f}(b_0), w, s)) = F(\tilde{f}(b_0), w, s), \]
hence
\[ F(\tilde{f}(b_0), w, s) = F(\tilde{f}(b_0), w, s) \]
\[ w \in S^{-1}, \quad s \in R. \]
It follows that \( \tilde{f}(b_0) = \tilde{f}(b_0) \) and by (8, 5), (7, 5)
\[ \tilde{f}(b) = r(\cdot)(\psi(\tilde{f}(b_0), w, s)) \]
\[ = r(\cdot)(\tilde{f}(\psi(b_0, w, s))) = \tilde{f}(r(\cdot)(\psi(b_0, w, s))) \]
\[ = f(b), \]
that is
\[ (8, 10) \quad \lim \chi(f_m)(b) = \chi(\lim f_m)(b), \quad b \in B. \]

For any \( V_n \) which is not complete, we can carry the same argument by means of a finite number of points of \( B \) such as \( b_0 \). Thus, we obtain.

**Theorem 4.** Let \( V_n \) be a Riemannian manifold and let \( \{f_m\}, m = 1, 2, \ldots, \) be a sequence of motions of \( V_n \). Then the sequence \( \{\chi(f_m)\} \) is simultaneously convergent or do not convergent at every point of \( B \). In the first case, we have
\[ \lim \chi(f_m)(b) = \chi(\lim f_m)(b), \quad b \in B. \]

In the next place, we suppose that for a sequence \( \{f_m\} \) of motions of \( V_n \), \( \lim f_m(x_0) = x_0 \). For a subsequence \( \{f_{m_{\lambda}}\} \) of \( \{f_m\} \), we may suppose that \( \lim f_{m_{\lambda}}(b_0) = b_0 \), where \( b_0 \) is a fixed element in \( p^{-1}(x_0) \). Then, by means of the above theorem there exists a \( f \in M(V_n) \) such that \( f(x) = \lim \lambda \to \lambda f_{m_{\lambda}}(x) \) and \( \chi(f)(b) = \lim \lambda \to \lambda \chi(f_{m_{\lambda}})(b), \quad x \in V_n, \quad b \in B \). Accordingly, we see that if \( \lim f_m(x) = f(x) \), then \( \lim \chi(f_m)(b) = \chi(f)(b) \). Thus, we obtain.

**Theorem 5.** For any \( V_n \), \( \chi: M(V_n) \to M(V_N) \) is continuous in the sense of weakly convergence, that is, if \( \lim f_m(x) = f(x) \), then \( \lim \chi(f_m)(b) = \chi(f)(b), \quad x \in V_n, \quad b \in V_N \).

\[ \S 9. \] Tangent vector fields over \( V_n \) derived from sequences of motions of \( V_n \). For the sake of simplicity, let \( V_n \) be a complete Rie-
mannian manifold. For any $f \in M(V,)$, $x \in V$, since we have by Theorem 3

$$(b, \chi(f)(b)) = (r(\alpha)(b), r(\alpha)(\chi(f)(b)))$$

$$= (r(\alpha)(b), \chi(f)(r(\alpha)(b)))$$

$$b \in p^{-1}(x), \; \alpha \in O$$

we define a function $u_f : V \to R$ by

$$(9, 1) \quad u_f(x) = (b, \chi(f)(b)), \; b \in p^{-1}(x),$$

which is differentiable. If $f \neq 1$, then everywhere $u_f(x) \neq 0$ by (8, 5).

Now, let be given a sequence $\{f_m\}$, $m = 1, 2, \cdots$, of motions of $V$ which are mutually distinct and weakly converge to the identity transformation. For simplicity, we put

$$u_m(x) = u_{f_m}(x), \; x \in V.$$  

By Theorem 5, we have

$$\lim_{m \to \infty} \chi(f_m)(b) = b, \; b \in B.$$  

Then we define a tangent vector field $\gamma$ over $V$ by

$$(9, 2) \quad \gamma(x)(h) = \lim_{m \to \infty} \frac{(f_m^* h)(x) - h(x)}{u_m(x)},$$

where $x \in V$, and $h$ is any differentiable function defined on an open neighborhood of $x$. We shall show that $\gamma(x)$ can be defined by the right hand side of (9, 2) and $\gamma$ is a differentiable tangent vector field over $V$.

Now, we define a differentiable function by

$$(9, 3) \quad w(b, b', w, s) = (\Phi(b, w, s), \Phi(b', w, s)),$$

$$b, b' \in B, \; w \in S^{n-1}, \; s \in R.$$  

By Theorem 3, (8, 2), we get

$$(9, 4) \quad w(b, b', w, s) = w(r(\alpha)(b), r(\alpha)(b'), \alpha^{-1}w, s) \; \alpha \in O.$$  

For any point $x$, we can take a spherical neighborhood $U$, such that for a fixed $b, b \in p^{-1}(x)$, $f(b, w, s)$ gives a geodesic polar coordinate system on $U$. Then we can define a function $u$ by

$$(9, 5) \quad u(b, b, x) = w(b, b, w, s).$$
where \( x \in U_{p(b)} \), \( x = F_{(b_1, w, s)}, \ b_1 \in B \). By (8, 3), (8, 4), (9, 4), we get a relation as

\[
(9, 6) \quad u(b_0, b_1, x) = u(r(\alpha)(b_0), r(\alpha)(b_1), x).
\]

For a motion \( f \) on \( V_m \), we have

\[
u_f(F(b_0, w, s)) = (\Phi(b_0, w, s), \chi(f)(\Phi(b_0, w, s)))
\]

\[
= (\Phi(b_0, w, s), \Phi(\chi(f)(b_0), w, s))
\]

\[
= w(b_0, \chi(f)(b_0), w, s),
\]

hence

\[
(9, 7) \quad u_f(x) = u(b_0, \chi(f)(b_0), x), \ x \in U_{p(b)}, \ b_0 \in B.
\]

For a fixed \( w \in S^{m-1} \), a fixed \( s \in R \), the differentiable mapping \( \Phi_{w,s} : B \to B \) by

\[
(9, 8) \quad \Phi_{w,s}(b) = \Phi(b, w, s), \ b \in B
\]

is a differentiable homeomorphism on \( B \) and it is evident from the definition of \( \Phi \) that

\[
(9, 9) \begin{cases}
\Phi_{-w,s} = \Phi_{w,-s}, \\
\Phi_{w,s} \Phi_{-w,-s} = 1.
\end{cases}
\]

Accordingly, if for a point \( b_0 \in B \), the tangent unit vectors to elementary geodesic arcs \( \gamma(b_0, f_m(b_0)), f_m = \chi(f_m) \), from \( b_0 \) to \( f_m(b_0) \) at \( b_0 \), converge to a tangent vector to \( B \) at \( b_0 \), then for any point \( b \in B \), the same is true. Furthermore, since for the function \( u(b_0, b_1, x) \) which is differentiable with respect to \( b_0, b_1 \in B, x \in U_{p(b)}, \) we have

\[
u(b_0, b_1, x) = (1, b_1), \ x = p(b_0),
\]

we can take an open neighborhood of \( x \), \( U_{x_0} \subset \bar{U}_{x_0} \subset U_{x_0} \) such that

\[
(9, 10) \quad \lim_{m \to \infty} \frac{u(b_0, b_m, x)}{(b_0, b_m)} = \lim_{m \to \infty} \frac{U_m(x)}{(b_0, b_m)} \neq 0,
\]

\[
b_m = f_m(b_0), \ x \in U_{x_0}.
\]

In \( U_{x_0} \), we get from the equation \( x = F(b_0, w, s) \) the inverse mapping

\[
w = w(x), \ s = s(x), \ x \neq x_0,
\]

which are differentiable. Then, we have by (8, 6)
(9, 11) \[ f_m(x) = F(b_m, \omega(x), s(x)). \]

Therefore, we have by (9, 10), (9, 11)

\[
\begin{align*}
\lim_{m \to \infty} \frac{(f_m \ast h_h)(x) - h(x)}{u_m(x)} &= \lim_{m \to \infty} \frac{h(F(b_m, \omega(x), s(x))) - h(x)}{(b_m, b_m)} \\
&= \lim_{m \to \infty} \frac{u(b_m, b_m, x)}{(b_m, b_m)}
\end{align*}
\]

This equation shows that \( h(x) \) is defined and differentiable on \( U_{x_0'} - x_0 \).

We have proved that we can define a vector field \( \gamma \) over \( V_v \) by (9, 2) and it is differentiable on it.

On the other hand, from the above consideration, we can define a differentiable scalar field \( \sigma_{x_0} \) over \( V_v \) for a point \( x_0 \in V_v \) by

(9, 12) \[ \sigma_{x_0}(x) = \lim_{m \to \infty} \frac{u_m(x)}{(b_m, \omega(f_m(b_m)))}, \quad x \in V_v, \quad b_m \in p^{-1}(x_0), \]

which is everywhere positive. Then, we can define a differentiable covariant vector field \( \tau \) over \( V_v \) as follows: in local coordinates \( x^1, \ldots, x^v \) on \( U \subset V_v \),

(9, 13) \[ \tau_i(x) = \frac{\partial}{\partial x^i} \log \sigma_{x_0}(x) \left( = \frac{\partial}{\partial x^i} \log \sigma_{x_1}(x) \right) \]

\[ x_0, x_1 \in V_m, \quad x \in U, \]

which does not depend on the point \( x_0 \).

Now, in the coordinate neighborhood \( U_{x_0'} \) for sufficiently large \( m \), we must have

(9, 14) \[ g_{ij}(x) = g_{kk}(f_m(x)) \frac{\partial f_m \ast x^k}{\partial x^i} \cdot (x) \cdot \frac{\partial f_m \ast x^k}{\partial x^j} \]

where \( g_{ij}(x) \) are the components of the fundamental tensor of \( V_v \) with respect to the coordinates \( x^1, \ldots, x^v \). Taking a suitable neighborhood \( W \) of \( b_0 \) in \( V_v \), we can consider differentiable functions \( H_{ij}(b, x) \) defined on \( W \times U_{x_0'} \) by

\[ H_{ij}(b, x) = -g_{ij}(x) \]

(9, 15) \[ + g_{kk}(F(b, \omega(x), s(x)) \cdot \frac{\partial}{\partial x^i} x^i(F(b, \omega(x), s(x))) \times \\
\times \frac{\partial}{\partial x^j} x^j(F(b, \omega(x), s(x))), \quad b \in W, \quad x \in U_{x_0'}. \]
where we use $\frac{\partial}{\partial x^\ell}$ conventionally but there will be no confusion.

By (9, 11), (9, 14), we have for sufficiently large $m$ the equation

$$H_{ij}(b_m, x) = 0.$$ 

Then we get easily the equation

$$0 = \lim_{m \to \infty} \frac{H_{ij}(b_m, x)}{u_n(x)} = \gamma_{i, j}(x) + \gamma_{j, i}(x) + \gamma_i(x)\gamma_j(x) + \gamma_j(x)\gamma_i(x),$$

that is

(9, 16) $\gamma_{i, j}(x) + \gamma_{j, i}(x) + \gamma_i(x)\gamma_j(x) + \gamma_j(x)\gamma_i(x) = 0,$

where $\gamma_i(x) = g_{ij}(x)\gamma^j(x)$ and a comma denotes the covariant differentiation of $V_n$. This relations are clearly true on any coordinate neighborhood since the fields $\gamma, \gamma$ are defined on $V_n$ and do not depend on the point $x_m$.

If we define a differentiable contravariant vector field $\xi$ by

$$\xi = \sigma_n \gamma,$$

that is

(9, 17) $\xi(x)(h) = \lim_{m \to \infty} \frac{(f_m \ast h)(x) - h(x)}{(b_0, \chi(f_m)(b_0))},$

where $x \in V_n$, $h$ is any differentiable function defined on an open neighborhood of $x$ and $b_n$ is a fixed point of $B$.

Then we get by (9, 13), (9, 16)

$$\xi_{i, j}(x) + \xi_{j, i}(x) = 0$$

in any local coordinate neighborhood. This is the equation of Killing.

Since we can omit, in the above consideration, the condition that $V_n$ is complete as in §8 by means of a finite number of points in $B$ such as $b_0$, we obtain the classical theorem [6].

**Theorem 6.** Let $V_n$ be a Riemannian manifold and let \{f_m\} be a sequence of motions of $V_n$ which are mutually distinct and weakly converge to the identity transformation. If the tangent unit vector to elementary geodesic arcs $\gamma(b_n, \chi(f_m)(b_n))$ from $b_n$ to $\chi(f_m)(b_0)$ at a fixed point $b_0 \in B$ converge to a tangent vector, then we can obtain a
differentiable tangent vector field which represents an infinitesimal transformation of motion by (9, 17).

§ 10. \( \Phi (b, \xi) \) and holonomy groups. In this paragraph, we shall investigate the automorphisms on \( V^\omega \) which are generalizations of \( \Phi (b, w, s) \) in §8 and the holonomy group of \( V^\omega \).

Let \( W \) be the set of piecewise differentiable arcs parameterized with arclengths in an \( n \)-dimensional Euclidean space. We shall classify the elements of \( W \) as follows: \( W \ni \gamma : 0 < s < l \to \mathbb{R}^n, a = 1, 2, \) are \( \gamma_1 = \gamma_2 \), (1) if there exists a translation such that \( \gamma_i = \gamma \circ \gamma_i \), (2) if for some \( k, c > 0 \) such that \( 0 < k - c < k + c < l_i = l_2 + 2c \), and we have

\[
\begin{align*}
\gamma_1(s) &= \gamma_2(s) & \text{for } 0 < s < k - c, \\
\gamma_1(s) &= \gamma_2(2k - s) & \text{for } k - c < s < k, \\
\gamma_1(s) &= \gamma_2(s - 2c) & \text{for } k - c < s < l_i
\end{align*}
\]

or (3) if there exists a relation between \( \gamma_1 \) and \( \gamma_2 \) exchanged \( \gamma_1 \) and \( \gamma_2 \) in (2).

Let \( \mathbb{W} \) be the set of equivalent classes of \( W \) by the above equivalent relation.

For \( \gamma_1, \gamma_2 \in W \) such that the end point of \( \gamma_1 \) is the starting point of \( \gamma_2 \), \( \gamma = \gamma_1 \gamma_2 \) is usually defined by

\[
\gamma(s) =
\begin{cases}
\gamma_1(s) & 0 < s < l_1, \\
\gamma_2(s - l_1) & l_1 < s < l_1 + l_2.
\end{cases}
\]

We define multiplication in \( \mathbb{W} \) as follows:

\( \xi_1, \xi_2 \in \mathbb{W} \), we take \( \gamma_1 \in \xi_1, \gamma_2 \in \xi_2 \) such that the end point of \( \gamma_1 \) is the starting point of \( \gamma_2 \) and we denote the class containing \( \gamma_1 \gamma_2 \) by \( \xi_1 \cdot \xi_2 \). Clearly \( \xi_1 \cdot \xi_2 \) does not depend on the choice of \( \gamma_1 \in \xi_1 \) and \( \gamma_2 \in \xi_2 \).

We can easily prove that \( \mathbb{W} \) is a group with respect to this multiplication. \( \mathbb{W} \) contains the \( n \)-dimensional translation group \( \mathbb{Z}_n \) of \( \mathbb{R}^n \) as a subgroup.

We define a homomorphism \( \sigma : \mathbb{W} \to \mathbb{Z}_n \) as follows: For any \( \xi \in \mathbb{W} \), let \( \gamma \) be a representative with the minimum length in \( \xi \), and let \( \sigma(\cdot) \) be the translation corresponding to the sensed segment from the starting point to the end point of \( \gamma \). We can easily see that \( \sigma(\cdot) \) does not depend on the choice of \( \gamma \) and \( \sigma \) is a homomorphism onto. Let \( \mathcal{W}_0 \) be the kernel of \( \sigma \). We obtain easily the relations.
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(10, 1) \[ \mathbb{W}_0 \cap \mathbb{I}_n = 1, \]

(10, 2) \[ \mathbb{W} = \mathbb{W}_0 \cdot \mathbb{I}_n = \mathbb{I}_n \cdot \mathbb{W}_0, \]

(10, 3) \[ \sigma(\mathbb{I}_n) = \mathbb{I}_n. \]

Now, for any \( \xi \in \mathbb{W} \), we define a homeomorphism \( \eta'_{\xi} : V_\kappa \to V_\kappa \) as follows: Let \( \eta \in \xi \) be a representative with its end point at the origin \( O \) of \( R^n \). For any point \( b \in V_\kappa \), we take a curve \( C \) in \( V_\kappa \) and a curve \( \overline{C} \) in \( B \) such that

(i) \( \eta(\overline{C}) = C, \)

(ii) the points of \( \overline{C} \) are the parallel displaced frames along \( C \).

(iii) the point \( b \) is the end point of \( \overline{C} \).

(iv) by the linear mapping \( I_b : T_{\eta(b)}(V_\kappa) \to R^n, I_b(e_i(b)) = \mathbf{w}_i \), the tangent unit vector \( C \) at \( \eta(b) \) is transformed to the tangent unit vector to \( \eta \) at \( O \), where \( \mathbf{w}_i \) is the \( i \)-th unit vector at \( O \) of \( R^n \),

and

(v) the development of \( C \) on \( R^n \) so that the condition in (iv) is satisfied at \( \eta(b) \) is \( \eta' \).

As is well known, for \( \eta \) and \( b, C, \overline{C} \) are uniquely determined under these conditions (i)—(v).

Let \( b' \) be the starting point of \( \overline{C} \) which depends only on \( \xi, b \) and put \( b' = \eta'(b) \). \( \eta'_{\xi} \) is clearly a homeomorphism on \( V_\kappa \) and from the above definition it follows that

(10, 4) \[ \eta'_{\xi_1} \circ \eta'_{\xi_2} = \eta'_{\xi_1 \cdot \xi_2}, \quad \xi_1, \xi_2 \in \mathbb{W} \]

(10, 5) \[ r(\alpha) \circ \eta'_{\xi} = \eta'(r^{-1}(\xi)) \circ r(\alpha), \quad \alpha \in O_n. \]

The set \( \mathfrak{K} \) of all the \( \eta'_{\xi}, \xi \in \mathbb{W} \), is a group of automorphism on \( V_\kappa \) and the correspondence \( \eta' : \mathbb{W} \to \mathfrak{K} \) by \( \eta'(\cdot) = \eta'_{\cdot} \) is a homomorphism by (10, 4).

For any \( \mathbf{w} \in S^{n-1}, s \in R \), we get easily the relation

(10, 6) \[ \eta'(b, \mathbf{w}, -s) = \eta'(b, -\mathbf{w}, s). \]

By means of (10, 2), putting

\[ \mathfrak{K}_0 = \eta'(W_0), \quad S_n = \eta'(T_n), \]

\( \mathfrak{K}_0 \) is an invariant subgroup of \( \mathfrak{K} \) and

(10, 7) \[ \mathfrak{K} = \mathfrak{K}_0 \cdot S_n = S_n \cdot \mathfrak{K}_0. \]

Now, for a fixed point \( x \in V_\kappa \), let \( \Omega_x \) be the set of piecewise differen-
tiable closed curves in $V$, starting and ending at $x$ and parameterized with arclength. Classifying the elements of $O_n$ by the equivalent relation (2) which was used when we derived $B$ from $W$ in the beginning of this paragraph, we define a group $H_x$ with multiplication by the usual method in it.

For any $b \in p^{-1}(x)$, $C \in O_n$, we obtain $\bar{C} \subset B$, $\gamma \subset R^n$ such that $C$, $\bar{C}$, $\gamma$ have the above mentioned properties (i)—(v). Then, let $\psi_c : O_n(x) \rightarrow O_n(x)$ be defined by

\[[10,8] \] $\psi_c(b) = \gamma b$, \hspace{1cm} $\xi = \xi(C, b)$

where $\xi$ denotes the class containing $\gamma$ depending on $b$ and $C$. Since by a right translation $r(\alpha)$, $\alpha \in O_n$, a system $\{C, \bar{C}, \xi\}$ is transformed to $\{C, r(\alpha)(\bar{C}), \alpha^{-1}(\xi)\}$, we get

\[[10,9] \] $r(\alpha) \circ \psi_c = \psi_{c} \circ r(\alpha)$.

By definition, we get easily

\[[10,10] \] $\psi_{c_1} \circ \psi_{c_2} = \psi_{c_1 c_2}$.

Since $\psi_c$ depends only on the element in $H_x$ containing $C$, it define a homomorphism of $H_x$ onto a group of automorphisms on $O_n(x)$ by means of (10,10). For any $b \in O_n(x)$, $C \in \xi \in H_x$, let $\beta_\xi$ be defined by

$\psi_c(b) = r(\beta_\xi)(b)$,

then for any $\xi, \xi_2 \in H_x$, we have by (11,9), (11,10)

\[ r(\beta_{\xi_1}(\xi_2))b = r(\beta_{\xi_2}(\xi_1))rb = r(\beta_{\xi_2}(\xi_1))r(\beta_{\xi_1}(\xi_2))b = r(\beta_{\xi_2}(\xi_1))\psi_{c_1}(b) = \psi_{c_2}(r(\beta_{\xi_2}(\xi_1))b) = \psi_{c_1}(\psi_{c_2}(b)) = \psi_{c_1 \circ c_2}(b) = r(\beta_{\xi_2}(\xi_1))b, \quad C_1 \in \xi_1, C_2 \in \xi_2,

hence

\[[10,10] \] $\beta_{\xi_1}(\xi_2) \beta_{\xi_1}(\xi_2) = \beta_{\xi_1 \bullet \xi_2}$.

The transformation $\beta_\xi : H_x \rightarrow O_n$ is a homomorphism. For $b = r(\alpha_1)b$, we have

$\psi_c(b)$

\[ = r(\beta_\xi(\xi))b = r(\beta_\xi(\xi)) (r(\alpha_1)b) = r(\alpha_1 \beta_\xi(\xi))b
\]
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\[ r(a_1) \psi_e(b) = r(a_1) (r(\beta_n(\xi))b) \]
\[ = r(\beta_n(\xi)\alpha_1)b, \]

hence

\[ \beta_{r(a)}(\xi) = \alpha^{-1}\beta_\alpha(\xi)\alpha, \quad \xi \in \mathbb{H}_\alpha, \quad \alpha \in O_n. \]

\[ H_{n,b} = \beta_n(\mathbb{H}_a) \] is the holonomy group of \( V_n \) at \( x \) with respect to \( b \).

With regards to \( \xi(C, b) \), we get analogously the formulas:

\[ \xi(C, r(a)b) = \alpha^{-1}(\xi(C, b)), \quad (10, 12) \]
\[ \xi(C_1C_2, b) = \xi(C_1, \psi_e(b)) \cdot \xi(C_2, b) \]
\[ = (\beta_n(\xi_2^{-1}) \cdot \xi(C_1, b)) \cdot \xi(C_2, b), \quad (10, 13) \]
\[ C_a \in \mathbb{H}_\alpha, \quad a = 1, 2. \]

BIBLIOGRAPHY


(Received June 14, 1955)