On rings satisfying the identity $X^{2k} = X^k$

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Throughout the present paper, $R$ will represent a ring, $E$ the set of idempotents in $R$, and $N$ the set of nilpotents in $R$. Our present objective is to give the conditions for $R$ to satisfy the identity $x^{2k} = x^k$ and to reprove all the results obtained in the previous paper [5], without the extra hypothesis that $R$ is left s-unital.

First, careful scrutiny of the proof of [1, Lemma 1] shows the next

Lemma 1. Let $m$ and $q$ be positive integers, and let $k = q^m$. Suppose that $R$ satisfies the identity $f(x) = 0$, where $f(t)$ is a co-monic polynomial in $tZ[t]$ with degree $\leq m$. If $qR = 0$ then $R$ satisfies the identity $x^{k+1} = x^k$, and therefore $x^{2-k} = x^k$.

Next, we shall prove

Lemma 2. Suppose that $R$ satisfies the identity $f(x) = 0$, where $f(t)$ is a primitive polynomial in $tZ[t]$. Then there exist positive integers $q$ and $h$ such that $(qr)^h = 0$ for all $r \in R$.

Proof. Consider the direct product $S = R^k$, which satisfies the same identity $f(x) = 0$. In case $S$ coincides with its prime radical $P(S)$, $R$ is a nil ring of bounded index. In what follows, we assume that $S$ contains a proper prime ideal $P$, and choose an integer $n_o$ such that $q = |f(n_o)| > 0$. By [2, Theorem 7 (6)], the classical quotient ring of $S/P$ is an Artinian simple ring satisfying the same identity $f(x) = 0$. Hence $qS \subseteq P$, which proves that $qS \subseteq P(S)$. Thus we can find a positive integer $h$ such that $(qr)^h = 0$ for all $r \in R$.

Corollary 1. Suppose that $R$ satisfies the identity $f(x) = 0$, where $f(t)$ is a co-monic polynomial in $tZ[t]$. Then $R$ satisfies the identity $x^{2k} = x^k$ for some positive integer $k$.

Proof. In view of Lemma 2, there exist positive integers $q$ and $h$ such that $(qr)^h = 0$ for all $r \in R$. Let $T$ be the subring of $R$ generated by $|r^h| r \in R$. Then $T$ satisfies the identity $f(x) = 0$ and $q^hT = 0$. Hence,
by Lemma 1, there exists a positive integer $k$ such that $r^{kk} = r^k$ for all $r \in R$.

Now, we can prove our first theorem.

**Theorem 1.** The following conditions are equivalent:

1) There exists a primitive polynomial $f(t)$ in $\mathbb{Z}[t]$ such that $R$ satisfies the identity $f(x) = 0$.

2) There exists a monic polynomial $f(t)$ in $\mathbb{Z}[t]$ such that $R$ satisfies the identity $f(x) = 0$.

3) There exists a co-monadic polynomial $f(t)$ in $\mathbb{Z}[t]$ such that $R$ satisfies the identity $f(x) = 0$.

4) There exists a positive integer $k$ such that $R$ satisfies the identity $x^{2k} = x^k$.

5) $qE = 0$ for some positive integer $q$, and there exists a positive integer $m$ with the following property: For every $r \in R$, there exists a co-monadic polynomial $g(t)$ in $\mathbb{Z}[t]$ with $\deg g(t) \leq m$ such that $g(r) = 0$.

6) The (Jacobson) radical $J$ of $R$ is a nil ideal of bounded index, and there exists a positive integer $k$ such that every primitive homomorphic image of $R$ contains at most $k$ elements.

In case $R$ contains 1, the next is equivalent to each of the above equivalent conditions:

7) The addition of $R$ is equationally definable in terms of the multiplication and the successor operation.

**Proof.** Obviously, 4) $\iff$ 2) $\iff$ 1), and 4) $\iff$ 3) $\iff$ 1).

1) $\iff$ 4). Consider the direct product $S = R^k$, which satisfies the same identity $f(x) = 0$. In case $S$ coincides with its prime radical $P(S)$, there is nothing to prove. Thus, henceforth, we may assume that $S$ contains a proper prime ideal $P$. Choose an integer $n_0$ such that $q = |f(n_0)| > 0$. By [2, Theorem 7 (6)], the classical quotient ring of $S/P$ is an Artinian simple ring satisfying the same identity $f(x) = 0$. Hence the characteristic of $S/P$ is a factor of $q$. Noting that $f(t)$ is primitive, we can easily see that there exists a co-monadic polynomial $g(t)$ in $\mathbb{Z}[t]$ with $\deg g(t) \leq m = \deg f(t)$ such that $S/P$ satisfies the identity $g(x) = 0$. Then, by Lemma 1, there exists a positive integer $l = l(q, m)$ such that $S/P$ satisfies the identity $x^{2l} = x^l$. This proves that $S/P(S)$ satisfies the identity $x^{2l} = x^l$. Then there exists a positive integer $h$ such that $R$ satisfies the identity $(x^l - x^{2l})^h = 0$. Now, by Corollary 1, there exists a positive integer $k$ such that $R$
satisfies the identity $x^{2k} = x^k$.

3) $\Rightarrow$ 5). Let $q = |f(2)|$, and let $g(t) = f(t)$ for all $r \in R$.

5) $\Rightarrow$ 3). Let $f(t) = \prod_{p} \prod_{\alpha = 1}^{m} (t - t^\alpha)^m$, where $p$ ranges over all the prime factors of $q$. We shall show that $R$ satisfies the identity $f(x) = 0$. Now, let $r$ be an arbitrary element of $R$, and let $\langle r \rangle$ be a subdirect sum of subdirectly irreducible rings $R_\alpha$. By 5), there exists a co-monic polynomial $g(t)$ in $tZ[t]$ with $\deg g(t) \leq m$ such that $g(r) = 0$. Let $N_\alpha$ be the set of nilpotents in $R_\alpha$. Then it is easy to see that $a^m = 0$ for all $a \in N_\alpha$, and so $N_\alpha$ satisfies the identity $f(x) = 0$. Now, assume that $R_\alpha$ is not nil. Then, as is easily seen, $R_\alpha$ is a local ring whose radical is $N_\alpha$ and $R_\alpha/N_\alpha = GF(p^a)$ with some prime factor $p$ of $q$ and $a \leq m$. Hence $f(r) = 0$.

4) $\Rightarrow$ 6). This is an easy consequence of Kaplansky's theorem (see, e.g., [2, Theorem 1]).

6) $\Rightarrow$ 3). As is easily seen, every primitive homomorphic image of $R$ satisfies the identity $x^{2k} = x^k$, and so $R/J$ satisfies the same. Hence $R$ satisfies the identity $(x^{k_1} - x^{2-k_2})^h = 0$ for some positive integer $h$.

The latter assertion is clear by [6, Theorem 1].

Following [7], a ring $R$ is called a $\delta$-ring if $R$ contains a finite subset $S$ with the following property: For every $x \in R$, there exists a $p(t) \in Z[t]$ such that $x - x^2 p(x) \in S$. As an application of Theorem 1, we shall prove the following theorem.

**Theorem 2.** Let $R$ be a $\delta$-ring. If there exists a positive integer $q$ such that $|K| \leq q$ for every field $K$ which is a homomorphic image of $R$, then there exists a positive integer $k$ such that $R$ satisfies the identity $x^{2k} = x^k$.

In preparation for proving Theorem 2, we state the next lemma.

**Lemma 3.** Suppose that $R$ contains a finite subset $S$ with the following property: For every $x \in R$, there exists a $p(t) \in Z[t]$ such that $x - x^2 p(x) \in S$. Let $s = |S|$. Then there holds the following:

1. $R$ is a periodic ring and $N$ is finite.
2. There is a positive integer $n$ such that for every $x \in R$ there exists an $f(t) \in Z[t]$ with $x^n = x^{n+1} f(x)$, and then $|N| \leq (s!)^{n-1}s$.

**Proof.** Let $x$ be an arbitrary element of $R$. For each positive integer $i \leq s + 1$, there exists $g_i(t) \in Z[t]$ such that $x^i - x^{2i} g_i(x) \in S$. Then we...
can easily see that there exists a positive integer \( i' \) and \( g(t) \in \mathbb{Z}[t] \) such that \( x^{i'} = x^{i' + 1} g(x) \). Hence \( R \) is periodic by Chacron's theorem (see, e.g., [3, Theorem 1]). Now, let \( a \in N; \ a^k = 0 \). Choose a positive integer \( m \) such that \( 2^m \geq k \). By hypothesis, there exist \( p_1(t), \ldots, p_m(t) \) in \( \mathbb{Z}[t] \) such that \( a_1 = a - a^2 p_1(a) \) and \( a_j = a^{2^{j-1}} p_{j-1}(a) - a^{2^j} p_j(a) \) are in \( S \cap N \) (\( j = 2, \ldots, m \)). Then \( a = a_1 + a_2 + \ldots + a_m \). Again by hypothesis, for each positive integer \( i \leq s + 1 \), there exists \( q_i(t) \in \mathbb{Z}[t] \) such that \( iq_i(t) \in \mathbb{Z}[t] \) and \( a^{s!} q_i(a) \in S \). Then we can easily see that \( (s!) a = a^{s!} q(a) \) with some \( q(t) \in \mathbb{Z}[t] \). This implies that \( a^{s!} q(a) = a^{s!} q(a) \) is \( k \)-nilpotent, and hence the additive order of every element in \( N \) is finite. Combining this with the fact that every element is a sum of elements in \( S \cap N \), we see that \( N \) is finite. Now, we can choose a positive integer \( n \) such that \( a^n = 0 \) for all \( a \in N \). Since \( x - x^{i'} g(x) \in \mathbb{Z}[t] \), we get \( 0 = (x - x^{i'} g(x))^n = x^n - x^{n + 1} f(x) \) with some \( f(t) \in \mathbb{Z}[t] \).

Proof of Theorem 2. Let \( S, s \) and \( n \) be as in Lemma 3. If \( R' \) is an arbitrary homomorphic image of \( R \) and \( N' \) is the set of nilpotents in \( R' \), then \( |N'| \leq (s!)^{n-1}\) by Lemma 3. This together with the structure theorem of primitive rings shows that every primitive homomorphic image of \( R \) is either a periodic field or the full matrix ring \( M_m(K) \), where \( 1 < m \leq n \) and \( K \) is a field with \( |K| \leq (s!)^{n-1} \). Hence, by Theorem 1 6), \( R \) satisfies the identity \( x^{sk} = x^k \) for some positive integer \( k \).

By the proof of Theorem 2, we can easily see the following

Corollary 2. Let \( R \) be a \( \delta \)-ring. If \( R = \langle E \cup N \rangle \) and \( qE = 0 \) for some positive integer \( q \), then there exists a positive integer \( k \) such that \( R \) satisfies the identity \( x^{3k} = x^k \).

Next, by making use of Theorem 1, we shall improve [5, Theorems 1 and 2].

Theorem 3. Suppose that \( R \) satisfies the identity \( f(x) = 0 \), where \( f(t) \) is a primitive polynomial in \( t \mathbb{Z}[t] \).

1. If either \( R \) is normal or \( N^* = \{ x \in R \mid x^2 = 0 \} \) is commutative, then \( N \) is a nil ideal and \( R/N \) satisfies the identity \( x = x^{k+1} \) for some \( k > 1 \).

2. If \( N \) is commutative then \( N \) is a commutative nil ideal and \( R/N \) satisfies the identity \( x = x^{k+1} \) for some \( k > 1 \). If, furthermore, \( [a, x] \cdot x = 0 \) for all \( a \in N \) and \( x \in R \), then \( R \) is commutative.

Proof. By Theorem 1, there exists a positive integer \( k \) such that \( R \)
satisfies the identity $x^{2k} = x^k$.

(1) If $R$ is normal, then $R$ satisfies the identity $[x^k, y] = 0$, and therefore [4, Proposition 2] shows that $N$ is a nil ideal of $R$. On the other hand, if $N^*$ is commutative, then [5, Lemma 2 (2)] shows that $N$ is a nil ideal of $R$. Needless to say, $R/N$ satisfies the identity $x = x^{k+1}$, in either case.

(2) The former assertion is clear by (1), and the latter is immediate by [8, Theorem 1]. (If $a \in N$ and $x \in R$, then $[a, x]^2 = [a, [a, x]] = 0$. Hence, in [5, Theorem 2 (3)], the hypothesis (iv) implies (iii).)

Given $x \in R$, we define inductively $x^{11} = x$, $x^{(k_i)} = x^{k_i-1} \circ x$, where $x \circ y = x + y + xy$. In [5], we introduced the following conditions:

(i) $$(x + x^2 + \cdots + x^n)^m = 0 \text{ for all } x \in R.$$

(*) For any $x, y \in R$, $(x + xy) \circ (y + yx) = 0$ if and only if $x = y$.

In what follows, we shall reprove [5, Theorems 3, 4 and 5] without the hypothesis that $R$ is a left s-unital ring.

**Lemma 4.** Suppose that $R$ satisfies $(i)_{\geq m}$. Then either $R$ is a nil ring of bounded index or there exists a positive integer $q$ such that $qR = 0$.

**Proof.** There exist positive integers $q'$ and $h$ such that $(q'x)^h = 0$ for all $x \in R$, by Lemma 2. If $h > 1$ then $|(q'x)^{h-1}|^2 = 0$, and so $(i)_{\geq m}$ implies that $2^m(q'x)^{h-1} = 0$; hence $(2^m q'x)^{h-1} = 0$. Repeating the same argument, we obtain eventually $2^{m(h-1)}q'x = 0$ for all $x \in R$.

Now, we can improve [5, Theorems 3 and 4] as follows:

**Theorem 4.** Suppose that $R$ satisfies $(i)_{\geq m}$. Then $N$ is a nil ideal and $R = R_1 \oplus R_2$, where $R_1$ is either 0 or a ring of odd characteristic satisfying the identity $x = x^{k+1}$ for some $k > 1$. $R_2 \supseteq N$, and $R_2/N$ is a Boolean ring. If, furthermore, $R$ is normal and $N$ is commutative then $R$ is commutative.

**Proof.** Take Lemma 4 into account and follow the proof of [5, Theorems 3 and 4].

Finally, we shall reprove [5, Theorem 5] without assuming that $R$ is left s-unital.

**Lemma 5.** Let $f(t) = k_1t + k_2t^2 + \cdots + k_mt^m$ be a polynomial in $t\mathbb{Z}[t]$ with $(k_1, k_2) = 1$. If $N$ satisfies the identity $f(x) = 0$, then $N$ satisfies the identities $x^3 = 0 = k_1x + (k_2 - k_1)x^2$. 

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Proof. Let \( a \) be an arbitrary element of \( N \). To see that \( a^3 = 0 \), it suffices to show that if \( a^n = 0 \) with \( n \geq 4 \) then \( a^{n-1} = 0 \). Obviously, \( 0 = f(a^{n-2}) = k_1a^{n-2} + \cdots + k_ma^m = k_1a^{n-1} \). Since \( (k_1, k_2) = 1 \), we obtain \( a^{n-1} = 0 \). Hence \( a^3 = 0 = k_1a + k_2a^2 \), and therefore \( k_1a + (k_2 - k_1)a^2 = k_1a + k_2a^2 - (k_1a + k_1a^2)a = 0 \).

Combining Lemma 5 with Theorem 1, we readily obtain

**Corollary 3.** Let \( f(t) = k_1t + k_2t^2 + \cdots + k_mt^m \) be a polynomial in \( t \mathbb{Z}[t] \) with \( (k_1, k_2) = 1 \). If \( R \) satisfies the identity \( f(x) = 0 \), then \( R \) satisfies the identity \( (x-x^k)^3 = 0 \) for some \( k > 1 \).

**Lemma 6.** Suppose that \( R \) satisfies (i). Then \( N \) is a nil ideal of \( R \) and \( R/N \) is a Boolean ring.

**Proof.** Since \( 6x^2 + 2x^4 = (x+x^2)^2 + (-x + (-x)^2)^2 = 0 \) and \( 4x + 4x^3 = (x+x^2)^2 - (-x + (-x)^2)^2 = 0 \), we get \( 2x^2 - 2x^4 = (6x^2 + 2x^4) - (4x^2 + 4x^3)x = 0 \), and therefore \( 8x^2 = (6x^2 + 2x^4) + (2x^2 - 2x^4) = 0 \). Hence \( 2^3x = 8x - 2(4x + 4x^3) = -8x^2 = 0 \), and therefore \( N \) is a nil ideal and \( R/N \) is a Boolean ring by [5, Lemma 3].

**Lemma 7.** If \( R \) satisfies (\( \ast \)), then \( R \) is normal.

**Proof.** The assertion has been proved in the proof of [5, Theorem 5].

We are now ready to prove the following

**Theorem 5.** A ring \( R \) satisfies the condition (\( \ast \)) if and only if 1) \( R \) is commutative and \( R/N \) is a Boolean ring, and 2) \( a^{(2)} = 0 \) for all \( a \in N \).

**Proof.** Since the "if" part has been proved in the proof of [5, Theorem 5], it remains only to prove the "only if" part. Obviously, \( (\ast) \) implies \( (i) \), and so \( N \) is a nil ideal of \( R \) and \( R/N \) is a Boolean ring by Lemma 6. Noting that \( R \) satisfies the identity \( 2x + 3x^2 + 2x^3 + x^4 = (x + x^2)^2 = 0 \), we can conclude that \( a^{(2)} = 0 \) for all \( a \in N \) (Lemma 5). Therefore, for any \( a, b \in N \), we get \( a \circ b = a \circ (a \circ b)^{(2)} \circ b = b \circ a \), which shows that \( N \) is commutative. Furthermore, \( R \) is normal by Lemma 7, and so \( R \) is commutative.

**References**

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