A note on universally going-down

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We shall assume throughout that all rings and algebras are commutative with identity and that all homomorphisms are unital. Recall that a ring-homomorphism $R \to T$ is said to be universally going-down in case $S \to S \otimes_{R} T$ satisfies going-down (henceforth abbreviated GD) for each change of base $R \to S$. This concept was introduced in [7] and studied extensively in [4]. The most natural examples of universally going-down homomorphisms $R \to T$ arise when dim $(R) = 0$ or $T$ is $R$-flat [4, Proposition 3.3]. It is interesting that, in some cases, universally going-down reduces to one of these two archetypes. For zero-dimensionality, this is essentially well known and summarized in Proposition 1 below. Our main result, Theorem 3, is that any universally going-down overring extension of an integrally closed domain must be flat. This will follow easily from the main results of [4] and [5], with which we assume familiarity.

Proposition 1. For a ring $R$, the following five conditions are equivalent:

1. $R \to T$ is universally going-down for each $R$-algebra $T$;
2. The canonical map $R \to R/P$ is universally going-down for each nonminimal $P \in \text{Spec} (R)$;
3. The canonical map $R \to R/P$ satisfies GD for each non-minimal $P \in \text{Spec} (R)$;
4. $R/P$ is $R$-flat for each nonminimal $P \in \text{Spec} (R)$;
5. $\text{dim} (R) = 0$.

Proof. The equivalence of $(3), (4), and (5)$ was observed in [3, Proposition 2.1]. As noted above, [4, Proposition 3.3] yields that $(5) \Leftrightarrow (1)$; and $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ trivially, to complete the proof.

The above theme that (universally) GD-behavior often entails flatness was also noted in [3, Remark 2.6 (c)]. It was shown there that if $R$ is a reduced ring, then the canonical map $R \to R/P$ satisfies (universally) GD for each nonmaximal $P \in \text{Spec} (R)$ (if and) only if each such $R/P$ is $R$-flat. As this result is false without the "reduced" hypothesis [3, Remark 2.6 (b)], we are motivated to consider the "reduced" case of Proposition 1.
Corollary 2. For a ring $R$, the following six conditions are equivalent:

(1) $R$ is reduced and $R \to T$ is universally going-down for each $R$-algebra $T$;

(2) Each $R$-algebra is $R$-flat;

(3) $R$ is reduced and the canonical map $R \to R/P$ is universally going-down for each maximal ideal $P$ of $R$;

(4) $R$ is reduced and the canonical map $R \to R/P$ satisfies GD for each maximal ideal $P$ of $R$;

(5) $R/P$ is $R$-flat for each maximal ideal $P$ of $R$;

(6) $R$ is von Neumann regular (i.e., absolutely flat).

Proof. As cited in [3], the equivalence $(2) \Leftrightarrow (6)$ is in well known work of Harada and Auslander. Also, Akiba [1, Corollary 4] (and, much later, [3, Remark 2.6 (e)]) established $(6) \Leftrightarrow (5)$. Moreover, since $(6)$ is well known to be equivalent to the condition that $R$ be reduced and zero-dimensional, Proposition 1 yields $(1) \Leftrightarrow (6)$. Similarly, $(4) \Leftrightarrow (6)$, as it is easy to see that the GD condition in $(4)$ implies $\dim(R) = 0$. Finally, $(1) \Leftrightarrow (3) \Leftrightarrow (4)$ trivially, completing the proof.

Before stating our main result, we recall a definition from [4]. A ring-homomorphism $f: R \to T$ is said to be quasi-going-up (in short, QGU) if, for each pair of primes $P_1 \subset P_2$ of $R$ such that $f(P_1) \not= T$ and each $Q_1 \in \text{Spec}(T)$ such that $f^{-1}(Q_1) = P_1$, there exists $Q_2 \in \text{Spec}(T)$ such that $Q_1 \subset Q_2$ and $f^{-1}(Q_2) = P_2$. The key fact used in the next proof is that universally going-down overring extensions of domains satisfy this weak form of going-up, even after change of base.

Theorem 3. Let $T$ be an overring of a domain $R$ such that $R$ is integrally closed in $T$. Then the inclusion map $R \to T$ is universally going-down (if and only if $T$ is $R$-flat.

Proof. The parenthetic assertion holds since flat implies universally going-down. Conversely, to show $T$ is $R$-flat, a criterion of Richman (cf. proof of [8, Theorem 2]) reduces us to verifying the following: if $P \in \text{Spec}(R)$ and $PT \neq T$, then $T_P = R_P$. (As usual, $T_P$ denotes the ring of fractions $T_{h, P}$. For any such $P$, the hypothesis yields that $R_P$ is integrally closed in $T_P$. Thus, it suffices to verify that, for each $P \in \text{Spec}(R)$ such that $PT \neq T$, one has that $T_P$ is integral over $R_P$. In the terminology of [5], our task is thus to show that the inclusion map $f: R \to T$ is quasi-
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integral. By the main result of [5], namely [5, Theorem 3.2], this is equivalent to showing that \( f \) is universally QGU, in the sense that \( S \rightarrow S \otimes T \) is QGU for each change of base \( R \rightarrow S \). However, the main result of [4], namely [4, Theorem 3.17], assures that each universally going-down overring extension of a domain is universally QGU. The proof is complete.

**Remark 4.** (a) A result of Papick (cf. [6, (3.14)]) asserts that if \( T \) is an overring of a coherent domain \( R \) such that \( R \) is integrally closed in \( T \), then \( R \subseteq T \) satisfies GD (if and) only if \( T \) is \( R \)-flat. One may regard the assertion of Theorem 3 in the same vein, where the finiteness hypothesis of coherence has been eliminated, at the expense of enhancing the GD hypothesis to universally going-down.

(b) One way to motivate the "integrated closed in" hypothesis in Theorem 3 is via Corollary 2, for any von Neumann regular ring is trivially integrally closed. Another way is to note that the "dual" situation, that of an integral overring extension of domains that is universally going-down, has been extensively characterized (cf. [4, Corollaries 3.19 and 3.20]).

(c) The above "flat" impact of universally going-down should be contrasted with the effect of another type of "enhanced GD" condition considered in some of our recent work. Let \( R \) be a domain such that \( A \subseteq B \) satisfies GD for all pairs \( A \subseteq B \) of subrings of \( R \). Then by [2, Theorem 2.1 and Proposition 2.5], \( \dim(A) \leq 1 \) and \( \dim(B) \leq 1 \) for all subrings \( A \subseteq B \) of \( R \), but it need not follow that \( B \) is \( A \)-flat.

**References**

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