Operations associated with the G-equivariant unitary cobordism theory

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OPERATIONS ASSOCIATED WITH
THE G-EQUIVARIANT
UNITARY COBORDISM THEORY

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Introduction. Let $G$ be a compact abelian Lie group. In the previous paper [4] we have introduced a $G$-equivariant cohomology theory which is concerned with the $G$-equivariant unitary cobordism theory. In the equivariant cohomology theory there is the splitting principle and Chern classes are defined for complex $G$-vector bundles.

In this paper we shall study on cohomology operations in the equivariant cohomology theory. In §1 we consider Landweber-Novikov operations in our equivariant cohomology theory. And, in §2 we observe mod $p$ Steenrod operations in the $G$-equivariant unitary cobordism theory and research on them in connection with the Landweber-Novikov operations introduced in §1.

1. Landweber-Novikov operations. Let $G$ be a compact abelian Lie group. Let $U^*_c(-)$ and $K^*_c(-)$ be the $G$-equivariant unitary cobordism theory and the $G$-equivariant complex $K$-theory, respectively. By making use of Thom classes in $K^*_c$-theory, we can get a natural multiplicative transformation

$$\mu_c: U^*_c(-) \to K^*_c(-)$$

of the cohomologies (cf. [3], [4]). We take up a multiplicative set $T_k$ consisting of all one dimensional representations in the representation ring $R(G) \cong K_c(pt)$ and we consider a multiplicative system $T = \mu_c^{-1}(T_k)$ in $U^*_c$. Then our multiplicative $G$-equivariant cohomology theory $h^*_c(-)$ is defined by

$$h^*_c(-) = T^{-1}U^*_c(-),$$

where $T^{-1}U^*_c(-)$ means a ring localized by the multiplicative system $T$.

Using the local triviality of complex $G$-vector bundles [5] and Theorem 4.5 in [4] we obtain the following splitting principle:

Proposition 1.1. Let $\xi$ be an $n$-dimensional complex $G$-vector bundle over a compact $G$-space $X$. Then there exist a compact $G$-space $F(\xi)$, a $G$-map $\pi: F(\xi) \to X$ and $n$ complex $G$-line bundles $\xi_1, \ldots, \xi_n$ over $F(\xi)$ satisfying the following conditions:

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1) \( \pi^*: h^*_c(X) \to h^*_c(F(\xi)) \) is a monomorphism.
2) \( \pi^*(\xi) \) is isomorphic to the sum \( \xi_1 \oplus \cdots \oplus \xi_n \).

**Proposition 1.2.** Let \( \xi \) and \( \eta \) be \( n \) and \( m \)-dimensional complex \( G \)-vector bundles over a compact \( G \)-space \( X \), respectively. Then there exist a compact \( G \)-space \( F \) and a \( G \)-map \( \pi: F \to X \) satisfying the following conditions:

1) \( \pi^*: h^*_c(X) \to h^*_c(F) \) is a monomorphism.
2) \( \pi^*(\xi) \) and \( \pi^*(\eta) \) are isomorphic to the sums of \( n \) and \( m \) \( G \)-line bundles over \( F \), respectively.

Furthermore we have \( G \)-equivariant Chern classes \( c_i^o(\xi) \in h^*_c(X) \), \( 0 \leq i \leq n \), \( c_i^o(\xi) = 1 \), of an \( n \)-dimensional complex \( G \)-vector bundle \( \xi \) over a compact \( G \)-space \( X \).

We now define Landweber-Novikov operations \([1, 9]\) in the cohomology theory \( h^*_c(-) \) and call up their basic properties. Let \( t = (t_1, t_2, \ldots) \) be a sequence of indeterminates. Assigning \( \deg t_i = -2i \) for each \( i \geq 1 \), \( U^*_c(-)[[t]] \) and \( h^*_c(-)[[t]] \) become multiplicative \( G \)-equivariant cohomology theories.

Let \( \xi \) be an \( n \)-dimensional complex \( G \)-vector bundle over a compact \( G \)-space \( X \) and let \( \pi: F(\xi) \to X \) and \( \xi_1, \ldots, \xi_n \) be ones of Proposition 1.1. Consider the following

\[
\prod_{i=1}^n (1 + e(\xi_i)t_1 + \cdots + e(\xi_i)^k t_k + \cdots) \in U^*_c(F(\xi))[[t]]
\]

and

\[
\prod_{i=1}^n (1 + c_1^o(\xi_i)t_1 + \cdots + c_1^o(\xi_i)^k t_k + \cdots) \in h^*_c(F(\xi))[[t]],
\]

where \( e(\xi_i) \in U^*_c(F(\xi)) \) is the Euler class of \( \xi_i \) and \( c_1^o(\xi_i) = \frac{e(\xi_i)}{1} \in h^*_c(F(\xi)) \) is the first Chern class of \( \xi_i \).

Given an \( n \)-tuple \( \iota = (i_1, \ldots, i_n) \) of non-negative integers, denote by

\[
\sum x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}
\]

the least symmetric polynomial in variables \( x_1, \ldots, x_n \) which contains the term \( x_1^{i_1} \cdots x_n^{i_n} \). The symmetric polynomial can be written as a polynomial \( P(\sigma_1, \ldots, \sigma_n) \) in the elementary symmetric functions \( \sigma_1, \ldots, \sigma_n \) of the variables \( x_1, \ldots, x_n \):
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$P(\sigma_1, \ldots, \sigma_n) = \sum x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$.

As for the coefficient of $t_1 t_2 \cdots t_n$ in the sequence (1), by making use of the splitting principle, we get the equality

$$\sum c_i(\xi_1)^{i_1} c_i(\xi_2)^{i_2} \cdots c_i(\xi_n)^{i_n} = P(\sigma_1, \ldots, \sigma_n) = \pi^* P(c_i(\xi), \ldots, c_n(\xi)).$$

where $\sigma_k = \sigma_k(c_i(\xi_1), \ldots, c_i(\xi_n))$ is the $k$-th elementary symmetric function of the classes $c_i(\xi_1), \ldots, c_i(\xi_n)$.

Let us define the total Chern class $c(\xi)$ of $\xi$ in the theory $h_\xi^*(\mathbb{C})[[t]]$ by

$$c(\xi) = \sum_i P(c_i(\xi), \ldots, c_n(\xi)) \ t_i \in h_\xi^*(X)[[t]]$$

where $t_i = t_1 t_2 \cdots t_n$. Then, in virtue of the naturality of Euler classes of $G$-line bundles, the splitting principle and the external product we obtain

Proposition 1.3. The total Chern classes satisfy the following properties:

1. (naturality) $c(\xi^* (\eta)) = f^* (c(\xi))$.

2. (multiplicativity) $c(\xi \times \eta) = c(\xi) \times c(\eta)$.

3. (normality) $c^0(\xi) = 1$.

where $\xi : pt \times C \to pt$ is the trivial $G$-line bundle over a point.

4. If $\xi$ is a complex $G$-line bundle, then

$$c(\xi) = 1 + c_i(\xi) t_i + \cdots + c_n(\xi) t_n + \cdots.$$ 

Let $T(\xi)$ be the Thom space of an $n$-dimensional complex $G$-vector bundle $\xi$ over a compact $G$-space $X$. Then, by making use of the Thom isomorphism

$$\phi(\xi) : h_\xi^*(X) \to \overline{h}_\xi^{*+2n}(T(\xi)),$$

we obtain the Thom isomorphism

$$\phi(\xi) : h_\xi^*(X)[[t]] \to \overline{h}_\xi^{*+2n}(T(\xi))[[t]],$$

which is defined by

$$\phi(\xi)(\sum a_{i_1 \cdots i_k} t_1^{i_1} \cdots t_k^{i_k}) = \sum \phi(\xi)(a_{i_1 \cdots i_k}) t_1^{i_1} \cdots t_k^{i_k}.$$ 

Put

$$s_i(\xi) = \phi(\xi)(c_i(\xi)) \in \overline{h}_\xi^*(T(\xi))[[t]].$$
Then we have

**Proposition 1.4.** The classes \( s_i(\cdot) \) satisfy the following properties:

1. (naturality) \( s_i(f^*(\xi)) = f^*(s_i(\xi)) \).
2. (multiplicativity) \( s_i(\xi \times \eta) = s_i(\xi) \times s_i(\eta) \).
3. (normality) \( s_i(\epsilon) = t_h(\epsilon) \in \mathbb{H}_c^i(S^2) \).
4. If \( \xi \) is a complex \( G \)-line bundle, then

\[
s_i(\xi) = t_h(\xi) + t_h(\xi)^2 t_1 + \cdots + t_h(\xi)^k t_{k-1} + \cdots
\]

where \( t_h(\xi) \) is the Thom class of \( \xi \) in the theory \( h^*_c(\cdot) \).

Let \( \gamma^n_c \) be the universal complex \( G \)-vector bundle and denote by \( M_n(G) \) the Thom space of \( \gamma^n_c \). Let \( W \) be a complex \( G \)-module and \( G_n(W) \) the Grassmann manifold of complex \( n \)-planes. Then \( \gamma^n_c \) and \( M_n(G) \) are the limit of the canonical \( n \)-dimensional \( G \)-vector bundle

\[
\gamma^n_c(W) = (E_n(W), \pi, G_n(W))
\]

and the Thom space \( M_n(W) = T(\gamma^n_c(W)) \), respectively.

Let \( x \in U^n_c(X) \) be represented by \( f: V^c \times X^+ \to M_{1+n}(W) \subset M_{1+n}(G) \), where \( X^+ = X \cup \{ \infty \} \) (disjoint union), \( V^c \) means the one point compactification of a complex \( G \)-module \( V \) and \( \| V \| = \dim_c V \). Defining

\[
s_i: U^n_c(X) \to h^*_c(X)[[t]]
\]

by

\[
s_i(x) = \phi_i(V)^{-1} f^*(s_i(\gamma^n_c(W))),
\]

we obtain a natural transformation

\[
s_i: U^n_c(-) \to h^*_c(-)[[t]]
\]

of \( G \)-equivariant cohomology theories.

**Proposition 1.5.** The natural transformation \( s_i \), has the following properties:

1. (naturality) \( s_i(g^*(x)) = g^*(s_i(x)) \).
2. (multiplicativity) \( s_i(xy) = s_i(x)s_i(y) \).
3. (normality) i) \( s_i(t(\xi)) = s_i(\xi) \) for the Thom class \( t(\xi) \in \tilde{U}^n_c(T(\xi)) \) of an \( n \)-dimensional complex \( G \)-vector bundle \( \xi \), ii) \( s_i(1) = 1 \), and

\[
iii) s_i(V) = t_h(V).
\]
Let $\omega = (\omega_1, \omega_2, \ldots)$ be a sequence of non-negative integers with $\omega_1 = 0$ except for a finite number of terms. $|\omega| = \sum \omega_i$ and $t^\omega = \prod_i t_i^{\omega_i}$. Put
\[
s_\omega(x) = \sum_{\omega} s_\omega(x) t^\omega
\]
for $x \in \mathcal{U}_C^*(X)$. Then, from the properties of $s_\omega$, it follows

**Theorem 1.6.** For each sequence $\omega = (\omega_1, \omega_2, \ldots)$ there exists an operation
\[
s_\omega: \mathcal{U}_C^*(-) \to \mathcal{H}_C^{*+2|\omega|-}(*)
\]
with the following properties:

1. (natural) $s_\omega(g^*(x)) = g^*(s_\omega(x))$.
2. (multiplicative) $s_\omega(xy) = \sum_{\alpha + \beta = \omega} s_\alpha(x) s_\beta(y)$

where $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots)$.

3. $s_\omega(x) = \frac{x}{1}$ for $0 = (0, 0, \ldots)$.

4. (stable) $s_\omega(\sigma(V)) = \sigma_\omega(V) s_\omega$, where $\sigma(V)$ and $\sigma_\omega(V)$ are suspension isomorphisms in the theories $\mathcal{U}_C^*(-)$ and $\mathcal{H}_C^*(-)$.

5. If $\xi$ is 1-dimensional, then
\[
s_\omega(t(\xi)) = \begin{cases} t_\alpha(\xi)^{\alpha_{i+1}} & \text{for } \alpha = (0, \ldots, 0, \alpha_i, 0, 0, \ldots) \\ 0 & \text{otherwise.} \end{cases}
\]

2. **Steenrod operations.** In this section we observe the mod $p$ Steenrod operations in the theory $\mathcal{U}_C^*(-)$ and reseach on them in connection with the Landweber-Novikov operations which are introduced in the previous section.

Let $G$ be a compact Lie group and $Z_p$ a cyclic group of order $p$ with a generator $\rho$. By a $(G, Z_p)$-space $X$ we mean a Hausdorff space $X$ having both actions of $G$ and $Z_p$ which commute. Let $V$ be a complex $G$-module. Throughout this section we only treat finite dimensional complex $G$-modules. We consider the $G$-module $V$ a $(G, Z_p)$-space with a $Z_p$ action defined by $\rho^k v = e^{\frac{2\pi i}{k} v} (v \in V)$. Then $S(V)^* = S(V) \cup \{\infty\}$ is a pointed $(G, Z_p)$-space with a fixed base point $\infty$, where $S(V)$ is the unit sphere in $V$.

**Example 1.** For a pointed $G$-space $X$, the $p$-fold reduced join $\overline{\bigwedge} X = X \wedge \cdots \wedge X$ is a pointed $(G, Z_p)$-space with a $Z_p$-action defined by $\rho(x_1 \wedge \cdots \wedge x_p)$.
\( \wedge x_\rho = x_1 \wedge \cdots \wedge x_\rho \wedge x_1 \). We consider the \( p \)-fold product \( \overset{\rho}{\bigotimes} X = X \times \cdots \times X \) a \((G, Z_\rho)\)-space for a \( G \)-space \( X \), too.

**Example 2.** Let \( \xi : E \to X \) be a complex \( G \)-vector bundle and denote by \( \overset{\rho}{\bigotimes} \xi \) the \( p \)-fold product bundle of \( \xi \). Then the total space \( E(\overset{\rho}{\bigotimes} \xi) = E \times \cdots \times E \) of \( \overset{\rho}{\bigotimes} \xi \) is a \((G, Z_\rho)\)-space with a \( Z_\rho \)-action defined by \( \rho(v_1, \ldots, v_\rho) = (v_2, \ldots, v_\rho, v_1) \).

Let us define a \( G \)-space and a pointed \( G \)-space as follows:

\[ E_v(X) = (S(V) \times X)/Z_\rho \quad \text{for \((G, Z_\rho)\)-space} \ X, \]

and

\[ \tilde{E}_v(X) = (S(V)^+ \wedge X)/Z_\rho \quad \text{for pointed \((G, Z_\rho)\)-space} \ X. \]

Then we have

**Proposition 2.1.** For a \((G, Z_\rho)\)-space \( X \), there holds

\[ \tilde{E}_v(X^+) = E_v(X)^+. \]

**Proposition 2.2.** For a complex \( G \)-vector bundle \( \xi \) over a compact \( G \)-space \( X \)

\[ E_v(\overset{\rho}{\bigotimes} \xi) : E_v(E(\overset{\rho}{\bigotimes} \xi)) \to E_v(\overset{\rho}{\bigotimes} X) \]

is a complex \( G \)-vector bundle.

Let \( \mathcal{C}(G, Z_\rho) \) be the category of pointed \((G, Z_\rho)\)-spaces and \( \mathcal{C}(G) \) the category of pointed \( G \)-spaces. Then \( \tilde{E}_v : \mathcal{C}(G, Z_\rho) \to \mathcal{C}(G) \) is a covariant functor.

Furthermore we have

**Proposition 2.3.** If \( \xi \) is a complex \((G, Z_\rho)\)-vector bundle over a compact \((G, Z_\rho)\)-space \( X \), then

\[ E_v(\xi) : E_v(E(\xi)) \to E_v(X) \]

is a complex \( G \)-vector bundle. And, as for the Thom spaces of them, it follows that
\[ T(E_v(\xi)) = \tilde{E}_v(T(\xi)). \]

**Proposition 2.4.** For a pair \((X, A)\) of a \((G, Z_p)\)-space and its subspace, there exist \(G\)-homeomorphisms
\[ \tilde{E}_v(X/A) \simeq E_v(X)/E_v(A) \simeq \tilde{E}_v(X^+)/\tilde{E}_v(A^+). \]

**Proposition 2.5.** For a pointed \(G\)-space \(X\) with the trivial \(Z_p\)-action and a pointed \((G, Z_p)\)-space \(Y\), there exists a \(G\)-homeomorphism
\[ \tilde{E}_v(Y \wedge X) \simeq \tilde{E}_v(Y) \wedge X. \]

**Proposition 2.6.** For a pointed \(G\)-space \(X\) with the trivial \(Z_p\)-action and a \(G\)-module \(W\), there exists a \(G\)-homeomorphism
\[ \tilde{E}_v((\mathcal{O} W^c) \wedge X) \simeq T(E_v(\mathcal{O} W) \times X)/T(E_v(\mathcal{O} W) \times \ast_x), \]
where \(W^c\) means the one point compactification of \(W\).

**Proof.** We have the following \(G\)-homeomorphisms
\[
\tilde{E}_v((\mathcal{O} W^c) \wedge X) \simeq \tilde{E}_v((\mathcal{O} W^c) \wedge X) \quad \text{(by 2.5)}
\]
\[ \simeq \tilde{E}_v((\mathcal{O} W^c)^+) \wedge X \]
\[ \simeq E_v((\mathcal{O} W^c)^+) \wedge X \quad \text{(by 2.1)}
\]
\[ \simeq T(E_v(\mathcal{O} W)) \wedge X \quad \text{(by 2.2)}
\]
\[ = T(E_v(\mathcal{O} W)) \wedge (X^+/\ast_x)
\]
\[ = T(E_v(\mathcal{O} W)) \wedge X^+/T(E_v(\mathcal{O} W)) \wedge \ast_x^\xi
\]
\[ = T(E_v(\mathcal{O} W) \times X)/T(E_v(\mathcal{O} W) \times \ast_x). \quad \text{q.e.d.} \]

By the same way as in the non-equivariant case we have the following Thom isomorphism theorem of a pair (cf. [2]):

**Theorem 2.7.** For an \(n\)-dimensional complex \(G\)-vector bundle \(\xi\) over a compact \(G\)-space \(X\) and a closed \(G\)-subspace \(A\) of \(X\), the Thom homomorphism
\[ \phi: U^*_v(X. A) \to U^{*+2n}_v(T(\xi). T(\xi | A)) \]
is an isomorphism.

In virtue of Proposition 2.3, for a \(G\)-module \(W\) and a pointed \(G\)-space
$X$ with the trivial $\mathbb{Z}_p$-action,

$$\times^\rho W : E_\nu((\times^\rho W) \times X) \to E_\nu((\star, \ldots, \star) \times X)$$

is a $G$-vector bundle. Therefore, by making use of Theorem 2.7 and Propositions 2.4 and 2.6, we obtain a Thom isomorphism

$$\phi : \bar{U}_G^\rho(\bar{E}_\nu(X)) \to \bar{U}_G^\rho(\bar{E}_\nu((\times^\rho W^c) \wedge X)).$$

We now would like to define the external mod $p$ Steenrod operation

$$P^p_V : \bar{U}_G^{2p}(X) \to \bar{U}_G^{2p}(E_\nu(X))$$

for each $G$-module $V$ and a pointed $G$-space $X$.

Let $x \in U_G^{2p}(X)$ be represented by $f : W^c \wedge X \to M_{1w^*+k} \subset M_{1w^*+k} (G)$. Consider the composition of $G$-maps

$$\bar{E}_\nu(\times^\rho f) : \bar{E}_\nu((\times^\rho (W^c \wedge X)) \to \bar{E}_\nu((\times^\rho M_{1w^*+k}(U)))$$

$$= \bar{E}_\nu(T(\times^\rho \gamma^{1w^*+k}_G(U)))$$

$$= T(E_\nu(\times^\rho \gamma^{1w^*+k}_G(U))) \quad (\text{by 2.3})$$

$$\mu_p \quad T(\gamma^{1w^*+k}_G) = M_{1w^*+k}(G),$$

where $\mu_p$ is the map of Thom spaces induced by the classifying map of the complex $G$-vector bundle $E_\nu((\times^\rho \gamma^{1w^*+k}_G(U)))$. The map $\mu_p$ represents the Thom class

$$[\mu_p] = t(E_\nu(\times^\rho \gamma^{1w^*+k}_G(U))) \in \bar{U}_G^{2p}(T(E_\nu((\times^\rho \gamma^{1w^*+k}_G(U)))).$$

Define a map $\tilde{d} : (\times^\rho W^c) \wedge X \to \times^\rho (W^c \wedge X)$ by $\tilde{d}(w_1, \ldots, w_p) \wedge x = (w_1 \wedge x) \wedge \cdots \wedge (w_p \wedge x)$. Then we get a $G$-map

$$\bar{E}_\nu(\tilde{d}) : \bar{E}_\nu((\times^\rho W^c) \wedge X) \to \bar{E}_\nu((\times^\rho (W^c \wedge X)).$$

Now we define $P^p_V(x)$ by

$$P^p_V(x) = \phi^{-1}\bar{E}_\nu(\tilde{d}) \star \bar{E}_\nu(\times^\rho f) \star t(E_\nu((\times^\rho \gamma^{1w^*+k}_G(U)))).$$

And we have the following properties:

**Proposition 2.8.** For a $G$-module $V$ there exists an operator

http://escholarship.lib.okayama-u.ac.jp/mjou/vol30/iss1/16
\[ P_v : U^*_v(-) \to U^*_v(E_v(-)) \]

with the following properties:

1. (naturality) \( P^{2k}_v(*)(x) = E_v(h)^*P^{2k}_v(x) \).

2. (multiplicativity) For \( x \in U^*_v(X) \) and \( y \in U^*_v(Y) \)

\[ P^{2k+2l}_v(x \times y) = P^{2k}_v(x) \times P^{2l}_v(y). \]

3. For the Thom class \( t(\xi) \in \widetilde{U}^*_v(T(\xi)) \) of a \( k \)-dimensional \( G \)-vector bundle \( \xi \),

\[ P^{2k}_v(t(\xi)) = \tilde{E}_v(\tilde{d})^*(t(E_v(\chi(x)))) = t(E_v(\xi)). \]

Let \( L \) be the canonical 1-dimensional \( Z_\rho \)-module and consider it a trivial \( G \)-module. Put

\[ \Delta = L \oplus L^2 \oplus \cdots \oplus L^{p-1}. \]

Then we obtain

**Proposition 2.9.** Let \( \xi \) be a complex \( G \)-vector bundle over a compact \( G \)-space \( X \). Consider the \( p \)-fold sum \( \xi \oplus \cdots \oplus \xi \) a \( (G, Z_\rho) \)-bundle over \( X \) with a \( Z_\rho \)-action defined by \( \rho(v_1, \ldots, v_p) = (v_2, \ldots, v_p, v_1) \) for \( (v_1, \ldots, v_p) \in E(\xi \oplus \cdots \oplus \xi) \). Then it follows that

1. the vector bundles \( \xi \oplus \cdots \oplus \xi \) and \( \xi \otimes (C \oplus \Delta) \) are \( (G, Z_\rho) \)-isomorphic, and

2. the diagram

\[
\begin{array}{ccc}
\xi & \xrightarrow{\tilde{d}} & \xi \oplus \cdots \oplus \xi \\
\downarrow^i & & \downarrow^{\cong} \\
\xi \otimes (C \oplus \Delta) & \cong & \xi \otimes (C \oplus \Delta)
\end{array}
\]

is \( G \)-homotopy commutative, where \( \tilde{d} \) is the diagonal map and \( i \) is the natural inclusion defined by \( i(v) = v \otimes 1 \in \xi \otimes C \).

**Proof.** (1) Let us consider a \((p, p)\)-matrix \( A \) and a unitary matrix \( U = (u_{ij}) \) such that

\[
A = \begin{pmatrix}
0 & \cdots & 0 & 1 \\
1 & \cdots & 0 \\
1 & & & \\
0 & \cdots & 1 & 0
\end{pmatrix}, \quad U^{-1}AU = \begin{pmatrix}
1 & 0 \\
\rho & 0 \\
0 & \rho^{p-1}
\end{pmatrix}
\]
Then a $(G, Z_ρ)$-bundle isomorphism
\[ h : \xi \oplus \cdots \oplus \xi \to \xi \otimes (C \oplus Δ) = \xi \otimes C \oplus \xi \otimes L \oplus \cdots \oplus \xi \otimes L^{p-1} \]
and its inverse $h^{-1}$ are given by
\[ h(v_1, \ldots, v_ρ) = (\sum_{j=1}^{ρ} u_{j1} v_j \otimes 1, \sum_{j=1}^{ρ} u_{j2} v_j \otimes 1, \ldots, \sum_{j=1}^{ρ} u_{jρ} v_j \otimes 1) \]
and
\[ h^{-1}(v_1 \otimes z_1, \ldots, v_ρ \otimes z_ρ) = (\sum_{j=1}^{ρ} u'_{j1} z_j v_j, \sum_{j=1}^{ρ} u'_{j2} z_j v_j, \ldots, \sum_{j=1}^{ρ} u'_{jρ} z_j v_j) \]
for $(v_1, \ldots, v_ρ) \in \xi \oplus \cdots \oplus \xi$ and $(v_1 \otimes z_1, \ldots, v_ρ \otimes z_ρ) \in \xi \otimes (C \oplus Δ)$, where $U^{-1} = (u'_{ij})$.

(2) Since there holds $h_ρ = ρh$ for the generator $ρ \in Z_ρ$, it follows that
\[
h_ρ(v) = h_ρ(v_1, \ldots, v_ρ) = (\sum_{j=1}^{ρ} u_{j1} v \otimes 1, \sum_{j=1}^{ρ} u_{jρ} v \otimes 1)
= ρ(\sum_{j=1}^{ρ} u_{j1} v \otimes 1, \sum_{j=1}^{ρ} u_{jρ} v \otimes 1)
= (\sum_{j=1}^{ρ} u_{j1} v \otimes 1, \sum_{j=1}^{ρ} u_{jρ} v \otimes ρ^{-1} \cdot 1).\]
Hence we have
\[
(\sum_{j=1}^{ρ} u_{jk})(v \otimes 1) = ρ^{k-1}(\sum_{j=1}^{ρ} u_{jk})(v \otimes 1) \quad \text{in } \xi \otimes L^{k-1}.
\]
This implies
\[ \sum_{j=1}^{ρ} u_{jk} = 0 \quad (k = 2, \ldots, ρ), \] that is,
\[ h_ρ(v) = (\sum_{j=1}^{ρ} u_{j1} v \otimes 1, 0, \ldots, 0). \]
Since $(u_{ji})$ is an eigenvector for 1 of $A$, we have $u_{i1} = \cdots = u_{iρ}$ and $|u_{i1}| = \frac{1}{\sqrt{ρ}}$. Hence a $G$-homotopy connecting $h_ρ$ and $i$ is given easily. q.e.d.

**Proposition 2.10.** For complex $G$-vector bundles $ξ$ and $η$ over a compact $G$-space $X$, let
be an inclusion given by \( i_\xi(v) = (v, 0) \) for \( v \in T(\xi) \). Then there holds
\[
i_\xi^*(t(\xi \oplus \eta)) = \phi_\xi(e(\eta))
\]
for the Thom class \( t(\xi \oplus \eta) \in \tilde{U}_c^*(T(\xi \oplus \eta)) \) and the Euler class \( e(\eta) \in \tilde{U}_c^*(X) \).

**Proof.** Consider the following commutative diagram
\[
\begin{array}{ccc}
\tilde{U}_c^*(T(\xi)) \otimes \tilde{U}_c^*(T(\eta)) & \xrightarrow{(1 \otimes s)^*} & \tilde{U}_c^*(T(\xi) \wedge T(\eta)) \\
\downarrow & & \downarrow \phi_\xi^* \\
\tilde{U}_c^*(T(\xi)) \otimes \tilde{U}_c^*(X) & \xrightarrow{(1 \otimes s)^*} & \tilde{U}_c^*(T(\xi) \wedge X) \\
\end{array}
\]

where \( s : X^+ \to T(\eta) \) is the 0-section and \( \tilde{d} : T(\xi \oplus \eta) \to T(\xi \times \eta) = T(\xi) \wedge T(\eta) \) is the map induced by the diagonal map. Then it follows that
\[
i_\xi^*(t(\xi \oplus \eta)) = i_\xi^*\tilde{d}^*(t(\xi) \times t(\eta))
= \tilde{d}^*(t(\xi) \times s^*(t(\eta))
= \tilde{d}^*(t(\xi)) \times e(\eta)
= \phi_\xi(e(\eta)).
\]

**Proposition 2.11.** For an \( n \)-dimensional complex \( G \)-vector bundle \( \xi \) over a compact \( G \)-space \( X \), there holds
\[
P_1^G(t(\xi)) = \phi_{E_{1}e}(e(E_1(\xi \otimes \Delta))).
\]

**Proof.** By Proposition 2.8 we have
\[
P_1^G(t(\xi)) = \tilde{E}_{1}(d)^*(t(E_1(\oplus \xi))).
\]
Since the commutative diagram
\[
\begin{array}{ccc}
T(\xi) & \xrightarrow{\tilde{d}} & T(\xi \oplus \cdots \oplus \xi) \\
\downarrow \tilde{d} \quad \downarrow \tilde{d} & & \downarrow \tilde{d} \\
T(\xi \times \cdots \times \xi)
\end{array}
\]
induces \( \tilde{E}_{1}(\tilde{d})^* = \tilde{E}_{1}(d)^*\tilde{E}_{1}(\tilde{d})^* \), we get
\[
\bar{E}_v(\bar{d})*(t(E_v(\xi))) = \bar{E}_v(\bar{d})*(t(E_v(\xi \oplus \cdots \oplus \xi))) \\
= \bar{E}_v(i_d)*(t(E_v(\xi \oplus \xi \otimes \Delta))) \quad \text{(by 2.9 (2))} \\
= i_{E_v, v}(t(E_v(\xi) \oplus E_v(\xi \otimes \Delta))) \\
= \phi_{E_v, v}(e(E_v(\xi \otimes \Delta))) \quad \text{(by 2.10)} \quad \text{q.e.d.}
\]

Let us consider a connection of the operations \( P_v \) with the Landweber-Novikov operations introduced in §1. Therefore, let us assume the compact Lie group \( G \) abelian hereafter.

Let \( V = L_1 \oplus \cdots \oplus L_m \) and \( W = L_1 \oplus \cdots \oplus L_n \) be complex \( G \)-modules, where \( L_i \) and \( L_j \) are 1-dimensional complex \( G \)-modules. Let \( P(V) \) be the complex projective space for the \( G \)-module \( V \) and \( \eta(V; C) \) the canonical complex \( G \)-line bundle over \( P(V) \). Then, according to [4, Theorems 4.2 and 4.5] we see that

\[
h^*_G(P(V) \times P(W)) = h^*_G(\text{pt})[x_v, y_w]/(\theta_v(x_v), \theta_w(y_w))
\]

where \( x_v = e_h(\eta(V; C) \otimes 1) \) and \( y_w = e_h(1 \otimes \eta(W; C)) \) are the Euler classes of the \( G \)-line bundles, and \( (\theta_v(x_v), \theta_w(y_w)) \) is an ideal generated by polynomials \( \theta_v(x_v) = (x_v - e_h(L_1)) \cdots (x_v - e_h(L_m)) \) and \( \theta_w(y_w) = (y_w - e_h(L_1)) \cdots (y_w - e_h(L_n)) \). As usual we put

\[
h^*_G(P_\infty \times P_\infty) = \lim h^*_G(P(V) \times P(W))
\]

where the limit depends on the inverse system defined by inclusion maps of \( G \)-modules. Then we get

\[
h^*_G(P_\infty \times P_\infty) = h^*_G(\text{pt})[[x, y]].
\]

As usual, by commutativity and associativity of tensor products of \( G \)-vector bundles, we obtain a commutative formal group

\[
F(x, y) = \sum a_{ij}x^iy^j \in h^*_G(P_\infty \times P_\infty)
\]

such that \( F(x, y)|P(V)\times P(W) = e_h(\eta(V; C) \otimes \eta(W; C)) \) and \( a_{10} = a_{01} = 1 \). And, for \( G \)-line bundles \( \xi \) and \( \eta \) over a compact \( G \)-space \( X \), we have

\[
e_h(\xi \otimes \eta) = F(e_h(\xi), e_h(\eta)) = e_h(\xi) + e_h(\eta) + \text{higher terms}.
\]

**Lemma 2.12.** For an \( n \)-dimensional complex \( G \)-vector bundle \( \xi \) over a compact \( G \)-space \( X \), there holds

\[
s_\xi P^*_\xi(t(\xi)) = \sum_{\alpha, \beta \in \mathbb{N}} e_h(E_v(\Delta))^{n-\alpha-\beta}b_{\alpha}(v)s_\alpha(t(\xi))
\]
where \( v = e_h(E_v(L)) \), \( |a| = \sum a_i \) for each sequence \( a = (a_1, a_2, \ldots) \) and \( b_\alpha(v) \in h_c^*(pt)[[v]] \) is a power series.

**Proof.** By Proposition 2.11 we have

\[
s_\alpha P_v^{2n}(t(\xi)) = s_\alpha \phi_{E_v}(e(E_v(\xi \otimes \Delta))) = \phi(E_v(\xi))(e_h(E_v(\xi \otimes \Delta))).
\]

1) When \( \xi \) is a sum of \( G \)-line bundles \( \xi_1, \ldots, \xi_n \), it follows that

\[
e_h(E_v(\xi \otimes \Delta)) = e_h(E_v(\xi_1 \otimes \Delta \oplus \ldots \oplus \xi_n \otimes \Delta))
= e_h(\xi_1 \otimes E_v(\Delta) \oplus \ldots \oplus \xi_n \otimes E_v(\Delta))
= e_h(\xi_1 \otimes E_v(\Delta)) \cdots e_h(\xi_n \otimes E_v(\Delta)).
\]

For each \( k \) we have

\[
e_h(\xi_k \otimes E_v(\Delta)) = e_h(\xi_k \otimes E_v(\Delta)) = e_h(\xi_k \otimes E_v(L^p) \oplus \ldots \oplus \xi_k \otimes E_v(L^{p-1}))
= F(e_h(\xi_k), e_h(E_v(L))) \cdots F(e_h(\xi_k), e_h(E_v(L^{p-1})))
= \prod_{k=1}^{p-1} (e_h(E_v(L')) + \sum_{j \geq 1} a_j(v)e_h(\xi_k)^j)
= e_h(E_v(\Delta)) + \sum_{j \geq 1} b_j(v)e_h(\xi_k)^j,
\]

where \( a_j(v) \) and \( b_j(v) \) are formal power series of \( v \). Hence we have

\[
e_h(E_v(\xi \otimes \Delta)) = \prod_{k=1}^{n} (e_h(E_v(\Delta)) + \sum_{j \geq 1} b_j(v)e_h(\xi_k)^j)
= \sum_{\alpha \in \mathbb{A}_n} e_h(E_v(\Delta))^{n-\alpha}b_\alpha(v)c_\alpha(\xi)
\]

where \( c_\alpha(\xi) = \sum e_h(\xi_1)^{\alpha_1} \cdots e_h(\xi_n)^{\alpha_n} \) and \( b_\alpha(v) \) is a formal power series of \( v \). Therefore we have

\[
s_\alpha P_v^{2n}(t(\xi)) = \phi(E_v(\xi))(\sum_{\alpha \in \mathbb{A}_n} e_h(E_v(\Delta))^{n-\alpha}b_\alpha(v)c_\alpha(\xi))
= \sum_{\alpha \in \mathbb{A}_n} e_h(E_v(\Delta))^{n-\alpha}b_\alpha(v)s_\alpha(\xi).
\]

2) General case is shown by making use of the splitting principle in the theory \( h_c^*(-) \). q.e.d.

Now we obtain an \( h_c^* \)-theoretic version of [15, Proposition 3.17].

**Theorem 2.13.** Let \( x \in \tilde{U}^{\text{fr}}_c(X) \) be represented by a map \( f : W^c \land X \rightarrow M_{\text{fr+}}(U) \subset M_{\text{fr+}}(G) \). Then there holds

\[
\text{\textbf{Theorem 2.13.} Let } x \in \tilde{U}^{\text{fr}}_c(X) \text{ be represented by a map } f : W^c \land X \rightarrow M_{\text{fr+}}(U) \subset M_{\text{fr+}}(G). \text{ Then there holds}
\]
where $b_\alpha(v) \in h_v^*(pt)[[v]]$ is a well defined power series.

Proof. There holds $x = \sigma_{W} f^*(t(\gamma^m_0(U)))$, where $m = \|W\| + n$ and $\sigma_w$ is the suspension isomorphism. Hence, by the previous lemma, the naturality of $s_w$ and $P^{2m}_v$ and the stability of $s_w$, we have

$$s_w P^{2m}_v(f^*(t(\gamma^m_0(U)))) = \tilde{E}_v(f)^*(s_w P^{2m}_v(t(\gamma^m_0(U)))) = \tilde{E}_v(f)^*(\sum_{n \leq m} e_h(E_v(\Delta))^{m-n} b_\alpha(v) s_\alpha(t(\gamma^m_0(U))))$$

$$= \sum_{n \leq m} e_h(E_v(\Delta))^{m-n} b_\alpha(v) s_\alpha(f^*(t(\gamma^m_0(U))))$$

$$= \sum_{n \leq m} e_h(E_v(\Delta))^{m-n} b_\alpha(v) s_\alpha(\sigma_w x)$$

$$= \sum_{n \leq m} e_h(E_v(\Delta))^{m-n} b_\alpha(v) s_w s_\alpha(x).$$

On the other hand we have

$$s_w P^{2m}_v(\sigma_w x) = s_w P^{2m}_v(\sigma_w(1) \times x) = s_w P_v(\sigma_w(1)) \times s_w P_v(x).$$

Since $\sigma_w(1) = t(W)$, by the previous lemma, we have

$$s_w P_v(\sigma_w(1)) = \sum_{n \leq m} e_h(E_v(\Delta))^{m-n} b_\alpha(v) s_\alpha(\sigma_w(1)).$$

Here

$$s_\alpha \sigma_w(1) = \sigma_w s_\alpha(1) = \begin{cases} \sigma_w(1) & \text{for } \alpha = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have

$$s_w P^{2m}_v(\sigma_w x) = e_h(E_v(\Delta))^{m-n} \sigma_w s_\alpha P^{2m}_v(x).$$

This completes the proof.

q.e.d.

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References


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