On the number of representations of an integer as the sum of a powerful and a squarefree integers

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ON THE NUMBER OF REPRESENTATIONS OF 
AN INTEGER AS THE SUM OF A POWERFUL 
AND A SQUAREFREE INTEGERS

Dedicated to Professor TAKESHI INAGAKI on the occasion 
of his sixtieth birthday

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The problem of finding the number of representations of a positive 
integer as the sum of the square of an integer and a squarefree integer 
has been discussed by Th. Estermann [1; § 1]. A positive integer \( n \) is 
called squarefree, if \( n \) is not divisible by the square of any prime. He has 
shown that if \( G(N) \) denotes the number of ways of representing a positive 
integer \( N \) as the sum of a square and a squarefree integer, then we have 
for any fixed \( \varepsilon > 0 \)

\[
G(N) = C(N)N^{1/2} \prod_{p \mid N} \left( 1 - \frac{1}{p} \right) + O(N^{(1/3)+\varepsilon}),
\]

where

\[
C(N) = \prod_{p \mid N} (1 - \nu_p(N) p^{-1})
\]

with

\[
\nu_p(N) = \begin{cases} 
1 + (-1)^{v_p(N)/2} & (p = 2) \\
1 + \left( \frac{N}{p} \right) & (p > 2)
\end{cases}
\]

for primes \( p \) not dividing \( N \), and where the constant implied in the symbol 
\( O \) depends only on \( \varepsilon \). In particular, every sufficiently large positive 
integer can be represented as the sum of a square and a squarefree integer.

In the present paper we shall consider the number of representations 
of a positive integer as the sum of a powerful and a squarefree integers. 
A positive integer \( n \) is a powerful integer, by definition, if \( p^2 \) divides \( n \) 
whenever the prime \( p \) divides \( n \) (cf. [2]).

We shall denote by \( P \) the set of all powerful integers and by \( S \) the
set of all squarefree integers. It is clear that the intersection $P \cap S$ consists of one element, the unity.

For every positive integer $N$ we denote by $E(N)$ the number of ways of representing $N$ in the form

$$N = a + b$$

with

$$a \in P, \ b \in S \quad \text{and} \quad (a, b) = 1.$$  

We shall prove the following

**Theorem.** For any fixed positive number $\varepsilon$ we have

$$E(N) = K(N) N^{\frac{11}{12}} \prod_{p \mid N} \left(1 - \frac{1}{p}\right) + O(N^{\varepsilon^{10^{10}}}) \quad (N > 0),$$

where

$$K(N) = \sum_{m=1}^{\infty} \frac{n_t(m)}{m^{3/2}} - C(mN)$$

and the $O$-constant may depend only on $\varepsilon$.

In particular, every sufficiently large positive integer $N$ admits a representation of the form

$$N = a + b$$

with

$$1 < a \in P, \ 1 < b \in S \quad \text{and} \quad (a, b) = 1,$$

since we have

$$K(N) \geq \prod_{p} \left(1 - \frac{2}{p^2}\right) > 0$$

and

$$\prod_{p \mid N} \left(1 - \frac{1}{p}\right) \geq \frac{c}{\log \log 3N}$$

for some absolute constant $c > 0$.

**1. Lemmata.** In order to establish the theorem stated above, we require some auxiliary results.

The letters $d, k, m, n, N$ denote positive integers and $p$ a prime
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number, and $\varepsilon$ is used to denote an arbitrarily small but fixed positive number. As usual, $\mu(n)$ is the Möbius function, $\varphi(n)$ is the Euler totient function and $\tau(n)$ is the divisor function giving the number of positive divisors of $n$. Also, $v(n)$ denotes the number of distinct prime divisors of $n$.

**Lemma 1.** Let $d \in S$ and suppose that $n$ is a quadratic residue (mod $d^3$). Then, the number $s(d)$ of the incongruent solutions $z$ of the congruence

$$z^2 \equiv n \pmod{d^3}$$

is given by

$$s(d) = 2^{\varsigma(d)}.$$

For a proof of this lemma one may refer e.g. to [3; Theorem 47]. We note that the integer $n$ is a quadratic residue (mod $d^3$) if and only if $n$ is a quadratic residue (mod $p$) for all prime factors $p$ of $d$, and further $n \equiv 1 \pmod{4}$ when $d$ is even. Thus, if we write

$$e_d(n) = 2^{-\varepsilon(n)} \prod_{p|d} \nu_p(n),$$

then $e_d(n) = 1$ or 0 according as $n$ is or is not a quadratic residue (mod $d^3$).

**Lemma 2.** Let $a$ and $b$ be positive integers and let $Q(N; a, b)$ denote the number of pairs of positive integers $m, n$ satisfying

$$am^2 + bn^2 = N.$$

Then, we have

$$Q(N; a, b) \leq 2\tau(N).$$

This is [1; Hilfssatz 1].

2. **Proof of the Theorem.** Using the relation

$$\mu^2(n) = \sum_{d|n} \mu(d)$$

and noticing that $(a, b) = 1$ is equivalent to $(a, N) = 1$ when $N = a + b$, we have

$$E(N) = \sum_{\alpha \in \mathbb{Z}} \mu^2(N - \alpha) = \sum_{\alpha \in \mathbb{Z}} \sum_{d|N - \alpha} \mu(d).$$

$$\text{with } (\alpha, N) = 1.$$
It is not difficult to see that every integer $a \in P$ can be uniquely written in the form

$$a = n^2 m^3$$

with $m \in S$ (cf. [2]). Hence, we may rewrite

$$E(N) = \sum_{d \leq 1} \mu(d) + \sum_{d \leq \sqrt[3]{N}} \mu(d) + \sum_{d \leq \sqrt[3]{N}} \mu(d),$$

where $\sum_1$, $\sum_2$, and $\sum_3$ respectively indicate the summation over the positive integers $d$, $m$, $n$ satisfying the conditions

\[
\begin{align*}
\begin{cases}
d \leq t, \quad m \leq x, \quad \mu(m) \neq 0, \quad n^2 m^3 \leq N, \\
(mn, N) = 1, \quad d^2 | N - n^2 m^3,
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
t < d \leq \sqrt[3]{N}, \quad m \leq x, \quad \mu(m) \neq 0, \quad n^2 m^3 \leq N, \\
(mn, N) = 1, \quad d^2 | N - n^2 m^3,
\end{cases}
\end{align*}
\]

where $t$ and $x$ are fixed real numbers such that

$$1 \leq t \leq \sqrt[3]{N} \quad \text{and} \quad 0 < x \leq \sqrt[3]{N}.$$  

Firstly we have

$$\sum_1 \mu(d) = \sum_{d \leq x} \mu(d) \cdot T_d,$$

where

$$T_d = \sum_{n \in \mathbb{Z}} \mu^2(m) \sum_{(n, dN) = 1} \sum_{n \in \mathbb{Z}, n \geq \frac{1}{2} \sqrt[3]{N}} \frac{1}{d^2 | N - n^2 m^3}$$

\[
= \sum_{n \in \mathbb{Z}} \mu^2(m) \sum_{x \leq n \leq \frac{1}{2} \sqrt[3]{N}} \left( \left[ \frac{1}{Nd^2} \left( \frac{N}{m^3} \right)^{1/3} \right] + 1 \right)
\]

\[
= \sum_{n \in \mathbb{Z}} \mu^2(m) e_n(mN) \frac{\psi(N) s(d)}{Nd^2} \left( \frac{N}{m^3} \right)^{1/3} + O \left( \sum_{n \in \mathbb{Z}} s(d) \right)
\]

\[
= N^{1/3} \frac{\varphi(N)}{N} \frac{s(d)}{d^2} \sum_{n = 1}^{\infty} \mu^2(m) e_n(mN) m^{3/2}
\]
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\[ + O\left( \frac{s(d)}{d^2} \cdot N^{1/2} \cdot x^{-1/2} \right) + O(s(d)x). \]

Hence, noticing that
\[ \sum_{d \leq t} \frac{s(d)}{d^3} = O(1) \quad \text{and} \quad \sum_{d \leq t} s(d) = O(t \log 2t), \]
we find
\[
\sum_{d \leq t} \mu(d) = N^{1/2} \frac{\varphi(N)}{N} \sum_{d \leq t} \frac{\mu(d)}{d^3} \sum_{n=1}^{\infty} \frac{\mu^2(m)e_\delta(mN)}{m^{3/2}} \\
+ O\left( N^{1/2} x^{-1/2} \sum_{d \leq t} \frac{s(d)}{d^2} \right) + O(x \sum_{d \leq t} s(d)) \\
= N^{1/2} \frac{\varphi(N)}{N} \sum_{m=1}^{\infty} \frac{\mu^2(m)}{m^{3/2}} \sum_{(d, mN) = 1}^{\infty} \frac{\mu(d)}{d^2} \sum_{n=1}^{\infty} \frac{s(d)e_\delta(mN)}{m^{3/2}} \\
+ O\left( N^{1/2} \sum_{d \leq t} \frac{1}{d^2} \right) + O(N^{1/2} x^{-1/2}) + O(x t \log 2t) \\
= N^{1/2} \frac{\varphi(N)}{N} \sum_{m=1}^{\infty} \frac{\mu^2(m)}{m^{3/2}} \prod_{p \mid mN} \left( 1 - \frac{\nu(pN)}{p^2} \right) \\
+ O(N^{1/2} x^{-1/2}) + O(N^{1/2} x^{-1/2}) + O(x t \log 2t),
\]
by Lemma 1.

Next, we have
\[
\sum_{d \leq t} \mu^2(d) = \sum_{x < d \leq t} \mu^2(m) \sum_{n \leq \sqrt{x/Nm^{1/2}}} \sum_{a \leq \sqrt{N-nm^3}} \mu(d) \\
= O\left( \sum_{n \leq x} \sum_{y \leq \sqrt{x/Nm^{1/2}}} \z(N-n^3m^3) \right) \\
= O\left( N^{1/2} \sum_{n \leq x} \sum_{y \leq \sqrt{x/Nm^{1/2}}} 1 \right) \\
= O(N^{(1/2)+e} x^{-1/2}),
\]
since
\[ \max_{n \leq x} \z(n) = O(N^e). \]

Finally, writing
for \( d^2 \mid N - n^2 m^3 \), we find
\[
\sum_{u \mid d} u^{\mu}(d) = O\left( \sum_{n \leq N^{1/2}} \sum_{m \leq n} Q(N; k, m^3) \right) = O(N^{1-\varepsilon} t^{-2} x)
\]
by Lemma 2.

We thus obtain
\[
E(N) = K(N) \frac{\varphi(N)}{N} + R(N),
\]
where
\[
R(N) = O(N^{1/2} t^{-1}) + O(x t \log 2t)
\]
\[
= O(N^{(1/2)+\varepsilon}) + O(N^{1+\varepsilon} t^{-2} x)
\]
\[
= O(N^{(2/3)+\varepsilon}),
\]
on taking
\[
t = N^{1/3} \quad \text{and} \quad x = N^{1/9}.
\]
This completes the proof of our theorem, since
\[
\frac{\varphi(N)}{N} = \prod_{p \mid N} \left(1 - \frac{1}{p}\right).
\]

3. A Dual Problem. Estermann [1; § 2] has also given an asymptotic formula for the number \( H(x) \) of squarefree integers not exceeding \( x \) and having the form \( n^3 + l \), where \( l \) is a given non-zero integer. Indeed, he has proved that
\[
H(x) = C(-l) x^{1/2} \prod_{p \mid l} \left(1 - \frac{1}{p}\right) + O(x^{1/2} \log 2x) \quad (x \geq 1),
\]
where the \( O \)-constant is dependent only on \( l \).

An analogous formula can be found for the number \( F(x) \) of positive integers \( a \leq x \) satisfying
\[
a \in P, \quad a + l \in S \quad \text{and} \quad (a, l) = 1,
\]
where \( l \) is again a fixed non-zero integer. By the method employed above, with [1; Hilfssatz 2] in place of Lemma 2, one may easily obtain
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for any positive number $\varepsilon$

$$F(x) = K(-l)x^{\frac{1}{l}} \prod_{p \neq l} \left(1 - \frac{1}{p} \right) + O(x^{\varepsilon(l^{1/2} + r)}) \ (x > 0),$$

where the $O$-constant depends on $l$ and $\varepsilon$.


REFERENCES


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