On the hyperspace of a quasi-uniform space

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ON THE HYPERSPACE OF A QUASI-UNIFORM SPACE

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1. Introduction

1.1. In [4], E. Michael studied various topologies on the hyperspace including the uniform topology and the finite topology; J. R. Isbell also studied the uniform topology on hyperspaces in [1].

Given a quasi-uniform space \( (X, \mathcal{U}) \), we define a quasi-uniformity \( 2^U \) on the hyperspace \( 2^X \) in a natural way and investigate some of its stable properties.

If we are given a topological space \( (X, \mathcal{D}) \) and let \( \mathcal{P}(\mathcal{D}) \) be Pervin's quasi-uniformity, then the quasi-uniform topology \( \mathcal{D}(2^{\mathcal{P}(\mathcal{D})}) \) is shown to coincide with \( 2^D \), the finite topology on \( 2^X \).

In § 3, we show that every quasi-uniform space has a compactification.

We shall use the definitions and properties of quasi-uniform spaces as developed by M. G. Murdeshwar and S. A. Naimpally in [5].

Finally, all spaces are presumed to be \( T_\infty \).

1.2. Let \( (X, \mathcal{D}) \) be a topological space; \( 2^X \) denotes the set of all nonempty closed sets and is called the hyperspace of \( X \). For each \( A \subseteq X \), let \( \langle A \rangle = \{ E : E \subseteq 2^X \text{ and } E \subseteq A \} \). \( 2^D \) is the topology for \( 2^X \) with \( \{ \langle 0 \rangle : O \subseteq \mathcal{D} \} \) as base and is called the upper-semi-finite topology. For each \( A \subseteq X \), let \( \langle X, A \rangle = \{ E : E \subseteq 2^X \text{ and } A \cap E \neq \emptyset \} \); \( 2^D \) is the topology for \( 2^X \) with \( \{ \langle X, O \rangle : O \subseteq \mathcal{D} \} \) as subbase and is called the lower-semi-finite topology. Finally, we let \( 2^D = 2^D \cup 2^D \); \( 2^D \) is called the finite topology for \( 2^X \). We note that \( \{ x \} \subseteq 2^X \) for each \( x \) in \( X \) since \( T_\infty \) is presumed.

1.3. Let \( (X, \mathcal{U}) \) be a quasi-uniform space (all axioms for a uniform space hold except perhaps the symmetry axiom). For each \( U \in \mathcal{U} \), we make the following definitions:

\[(i) \quad \overline{H}(U) = \{(A, B) : A, B \subseteq 2^X \text{ and } B \subseteq U[A]\}\\
(ii) \quad H(U) = \{(A, B) : A, B \subseteq 2^X \text{ and } A \subseteq U^{-1}[B]\}\\
(iii) \quad \overline{H}(U) = \overline{H}(U) \cap H(U) .\]

We note the asymmetry in the definitions of \( \overline{H}(U) \) and \( H(U) \).
Theorem 1.3.1. Let \((X, \mathcal{U})\) be a quasi-uniform space. Then \(\{\overline{H}(U) : U \subseteq \mathcal{U}\}\), \(\{H(U) : U \subseteq \mathcal{U}\}\) and \(\{H(U) : U \subseteq \mathcal{U}\}\) are bases for quasi-uniformities for \(2^X\) respectively denoted by \(2^\mathcal{U}\), \(\overline{2^\mathcal{U}}\) and \(2^\mathcal{U}\). These are called the upper-hyper-quasi-uniformity, the lower-hyper-quasi-uniformity and the hyper-quasi-uniformity respectively.

Proof. The fact that \(\{\overline{H}(U) : U \subseteq \mathcal{U}\}\) is a base for a uniformity for \(2^X\) follows from the identities:

(i) \(\overline{H}(U) \circ \overline{H}(V) \subseteq \overline{H}(U \circ V)\) when \(U \subseteq \mathcal{U}\)

(ii) \(\Delta \subseteq \overline{H}(U \cap \overline{H}(V)\) when \(U \subseteq \mathcal{V}, V \subseteq \mathcal{V}\).

Similarly for \(\{H(U) : U \subseteq \mathcal{U}\}\). We note that when \((X, \mathcal{U})\) is a separated uniform space, then \(2^\mathcal{U}\) is the uniformity studied by Michael and Isbell.

Theorem 1.3.2. Let \((X, \mathcal{U})\) be a quasi-uniform space and let \(\mathcal{U}^{-1} = \{U^{-1} : U \subseteq \mathcal{U}\}\). Then (i) \((X, \mathcal{U}^{-1})\) is a \(T_1\)-quasi-uniform space (ii) \(\overline{2^\mathcal{U}^{-1}} = (2^\mathcal{U})^{-1}\) (iii) \(\overline{2^\mathcal{U}^{-1}} = (\overline{2^\mathcal{U}})^{-1}\) (iv) \(2^\mathcal{U} = \overline{2^\mathcal{U}} \cup \overline{\mathcal{U}}\), \(\lor\) denoting supremum (v) \(2^\mathcal{U}^{-1} = (2^\mathcal{U})^{-1}\).

Proof. (i) is cited in [5]. (ii) follows from the fact that \(\overline{H}(U^{-1}) = (H(U))^{-1}\). (iii) follows from the identity \(H(U^{-1}) = (\overline{H}(U))^{-1}\). (iv) \(H(U) = \overline{H}(U) \cap \overline{H}(U)\) implies that \(2^\mathcal{U} = \overline{2^\mathcal{U}} \cup \overline{\mathcal{U}}\). (v) follows from (ii) and (iii) and the fact that \((\mathcal{U} \lor \mathcal{V})^{-1} = \mathcal{U}^{-1} \lor \mathcal{V}^{-1}\) when \(\mathcal{U}\) and \(\mathcal{V}\) are quasi-uniformities (see [5]).

1.4. In [4], Michael shows that when \((X, \mathcal{D})\) is a \(T_1\)-space, the function \(i : X \to 2^X\) defined by \(i(x) = \{x\}\) is a homeomorphism from \((X, \mathcal{D})\) into \((2^X, 2^D)\). For a separated uniform space the function \(i : X \to 2^X\) is a unimorphism from \((X, \mathcal{U})\) into \((2^X, 2^\mathcal{U})\). In this sense, \(2^\mathcal{D}\) is an admissible topology for \(2^X\) and \(2^\mathcal{U}\) is an admissible uniformity for \(2^X\).

We show next that in this sense, \(2^\mathcal{U}\), \(\overline{2^\mathcal{U}}\) and \(2^\mathcal{U}\) are admissible quasi-uniformities for \(2^X\) when \((X, \mathcal{U})\) is a \(T_1\) quasi-uniform space.

Theorem 1.4.1. Let \((X, \mathcal{U})\) be a quasi-uniform space. Then each of the following are unimorphisms:

(i) \(i : (X, \mathcal{U}) \to (i[X], \overline{2^\mathcal{U} \cap \overline{i[X]} \times \overline{i[X]}})\)

(ii) \(i : (X, \mathcal{U}) \to (i[X], \overline{2^\mathcal{U} \cap \overline{i[X]} \times i[X]})\)

(iii) \(i : (X, \mathcal{U}) \to (i[X], \overline{2^\mathcal{U} \cap i[X] \times i[X]})\).

Proof. (i) follows from the fact that \(U = (i \times i)^{-1}(\overline{H}(U))\) and
(i \times i)[U] = \overline{H}(U) \cap i[\mathcal{X}] \times i[\mathcal{X}] \text{ when } U \subseteq \mathcal{U}. \text{ Similarly for (ii) and (iii).}

Note that the assumption of \( T \) is vital here.

**Theorem 1.4.2.** Let \((X, \mathcal{U})\) be a quasi-uniform space and suppose that \( \mathcal{B} \subseteq \mathcal{U} \). The following are equivalent:

(i) \( \mathcal{B} \) is a base for \( \mathcal{U} \)

(ii) \( \{ \overline{H}(B) : B \subseteq \mathcal{B} \} \) is a base for \( \mathcal{U} \)

(iii) \( \{ \overline{H}(B) : B \subseteq \mathcal{B} \} \) is a base for \( \mathcal{U} \)

(iv) \( \{ H(B) : B \subseteq \mathcal{B} \} \) is a base for \( \mathcal{U} \).

**Proof.** (i) is equivalent to (ii) since \( B \subseteq \mathcal{U} \) is equivalent to \( \overline{H}(B) \subseteq \overline{H}(U) \). Similarly for the equivalence of (ii) and (iii), and (ii) and (iv).

The following example indicates that Theorem 1.4.2 cannot be generalized to subbase.

**Example 1.4.3.** Let \((X, \mathcal{U})\) be the unit interval with the usual uniformity. Let \( S = \{ U \cup \{ a \} \times X : U \subseteq \mathcal{U}, \ a=0 \text{ or } a=1 \} \). Then both \( S \) and \( S^{-1} \) are subbases for \( \mathcal{U} \). But \( \{ \overline{H}(S) : S \subseteq S \} \) is not a subbase for \( \mathcal{U} \), \( \{ H(S^{-1}) : S^{-1} \subseteq S^{-1} \} \) is not a subbase for \( \mathcal{U} \) and finally, \( \{ H(S) : S \subseteq S \} \) is not a subbase for \( \mathcal{U} \). To see this, let \( A = \{ 0, 1 \} \) and \( B = X \). Let \( S \subseteq S \). Then \( S[A] = X = S^{-1}B \) and \( S^{-1}[B] = X \) as the reader can easily see. Thus \( (A, B) \in H(S), (B, A) \in H(S^{-1}) \) and \( (A, B) \in H(V) \). Let \( V = \{ (x, y) : |x-y| < 1/2 \} \). Then \( V \subseteq \mathcal{U} \), but \( (A, B) \notin H(V), (A, B) \notin \overline{H}(V) \) and \( (B, A) \notin H(V) \).

**Theorem 1.4.4.** Let \((X, \mathcal{U})\) be a quasi-uniform space. The following are equivalent:

(i) \( \mathcal{U} \) is a uniformity

(ii) \( \overline{2U} = (\overline{2U})^{-1} \)

(iii) \( 2U \) is a uniformity.

**Proof.** (i) implies (ii). If \( \mathcal{U} \) is a uniformity, then \( \mathcal{U} = \mathcal{U}^{-1} \). Then \( \overline{2U} = \overline{2U}^{-1} = (\overline{2U})^{-1} \) by (ii) of Theorem 1.3.2.

(ii) implies (iii). \( 2U = \overline{2U} \cup \overline{2U} = (\overline{2U})^{-1} \cup (\overline{2U})^{-1} = (2U)^{-1} \) by (iv) of Theorem 1.3.2.

(iii) implies (i). If \( 2U \) is a uniformity, then \( 2U \cap i[\mathcal{X}] \times i[\mathcal{X}] \) is a uniformity and by (iii) of Theorem 1.4.1, \( \mathcal{U} \) is a uniformity.

2. **The Hyperspace of Pervin's Quasi-Uniformity**

2.1. For \( A \subseteq X \), let \( S(A) = A \times A \cup CA \times X \), \( C \) denoting the comp-
implem ent operator. In [6], Pervin showed that for a given topological space \((X, \mathcal{I})\), \(\{S(O) : O \in \mathcal{I}\}\) is a subbase for a quasi-uniform space \((X, \mathcal{P}(\mathcal{I}))\) with the property that \(\mathcal{I}(\mathcal{P}(\mathcal{I})) = \mathcal{I}\).

In this section, we will show that if \((X, \mathcal{I})\) is a topological space and \(\mathcal{P}(\mathcal{I})\) is Pervin's quasi-uniformity, then \(2^\mathcal{I} = \mathcal{I}(2^\mathcal{P}(\mathcal{I}))\).

Several properties of Pervin's quasi-uniformity were developed by Levine in [3]. Applications were also made in [5].

It is worth noting that \(S(A) = (S(CA))^{-1}\) for all sets \(A \subseteq X\).

**Theorem 2.1.1.** Let \((X, \mathcal{U})\) be a quasi-uniform space and \(\mathcal{I} = \mathcal{I}(\mathcal{U})\). Then

(i) \(\mathcal{I}(2^\mathcal{U}) \subseteq 2^\mathcal{I}\) and (ii) \(2^\mathcal{I} \subseteq \mathcal{I}(2^\mathcal{U})\).

**Proof.** (i). Let \(E \in O \in \mathcal{I}(2^\mathcal{U})\). There exists then a \(U \in \mathcal{U}\) such that \(\overline{H}(U)[E] \subseteq O\). But \(E \in \langle \text{Int}U[E] \rangle = \overline{H}(U)[E]\) as the reader can easily show.

(ii) It suffices to show that \(\langle X, O \rangle \in \mathcal{I}(2^\mathcal{U}) \) when \(O \in \mathcal{I}\). Let \(A \in \langle X, O \rangle\). Then \(A \cap O \neq \emptyset\); let \(a \in A \cap O\). There exists then a \(U \in \mathcal{U}\) such that \(U[a] \subseteq O\). We show now that \(A \subseteq \overline{H}(U)[A] \subseteq \langle X, O \rangle\). Let \(B \in \overline{H}(U)[A]\). Then \(A \subseteq U^{-1}[B]\) and hence \(\emptyset \neq \overline{U}[a] \cap B \subseteq B \cap O\). Thus \(B \in \langle X, O \rangle\).

**Theorem 2.1.2.** Let \((X, \mathcal{I})\) be a topological space and suppose that \(\mathcal{P}(\mathcal{I})\) is Pervin's quasi-uniformity. If \(S = \{S(O) : O \in \mathcal{I}\}\), then (i) \(\{\overline{H}(S) : S \in S\}\) is a subbase for \(2^\mathcal{P}(\mathcal{I})\), (ii) \(\{H(S) : S \in S\}\) is a subbase for \(2^\mathcal{P}(\mathcal{I})\) and (iii) \(\{H(S) : S \subseteq S\}\) is a subbase for \(2^\mathcal{P}(\mathcal{I})\).

**Proof.** (i) Let \(O_i \in \mathcal{I}\) for \(1 \leq i \leq n\) and for \(\emptyset \neq \delta \subseteq \{1, \cdots, n\}\), let \(O_\delta = \cup \{O_i : i \in \delta\}\). It suffices to show that \(\cap \{\overline{H}(S(O_i)) : \emptyset \neq \delta \subseteq \{1, \cdots, n\}\}\) \(\subseteq \overline{H}(S(O_\delta) \cap \cdots \cap S(O_n))\). Let \((A, B)\) be a member of the left side and take \(b \in B\). It suffices to show that there exists an \(a\) in \(A\) such that \((a, b) \in S(O_i)\) for \(1 \leq i \leq n\).

**Case 1.** \(b \in O_i\) for each \(i\). Then any \(a\) in \(A\) will do.

**Case 2.** \(b \notin \cap \{O_i : 1 \leq i \leq n\}\). Let \(\delta = \{i : b \notin O_i\}\). Then \((A, B) \in \overline{H}(S(O_\delta))\) and hence there exists an \(a \in A\) such that \((a, b) \in S(O_\delta)\). If \((a, b) \in S(O_i)\), then \(a \in O_i\) and \(b \notin O_i\), and hence \(a \in O_k\). It follows then that \(b \in O_k\), a contradiction.

(ii) Let \(O_i \in \mathcal{I}\) for \(1 \leq i \leq n\). For each \(\emptyset \neq \delta \subseteq \{1, \cdots, n\}\), let \(G_\delta = \cap \{O_i : i \in \delta\}\). It suffices to show that
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$\cap \{H(S(G_i)) : \emptyset \neq \delta \subseteq \{1, 2, \ldots, n\} \} \subseteq H(S(O_i) \cap \cdots \cap S(O_n))$. Since $\mathcal{C}G_i = \cup \{CO_i : i \in \delta\}$, it follows that $\cap \{H(S(\mathcal{C}G_i)) : \emptyset \neq \delta \subseteq \{1, \ldots, n\}\} \subseteq H(S(CO_i) \cap \cdots \cap S(CO_n))$ using the argument in (i) above. Recalling that $S(A) = (S(\mathcal{C}A))^{-1}$ (see §2.3) and $H(U^{-1}) = (H(U))^{-1}$, we have

$\cap \{H(S(G_i)) : \emptyset \neq \delta \subseteq \{1, \ldots, n\}\} = \cap \{H((S(\mathcal{C}G_i))^{-1}) : \emptyset \neq \delta \subseteq \{1, \ldots, n\}\}$

$= (\cap \{H(S(\mathcal{C}G_i)) : \emptyset \neq \delta \subseteq \{1, \ldots, n\}\})^{-1}$

$\subseteq (H(S(CO_i) \cap \cdots \cap S(CO_n))^{-1}$

$= H(S(O_i) \cap \cdots \cap S(O_n))$.

(iii) Let $O_i \in \mathcal{D}$ for $1 \leq i \leq n$. Let $O_i$ and $G_i$ be defined as in (i) and (ii) above. Then

$\cap \{H(S(O_i)) : \emptyset \neq \delta \subseteq \{1, \ldots, n\}\} \cap \{H(S(G_i)) : \emptyset \neq \delta \subseteq \{1, \ldots, n\}\}$

$\subseteq \cap \{H(S(O)) : \emptyset \neq \delta \subseteq \{1, \ldots, n\}\} \cap \{H(S(G)) : \emptyset \neq \delta \subseteq \{1, \ldots, n\}\}$

$\subseteq H(S(O)) \cap \cdots \cap S(O_n) \cap H(S(G)) \cap \cdots \cap S(G_n)$

$= H(S(O)) \cap \cdots \cap S(O_n)$.

**Theorem 2.1.3.** Let $(X, \mathcal{U})$ be a topological space and suppose that $O \in \mathcal{D}$. Then

(i) $H(S(O)) = S(\langle O \rangle)$

(ii) $H(S(O)) = S(\langle X, O \rangle)$

(iii) $H(S(O)) = S(\langle O \rangle) \cap S(\langle X, O \rangle)$.

**Proof.** (i) It suffices to show that $\bar{H}(S(O)) = \langle O \rangle \times \langle O \rangle \cup \langle X, CO \rangle \times 2^x$.

Let $A, B$ be in $2^x$.

Case 1. $A \subseteq O$. Then $(A, B) \in H(S(O))$ iff $B \subseteq S(O) [A]$ iff $B \subseteq O$ iff $(A, B) \in \langle O \rangle \times \langle O \rangle$ iff $(A, B) \in \langle O \rangle \times \langle O \rangle \cup \langle X, CO \rangle \times 2^x$.

Case 2. $A \nsubseteq O$. Then $(A, B) \in H(S(O))$ iff $B \subseteq S(O) [A]$ iff $B \subseteq X$ iff $(A, B) \in \langle X, CO \rangle \times 2^x$ iff $(A, B) \in \langle O \rangle \times \langle O \rangle \cup \langle X, CO \rangle \times 2^x$.

(ii) $H(S(O)) = (\bar{H}(S(CO)))^{-1} = (S(\langle CO \rangle))^{-1}$ (by (i))

$= S(\langle CO \rangle) = S(\langle X, O \rangle)$.

(iii) $H(S(O)) = H(S(O)) \cap \bar{H}(S(O))$

$= S(\langle X, O \rangle) \cap S(\langle O \rangle)$

**Corollary 2.1.4.** Let $(X, \mathcal{U})$ be a topological space. Then (i) $2^{\mathcal{D}} \subseteq \mathcal{P}(2^x)$ (ii) $2^{\mathcal{D}} \subseteq \mathcal{P}(2^x)$ and (iii) $2^{\mathcal{D}} \subseteq \mathcal{P}(2^x)$.

**Proof.** (i) Let $O \in \mathcal{D}$. By (i) of Theorem 2.1.2,

$\bar{H}(S(O))$ is subbasic in $2^{\mathcal{D}}$. But $\bar{H}(S(O)) = S(\langle O \rangle)$ by (i) of Theorem
2.1.3, and $S\langle O \rangle \in \mathcal{P}(\mathcal{O})$.

(ii) Let $O \in \mathcal{D}$. By (ii) of Theorem 2.1.2, $H(S(O))$ is subbasic in $\mathcal{P}(\mathcal{D})$. By (ii) of Theorem 2.1.3, $H(S(O)) = S\langle X, O \rangle \in \mathcal{P}(\mathcal{O})$.

(iii) Let $O \in \mathcal{D}$. By (iii) of Theorem 2.1.2, $H(S(O))$ is subbasic in $\mathcal{P}(\mathcal{D})$. By (iii) of Theorem 2.1.3, $H(S(O)) = S\langle X, O \rangle \cap S\langle X, O \rangle \in \mathcal{P}(\mathcal{O})$.

**Theorem 2.1.5.** Let $(X, \mathcal{D})$ be a topological space. Then

(i) $\mathcal{D}(\mathcal{P}(\mathcal{D})) = \mathcal{D}(\mathcal{P}(\mathcal{D}))$

(ii) $\mathcal{D}(\mathcal{P}(\mathcal{D})) = \mathcal{D}(\mathcal{P}(\mathcal{D}))$

(iii) $\mathcal{D}(\mathcal{P}(\mathcal{D})) = \mathcal{D}(\mathcal{P}(\mathcal{D}))$

**Proof.** (i) By (i) of Corollary 2.1.4, $\mathcal{D}(\mathcal{P}(\mathcal{D})) \subseteq \mathcal{D}(\mathcal{P}(\mathcal{D}))$. Hence it suffices to show that $\mathcal{D}(\mathcal{P}(\mathcal{D})) \subseteq \mathcal{D}(\mathcal{P}(\mathcal{D}))$ when $O \in \mathcal{D}$. Let $A \in \langle O \rangle$; by (i) of Theorem 2.1.3, $H(S(O)[A] = S\langle X, O \rangle[A] \subseteq \langle O \rangle$.

(ii) By (ii) of Corollary 2.1.4, $\mathcal{D}(\mathcal{P}(\mathcal{D})) \subseteq \mathcal{D}(\mathcal{P}(\mathcal{D}))$ and from (ii) of Theorem 2.1.1, $\mathcal{D}(\mathcal{P}(\mathcal{D})) \subseteq \mathcal{D}(\mathcal{P}(\mathcal{D}))$.

(iii) Since $\mathcal{P}(\mathcal{D}) = \mathcal{P}(\mathcal{D}) \cup \mathcal{P}(\mathcal{D})$, it follows that $\mathcal{D}(\mathcal{P}(\mathcal{D})) = \mathcal{D}(\mathcal{P}(\mathcal{D})) \cup \mathcal{D}(\mathcal{P}(\mathcal{D})) = \mathcal{D}(\mathcal{P}(\mathcal{D}))$.

3. A Compactification of a Quasi-Uniform Space

**Lemma 3.1.** Let $(X, \mathcal{D})$ be a topological space. Then $(\mathcal{D}(\mathcal{D}), \mathcal{D}(\mathcal{D}))$ is compact.

This fact is well known and the easy proof is omitted.

**Theorem 3.2.** Let $(X, \mathcal{U})$ be a $T_1$-quasi-uniform space. Then $(X, \mathcal{U})$ has a compactification.

**Proof.** By (i) of Theorem 1.4.1, $i : (X, \mathcal{U}) \to (i[X], \mathcal{D}(\mathcal{U}))$ is a quasi-uniformity and by (i) of Theorem 2.1.1, $\mathcal{D}(\mathcal{D}) \subseteq \mathcal{D}(\mathcal{D})$. Thus $\mathcal{D}(\mathcal{D})$ is a compact topology by Lemma 3.1 and hence $(i, c(i[X]))$ is a compactification of $(X, \mathcal{U})$.

**References**


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