On the bimodule structure of Galois extensions

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Throughout the present note, \( A \) will represent an (Artinian) simple ring which is finite Galois over a simple subring \( B \). Then, it is known that \( A \) is \( B \cdot V_s(B) \cdot A \)-irreducible. (See [4]. As to terminologies used without mention, we follow [4].) We use the following notations: \( C=V_s(A), Z=Z_s(B), V=V_s(A), H=V_s(V), C_s=V_s(V)=V \cap H, G = \text{the Galois group of } A/B, \{ \sigma_1, \sigma_2, \ldots, \sigma_n \} \) is a (fixed) representative system of \( G \) modulo the normal subgroup \( I \) consisting of all inner automorphisms, and \( \overline{G} = G/I \), which may and will be regarded as the Galois group of \( H/B \).

We set further \( S=V \cdot V_r \) and \( R=\text{Hom}(\sigma A B, \sigma A B) \), where \( V_i \) (resp. \( V_r \)) the left (resp. right) multiplication of \( V \). To be easily seen, \( R=GV_r=\bigoplus_{i=1}^{\infty} \sigma_i S \) and \( S=V^o \otimes \nu V \) is a Frobenius ring, where \( V^o \) is the opposite of \( V \). (See, for instance [3; Lemma 3].)

In this note, the main theme of our discussion will concern the bimodule structure of \( \sigma A_B \). We shall prove first that \( \sigma A_B \) is a direct sum of local submodules, where a module \( \sigma M_B \) is said to be local if it contains one and only one maximal submodule (Theorem 1). Next, we shall explain when \( \sigma A_B \) is completely reducible (Theorem 2) or local (Theorem 3).

The next lemma will play an essential role in our subsequent study.

**Lemma 1.** If \( T \) is a \( B \cdot B \)-submodule of \( A \) then the restriction \( T|_R \) of \( R \) to \( T \) contains a free \( V_r \)-basis that forms at the same time a free \( \sigma \)-basis of \( T|_R \cdot \sigma A_B \), \( \{ T|_R : V \} = \{ T|_R : \sigma A_B \} = \{ T : B \} \), and \( T|_R = \text{Hom}(\sigma A_B) \).

**Proof.** As is well-known, \( T \) is left (resp. right) \( B \)-free. Hence, the lemma is contained in [4; Lemma 5.8].

**Theorem 1.** \( R \) is a Frobenius ring and \( \sigma A_B \) is a direct sum of local modules.

**Proof.** To be easily seen, the map \( h \) defined by \( \sum_{i=1}^\infty \sigma_i s \mapsto s_i \) is a Frobenius homomorphism of the ring extension \( R/S \), namely, \( R/S \) is a free Frobenius extension. Since \( S \) is a Frobenius ring, so is \( R \) by [1;
Satz 10]. Next, we shall prove the latter part. To our end, it is enough to show that if \( e \) is a primitive idempotent of \( R \) then \( {}_nAe_n \) is local. Let \( T_1 \) and \( T_2 \) be arbitrary proper \( B \)-submodules of \( Ae \). Noting that 
\[
[T_1 : R : V_1] = [T_1 : B] < [Ae : B] = [Ae : R : V ] \quad \text{(Lemma 1)},
\]
the kernel of the restriction map \( h_1 : Ae \to R \to T_1 \) is non-zero. Since \( R \) is a Frobenius ring, \( \ker h_1 \cap \ker h_2 \) contains a submodule isomorphic to the unique minimal right subideal of \( eR \). Hence, the kernel of the restriction map \( Ae \to R \to T_1 + T_2 \) is non-zero, which implies \( T_1 + T_2 \neq Ae \) and that \( {}_nAe_n \) is local.

**Theorem 2.** The following conditions are equivalent:

1. \( {}_nA_B \) is completely reducible.
2. \( R \) is semisimple.
3. a) \( C_0/C \) is separable;
   b) \( \sum_{1}^{n} c_{\sigma_1} = 1 \) for some \( c \in C_0 \).

**Proof.** (1) \( \Rightarrow \) (2): This is evident by Lemma 1.

(2) \( \Rightarrow \) (1): It suffices to prove that if \( eR(e^2 = e) \) is a minimal right ideal then \( {}_nAe_n \) is irreducible. Let \( T' \) be an arbitrary non-zero \( B \)-submodule of \( Ae \). Noting that \( Ae \to R \) is \( R \)-irreducible, we obtain 
\[
[T' : B] = [T' : R : V ] = [Ae : R : V ] = [Ae : B], \quad \text{(Lemma 1)},
\]
namely, \( T' = Ae \).

(2) \( \Rightarrow \) (3): Evidently, the semisimplicity of \( R \) implies the semisimplicity of \( S \), equivalently, the separability of \( C_0/C \). Since \( {}_nA_B \) is then completely reducible, so is \( {}_nH_B \). Accordingly, \( \text{Hom}(\_nH_B, {}_nH_B) = \tilde{G}C_{\sigma} \) is semisimple. Hence, \( C_0 \) is a direct summand of \( H \) as \( \tilde{G}C_{\sigma} \)-module. Now, noting that \( H \) contains an element \( a \) with \( 1 = \sum_{x \in R} a \sigma = \sum_{x \in R} a \sigma_1 \), we readily obtain (iii) b).

(3) \( \Rightarrow \) (2): Let \( \sum_{1}^{n} c_{\sigma_1} = 1 \) for some \( c \in C_0 \). To be easily verified,
\[
\sum_{1}^{n} \sigma_i c_{\sigma} = 1 \quad \text{and} \quad \sum_{1}^{n} x_{\sigma} \sigma_i c_{\sigma} = \sum_{1}^{n} \sigma_i x_{\sigma} c_{\sigma} \quad \text{(in} \bigotimes_{R} R \text{)}
\]
for every \( x \in R \). This means that \( R \) is a separable extension of the semisimple ring \( S \). Then, \( R \) is semisimple by [2; Lemma 2.10 (1)].

**Corollary 1.** If \( C_0 \) is separable over \( C \) and \( n \) is not divisible by \( \text{char} A \), then \( {}_nA_B \) is completely reducible.

**Theorem 3.** The following conditions are equivalent:

1. \( {}_nA_B \) is local.
2. \( R \) is a local ring.
3. a) \( V = Z \) and is purely inseparable over \( C \);
b) either \( A/B \) is inner Galois or \( \bar{G} \) is a \( p \)-group and char \( A = p \).

**Proof.** (1) \( \Rightarrow \) (2): This is evident by Th. 1.

(2) \( \Rightarrow \) (3): Since \( R \) is local, so is the subring \( S \) which is isomorphic to \( V \otimes \sigma V \). Accordingly, \( C_0 \otimes \sigma C_0 \) is a local ring, namely, \( C_0/C \) is purely inseparable. If [\( [V : C_0] = m \)] then \( (C_0)_m(\cong V \otimes \sigma C_0) \) is a homomorphic image of the local ring \( S \). Hence, it follows \( V = C_0 \). Further, \( C_0/Z \) is Galois with \( C_0|\bar{G} \) as Galois group, and so if \( [C_0 : Z] = t \) then \( (Z) \), \( \cong (C_0|\bar{G}C_{m}) \) is homomorphic to \( R \). It follows therefore \( Z = C_0 = V \) and \( H|R \) is isomorphic to the group ring of \( \bar{G} \) over \( Z \). Hence, if \( n \geq 1 \) then \( \bar{G} \) is a \( p \)-group and \( p = \text{char} Z \) by [4; Lemma 13.4].

(3) \( \Rightarrow \) (2): Let \( \phi \) be the ring homomorphism of the local ring \( S = \mathbb{Z}_t \cdot \mathbb{Z}_r \) onto \( Z \) given by \( \sum_{i=1}^{n} z_i \cdot z_i' \mapsto \sum_{i=1}^{n} z_i z'_i \). Then, the kernel of \( \phi \) is the radical \( J \) of \( S \). Now, one will easily see that the kernel of the restriction map \( R \to H|R \) is \( \bigoplus_{i=1}^{n} \sigma_i J \) and nilpotent. Since \( H|R = \bar{G}Z \), is a local ring again by [4; Lemma 13.4], we can easily see that \( R \) is a local ring.

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