An equivalence relation in topology

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AN EQUIVALENCE RELATION IN TOPOLOGY

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1. Equivalent sets. Introduction

It seems reasonable to define equality or equivalence of sets in a topological space $X$ in some way which involves the topology. After some experimenting, we came upon the following:

Definition 1.1. In a space $X$, $A$ is equivalent to $B$ (written $A \equiv B$) iff for each open set $O$, $A \subseteq O$ iff $B \subseteq O$.

We shall make frequent use of

Lemma 1.2. In a space $X$, $A \equiv B$ iff $a \in A$ implies that $c(a) \cap B \neq \emptyset$ and $b \in B$ implies that $c(b) \cap A \neq \emptyset$, $c$ denoting the closure operator.

Proof. Let $A \equiv B$ and take $a \in A$. Then $A \subseteq c(a)$ and $c(a)$ is open, $C$ denoting the complement operator. Thus $B \subseteq c(a)$ and hence $B \cap c(a) \neq \emptyset$.

Conversely, suppose that $A \not\equiv B$. We may assume that there exists an open set $O$ such that $A \subseteq O$ and $B \not\subseteq O$; take $b \in B \cap C O$. Then $c(b) \subseteq C O \subseteq C A$ and hence $c(b) \cap A = \emptyset$.

Theorem 1.3. If $O$ and $U$ are open in $X$, then $O \equiv U$ iff $O = U$.

We shall often refer to

Example 1.4. Let $X = \{a, b\}$ with open sets $\emptyset$, $\{a\}$, $X$. Then $\{b\} \equiv X$ and both sets are closed, but equality fails (see Theorem 1.6). Note also that a set equivalent to an open set need not be open.

Definition 1.5. For each set $A \subseteq X$, let $A^* = \cap \{O : A \subseteq O$ and $O$ is open\}.

Theorem 1.6. In a space $X$, $A \equiv B$ iff $A^* = B^*$.

Proof. Let $A \equiv B$ and take $a^* \in A^*$. If $a^* \not\in B^*$, then $a^* \not\in O$ for some open set which contains $B$. But then $A \subseteq O$ and hence $a^* \not\in A^*$, a contradiction.

Conversely, let $A^* = B^*$ and suppose that $A \subseteq O$, $O$ being open. Then $B \subseteq B^* = A^* \subseteq O$ and hence $B \subseteq O$. It follows then that $A \equiv B$.

Theorem 1.7. * as defined in Definition 1.5 is a Kuratowski closure
operator.

Proof. $\emptyset^* = \emptyset$ and $A \subseteq A^*$ are clear. If $x \in (A \cup B)^*$, then $x \notin O$ for some open set such that $A \cup B \subseteq O$. Then $x \notin A^* \cup B^*$. Conversely, if $x \notin A^* \cup B^*$, then $x \notin O$ for some open set such that $A \subseteq O$ and $x \notin U$ for some open set such that $B \subseteq U$. Thus $A \cup B \subseteq O \cup U$ and $x \notin O \cup U$. Hence $x \notin (A \cup B)^*$.

It remains to show that $A^{**} \subseteq A^*$; suppose that $x \notin A^*$. Then there exists an open set $O$ such that $x \notin O$, $A \subseteq O$. Then $x \notin O$ and $A^* \subseteq O$ and hence $x \notin A^{**}$.

Theorem 1.8. In a space $X$, $A^*$ is the largest set which is equivalent to $A$.

Proof. Clearly, $A = A^*$. Suppose then that $B = A$. Then for each open set $O$ such that $A \subseteq O$, then $B \subseteq O$. It follows then that $B \subseteq A^*$.

In general, there is no smallest set which is equivalent to a given set. However, we have

Theorem 1.9. In a space $X$, let $A$ be closed and compact. There exists a smallest closed set $B$ which is equivalent to $A$.

Proof. Let $B = \bigcap \{A': A' \text{ is closed and } A' = A\}$. It suffices to show that $B = A$. Since $B \subseteq A$, it suffices to show that $A \subseteq O$ if $B \subseteq O$ and $O$ is open. $B \subseteq O$ implies that $\bigcap \{A_i: 1 \leq i \leq n\} \subseteq O$ and $A = A \cap \cdots \cap A_n$ (see Corollary 2.4). Thus $A \subseteq O$.

Equivalence of sets is an absolute property as shown in

Theorem 1.10. Let $Y$ be a subspace of $X$ and $A$, $B \subseteq Y$. Then $A = B$ (in $Y$) iff $A = B$ (in $X$).

Theorem 1.11. Let $f: X \to Y$ be continuous and suppose that $A = B$ in $X$. Then $f[A] = f[B]$ in $Y$.

Theorem 1.12. In a space $X$, all nonempty closed sets are equivalent iff $O \neq X$, $O$ open implies that $O$ has no nonempty closed subsets.

Proof. Suppose that $X \neq O$, $O$ open and that $O \supseteq E \neq \emptyset$, with $E$ closed. Then $\partial O$ and $E$ are nonempty closed sets which are not equivalent. Conversely, suppose that $E \notin O \neq F$, $E$ and $F$ being closed and non equivalent sets. We may assume that $E \subseteq O$ and $F \subseteq O$ for some open set $O$. Then $O \neq X$ and $O$ has a nonempty closed subset.

Corollary 1.13. Let $\tau$ be a chain topology for $X$. Then $E = F$ if $E$ and $F$ are nonempty closed sets.
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Proof. By Theorem 1.12, it suffices to show that $O$ has no nonempty closed subset if $O$ is open and $O \neq X$. If $O \supseteq E \neq \emptyset$, $E$ closed, then $O$ and $CE$ are non comparable open sets and $\mathcal{S}$ is not a chain topology, a contradiction.

The converse of Corollary 1.13 is false as shown in

Example 1.14. Let $X = \{a, b, c, d\}$ with open sets $\mathcal{S} = \emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X$. Then $\mathcal{S}$ is not a chain topology for $X$, but all nonempty closed sets contain $d$ and the only open set which contains $d$ is $X$. Thus all nonempty closed sets are equivalent.

2. The algebra of equivalent sets

Theorem 2.1. In a space $X$, let $A_\alpha \equiv B_\alpha$ for each $\alpha \in \Delta$. Then (1) $\cup \{A_\alpha: \alpha \in \Delta\} \equiv \cup \{B_\alpha: \alpha \in \Delta\}$ and (2) for each $A \subseteq X$, $A^* = \cup \{B: B \equiv A\}$.

We omit the easy proof.

If $A \equiv B$, it does not generally follow that $A \cap C \equiv B \cap C$. However, we have

Theorem 2.2. In a space $X$, let $A \equiv B$ and let $E$ be a closed subset of $X$. Then $A \cap E \equiv B \cap E$.

Proof. Let $A \cap E \subseteq O$, $O$ being an open set. Then $A \subseteq O \cup CE$ and hence $B \subseteq O \cup CE$. Thus $B \cap E \subseteq O$.

Theorem 2.3. In a space $X$, let $A \equiv E$ and $A \equiv F$, $E$ being closed. Then $A \equiv E \cap F$.

Proof. If $A \subseteq O$, $O$ open, then $E \subseteq O$ and hence $E \cap F \subseteq O$. Conversely, let $E \cap F \subseteq O$. Then $F \subseteq O \cup CE$ and since $F \equiv E$, we have $E \subseteq O \cup CE$. Then $E \equiv O$ implies that $A \subseteq O$.

Corollary 2.4. In a space $X$, let $A \equiv E_i$, $i = 1, \cdots, n$ where each $E_i$ is closed. Then $A \equiv E_1 \cap \cdots \cap E_n$.

Theorem 2.5. Let $X = \times \{X_\alpha: \alpha \in \Delta\}$ and suppose that $A_\alpha \neq \emptyset \neq B_\alpha$ for each $\alpha \in \Delta$. Then $A_\alpha \equiv B_\alpha$ for each $\alpha \in \Delta$ iff $\times \{A_\alpha: \alpha \in \Delta\} \equiv \times \{B_\alpha: \alpha \in \Delta\}$.

Proof. If $\times \{A_\alpha: \alpha \in \Delta\} \equiv \times \{B_\alpha: \alpha \in \Delta\}$, then $A_\alpha \equiv B_\alpha$ for each $\alpha \in \Delta$ by Theorem 1.11. Conversely, let $A_\alpha \equiv B_\alpha$ for each $\alpha \in \Delta$ and suppose that $x \in \times \{A_\alpha: \alpha \in \Delta\}$. Then $x(\alpha) \in A_\alpha$ for all $\alpha \in \Delta$ and hence
c(x(\alpha)) \cap B_\alpha \neq \emptyset \text{ by Lemma 1.2. It follows that } c(x) \cap \{B_\alpha : \alpha \in \Delta\} \neq \emptyset.

3. Separation. $R_0$, $T_0$, $T_1$ spaces

**Definition 3.1.** A space $X$ is called an $R_0$ space iff $x \in O$, $O$ open implies that $c(x) \subseteq O$.

**Theorem 3.2.** A space $X$ is an $R_0$ space iff $c(x) \subseteq \{x\}^*$ for each $x \in X$ (see Definition 1.5).

**Proof.** If $X$ is an $R_0$ space, then $x \in O$, $O$ open implies that $c(x) \subseteq O$ and hence $c(x) \subseteq \{O : x \in O, O \text{ open}\} = \{x\}^*$. Conversely, let $x \in O$, $O$ open. Then $c(x) \subseteq \{x\}^* \subseteq O$ and hence $X$ is an $R_0$ space.

**Theorem 3.3.** A space $X$ is a $T_0$ space iff $x \neq y$ implies that $\{x\}^* \neq \{y\}^*$.

**Proof.** Let $X$ be a $T_0$ space and suppose that $x \neq y$. We may assume that $x \in O$, $O$ open and $y \in O$. Then $y \notin \{x\}^*$ and hence $\{y\}^* \neq \{x\}^*$. Conversely, suppose that $x \neq y$ implies that $\{x\}^* \neq \{y\}^*$. Let $x \neq y$ and assume that $\{x\}^* \subseteq \{y\}^*$; take $z \in \{x\}^*$ and $z \notin \{y\}^*$. There exists then an open set $O$ containing $y$ such that $z \notin O$. Then $x \notin O$ and $X$ is a $T_0$ space.

**Theorem 3.4.** $X$ is a $T_1$ space iff equivalence and equality coincide.

**Proof.** Let $X$ be a $T_1$ space and suppose that $A = B$, but $A \not\subseteq B$. Let $a \in A$, $a \notin B$; then $B \subseteq C \{a\}$, $\{a\} \subseteq$ is an open set, but $A \not\subseteq C \{a\}$, a contradiction.

Conversely, suppose that equality and equivalence coincide, but that $\{x\} \neq c(x)$ for some $x \in X$. Then $c(x) - \{x\} \neq c(x)$ and hence $c(x) - \{x\} \neq c(x)$. There exists then an open set $O$ such that $c(x) - \{x\} \subseteq O$, but $c(x) \not\subseteq O$ and hence $x \in CO$. It follows then that $c(x) \subseteq CO$, a contradiction.

4. Compactness

**Theorem 4.1.** In a space $X$, let $A = B$ and suppose that $A$ is compact (Lindelöf, countably compact). Then $B$ is compact (Lindelöf, countably compact).

**Theorem 4.2.** In a space $X$, let $A = B$ and suppose that $A$ is
sequentially compact. Then $B$ is sequentially compact.

Proof. Let $\{b_i\}$ be a sequence in $B$. By Lemma 1.2, $c(b_i) \cap A \neq \emptyset$ for each $i$; take $a_i \in c(b_i) \cap A$ for each $i$. Then there exists an $a \in A$ and a subsequence $\{a_{n_i}\}$ which converges to $a$. Let $b \in c(a) \cap B$. Then lim $b_{n_i} = b$. If $b \in O$, $O$ open, then $a \in O$ and hence $a_{n_i} \in O$ for all $i \geq N$. Then $b_{n_i} \in O$ for all $i \geq N$.

Theorem 4.3. In a space $X$, let $A$ be locally compact and $CA$ compact. If $A \equiv B$ and $B$ is closed, then $B$ is locally compact (see Theorem 10.3).

Proof. Let $b \in B$. By Lemma 1.2, $c(b) \cap A \neq \emptyset$; take $a \in c(b) \cap A$. Then $a \in O \cap A \subseteq M \subseteq A$ for some open set $O$ and some compact set $M$. Then $b \in O \cap B \subseteq B \cap (M \cup CA)$ and $B \cap (M \cup CA)$ is a compact subset of $B$.

5. Uniform spaces

Theorem 5.1. Let $(X, \mathcal{U})$ be a uniform space and $A \equiv B$ in $X$. If $A$ is complete, then $B$ is complete.

Proof. Let $S : D \to B$ be a Cauchy net. Then by Lemma 1.2, $c(S(d)) \cap A \neq \emptyset$ for each $d \in D$; let $a_d \in c(S(d)) \cap A$ for each $d \in D$ and let $T : D \to A$ via $T(d) = a_d$. Then $T : D \to A$ is a Cauchy net. To see this, let $U \in \mathcal{U}$, $U$ closed. Then $(S(d'), S(d'')) \subseteq U$ for all $d'$, $d'' \geq d^*$ and hence $(a_{d'}, a_{d''}) \in c(S(d'))$, $S(d'') \subseteq U$ for all $d'$, $d'' \geq d^*$. Since $A$ is complete, there exists an $a \in A$ such that lim $T = a$. Let $b \in c(a) \cap B$. Then lim $S = b$; for if $b \in O$, $O$ open, then $a \in O$ and hence $T(d) \in O$ for all $d \geq M$. It follows that $S(d) \in O$ for all $d \geq M$.

Theorem 5.2. Let $(X, \mathcal{U})$ be a uniform space and $A \equiv B$ in $X$. If $A$ is totally bounded, so is $B$.

Proof. Let $U \in \mathcal{U}$, $U$ open. Then there exist $a_i \in A$ such that $A \subseteq U[a_1] \cup \cdots \cup U[a_n]$. By Lemma 1.2, $c(a_i) \cap B \neq \emptyset$; take $b_i \in c(a_i) \cap B$. Then $B \subseteq U[a_1] \cup \cdots \cup U[a_n]$ since $U[a_i]$ is open. $U[a_i] \subseteq U[b_i]$ implies that $B \subseteq U[b_1] \cup \cdots \cup U[b_n]$.

6. $R_e$-spaces. Introduction

Theorem 6.1. Let $X$ be an $R_e$-space (see Example 3.1). If $A \equiv B$
and $A$ and $B$ are closed, then $A=B$ (see Example 1.4).

Proof. Let $a \in A$ and suppose that $a \notin B$. Then $a \in CB$ and hence $c(a) \subseteq CB$. Thus $c(a) \cap B = \emptyset$, contrary to Lemma 1.2.

**Theorem 6.2.** Let $X$ be an $R_e$-space and $A=B$ in $X$. If $A$ is dense, then $B$ is dense.

Proof. Let $O$ be a nonempty open set. Then $A \cap O \neq \emptyset$. Take $a \in A \cap O$. Then $c(a) \cap B \neq \emptyset$ by Lemma 1.2 and $c(a) \subseteq O$. Thus $O \cap B \subseteq c(a) \cap B \neq \emptyset$.

In Example 1.4, $\{b\} = X$, but $\{b\}$ is not dense.

**Theorem 6.3.** Let $X$ be an $R_e$-space and $A=B$ in $X$. If $O$ is an open set, then $A \cap O \equiv B \cap O$ (see Theorem 2.2).

Proof. Let $a \in A \cap O$. Then $c(a) \cap B \neq \emptyset$ by Lemma 1.2. But $c(a) \cap B \cap O = c(a) \cap B \cap O$. Using Lemma 1.2 again, $A \cap O \equiv B \cap O$.

In Example 1.4, let $O = \{a\}$. Then $\{b\} = X$, but $\{b\} \cap O \not\subseteq X \cap O$ and $O$ is open.

**Theorem 6.4.** Let $X$ be an $R_e$-space and $A=B$ in $X$. If each closed set in $A$ is a $G_\delta$ in $A$, then each closed set in $B$ is a $G_\delta$ in $B$.

Proof. Consider $B \cap E$ where $E$ is closed in $X$. Then $A \cap E$ is closed in $A$ and hence $A \cap E = \cap \{A \cap O_i : i \geq 1\}$ where each $O_i$ is open in $X$. It suffices to show that $B \cap E = \cap \{B \cap O_i : i \geq 1\}$. By Theorem 2.2, $B \cap E = A \cap E$ and since $A \cap E \subseteq O_i$ for each $i$, it follows that $B \cap E \subseteq O_i$ for each $i$ and thus $B \cap E \subseteq \cap \{B \cap O_i : i \geq 1\}$. Conversely, let $b \in B \cap O_i$, for each $i$; it suffices to show that $b \in E$. By Lemma 1.2, $c(b) \cap A \neq \emptyset$; take $a \in c(b) \cap A$. Then $a \in c(b) \subseteq O_i$, and hence $a \in A \cap E$. But $b \in c(a) \subseteq E$ and hence $b \in E$ (in an $R_e$-space, $a \in c(b)$ implies that $b \in c(a)$).

In Example 1.4, $\{b\} = X$ and $\{b\}$ has the property that each closed set in $\{b\}$ is a $G_\delta$ in $\{b\}$. In $X$, $\{b\}$ is a closed set which is not a $G_\delta$.

**Theorem 6.5.** Let $X$ be an $R_e$-space and $A \subseteq X$. Then $\cap \{c(a) : a \in A\}$ is the largest set which is equivalent to $A$ (see Theorem 1.8).

Proof. By Theorem 1.8, it suffices to show that $\cup \{c(a) : a \in A\} = A^*$ (see Definition 1.5). Now $a \in A$ implies $c(a) \subseteq O$ when $A \subseteq O$ and $O$ is open. Thus $\cup \{c(a) : a \in A\} \subseteq \cap \{O : A \subseteq O, O$ open$\} = A^*$. Suppose next that $x \not\in \cup \{c(a) : a \in A\}$. Then $x \in Oc(a)$ for each $a \in A$ and hence $c(x) \subseteq Oc(a)$ since $Oc(a)$ is an open set. It follows then that $c(x) \cap A = \emptyset$. 

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and $A \subseteq Cc(x)$ and $Cc(x)$ is an open set such that $x \not\in Cc(x)$. Thus $x \not\in A^*$.

In Example 1.4, $\{b\} \equiv X$, and $\bigcup \{ c(b) : b \in B \} = B$; but $B$ is not the largest set equivalent to $B$.

7. $R_e$-spaces. Connectedness

Theorem 7.1. Let $X$ be an $R_e$-space and let $A \equiv B$. If $A$ is connected, then $B$ is connected.

Proof. Suppose $B$ is disconnected. Then there exist open sets $O_1$ and $O_2$ such that $B = (O_1 \cap B) \cup (O_2 \cap B)$, $B \cap O_1 \cap O_2 = \emptyset$ and $B \cap O_1 \neq \emptyset \neq B \cap O_2$. Since $B \subseteq O_1 \cup O_2$, it follows that $A \subseteq O_1 \cup O_2$ and hence $A = (A \cap O_1) \cup (A \cap O_2)$. If $A \cap O_1 = \emptyset$, then $A \subseteq O_2$ which implies that $B \subseteq O_2$. Then $\emptyset \neq B \cap O_1 = B \cap O_1 \cap O_1$ and $B \cap O_1 \cap O_2 \neq \emptyset$, a contradiction. Thus $A \cap O_1 \neq \emptyset \neq A \cap O_2$. Since $A$ is connected, it follows that $A \cap O_1 \cap O_2 = \emptyset$; let $a \in A \cap O_1 \cap O_2$. Since $X$ is an $R_e$-space, $c(a) \cap B \subseteq O_1 \cap O_2 \cap B = \emptyset$ and thus $c(a) \cap B = \emptyset$. Thus $A \not\equiv B$ by Lemma 1.2, a contradiction.

Example 7.2. Let $X = \{ a, b, c \}$ with open sets $\emptyset$, $\{ a \}$, $\{ a, b \}$, $\{ a, c \}$, $X$. Then $\{ b, c \} \equiv X$, $X$ is connected and $\{ b, c \}$ is disconnected. Thus the $R_e$ condition cannot be removed from Theorem 7.1.

Theorem 7.3. Let $X$ be a space ($R_e$ not assumed here) and $A \equiv B$ with $A \subseteq B$. If $A$ is connected, so is $B$.

Proof. Suppose that $B = (B \cap O_1) \cup (B \cap O_2)$ where $O_1$ is open and $B \cap O_1 \neq \emptyset \neq B \cap O_2$ and $B \cap O_1 \cap O_2 = \emptyset$. Now $A \subseteq B \subseteq O_1 \cup O_2$ and hence $A = (A \cap O_1) \cup (A \cap O_2)$. If $A \cap O_1 = \emptyset$, then $A \subseteq O_2$ and hence $B \subseteq O_2$; thus $B \cap O_1 \cap O_2 = B \cap O_1 \neq \emptyset \neq B \cap O_1 \cap O_2$, a contradiction. Thus $A \cap O_1 \neq \emptyset \neq A \cap O_2$. But $A \cap O_1 \cap O_2 \subseteq A \cap O_2 = \emptyset$ and hence $A$ is disconnected, a contradiction.

Note that in Theorem 7.3, if we assume that $B$ is connected, we cannot deduce that $A$ is connected (see Example 7.2). Note also in Example 7.2 that $X$ is path connected while $\{ b, c \}$ is not.

Lemma 7.4. Let $X$ be an $R_e$-space and suppose that $f : [0, 1] \rightarrow X$ is continuous. Then $g : [0, 1] \rightarrow X$ is continuous if $g(t) \in c(f(t))$ for all $t \in [0, 1]$.

Proof. Let $E \subseteq X$, $E$ closed. It suffices to show that $g^{-1}[E] = f^{-1}[E]$. Now $t \in g^{-1}[E]$ if $g(t) \in E$ iff $c(g(t)) \subseteq E$ iff $c(f(t)) \subseteq E$ iff $f(t) \in E$. 

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iff \( t \in f^{-1}[E] \). Note that in an \( R_\varepsilon \)-space \( X \), \( x \in c(y) \) implies that \( y \in c(x) \).

**Theorem 7.5.** Let \( X \) be an \( R_\varepsilon \)-space and let \( A \equiv B \). If \( A \) is path connected, then \( B \) is path connected.

**Proof.** Let \( b_1, b_2 \in B \). Take \( a_1 \in c(b_1) \cap A \) and \( a_2 \in c(b_2) \cap A \). There exists a continuous map \( f : [0, 1] \to A \) such that \( f(0) = a_1 \) and \( f(1) = a_2 \). Let \( g : [0, 1] \to B \) as follows: \( g(0) = b_1 \) and \( g(1) = b_2 \). \( g(t) \in c(f(t)) \cap B \) for \( 0 < t < 1 \). By Lemma 7.4, \( g \) is continuous on \( B \).

**Theorem 7.6.** In an \( R_\varepsilon \)-space \( X \), let \( A \equiv B \). If \( A \) is locally connected, then \( B \) is locally connected.

**Proof.** Let \( b \in O \cap B \), \( O \) being open in \( X \). By Lemma 1.2, \( c(b) \cap A \neq \emptyset \); let \( a \in c(b) \cap A \). Then \( a \in c(b) \cap A \subseteq O \cap A \). Hence there exists a set \( O^* \) open in \( X \) such that \( a \in O^* \cap A \subseteq O \cap A \) and \( O^* \cap A \) is connected. Now \( b \in O^* \cap B \) and \( O^* \cap A \equiv O^* \cap B \) by Theorem 6.3 and hence \( O^* \cap B \) is connected by Theorem 7.1. It suffices then to show that \( O^* \cap B \subseteq O \cap B \). Let \( x \in O^* \cap B \); then \( c(x) \subseteq O^* \) and \( c(x) \cap A \neq \emptyset \). Let \( y \in c(x) \cap A \). Then \( y \in O^* \cap A \subseteq O \cap A \) and hence \( x \in c(y) \subseteq O \). Thus \( x \in O \cap B \).

**Example 7.7.** Let \( (X, \mathcal{T}) \) be the rationals with the usual topology and \( y \notin X \); let \( Y = X \cup \{ y \} \). Let \( \mathcal{U} = \mathcal{T} \cup \{ Y \} \). Then \( \{ y \} \equiv Y, \{ y \} \) is locally connected, but \( Y \) is not. Note that \( Y \) is not an \( R_\varepsilon \)-space.

8. \( R_\varepsilon \) separation

**Theorem 8.1.** Let \( X \) be an \( R_\varepsilon \)-space and suppose that \( A \equiv B \). If \( A \) is regular, then \( B \) is regular.

**Proof.** Let \( b \in O \cap B \), \( O \) being open in \( X \). Then \( c(b) \cap A \neq \emptyset \) by Lemma 1.2; take \( a \in c(b) \cap A \). Then \( a \in O \cap A \) and hence there exists an open set \( O^* \) and a closed set \( E \) such that \( a \in O^* \cap A \subseteq E \cap A \subseteq O \cap A \). It is easy to show that \( b \in O^* \cap B \subseteq E \cap B \subseteq O \cap B \).

**Lemma 8.2.** Let \( X \) and \( Y \) be spaces, \( X \) being an \( R_\varepsilon \)-space. Suppose that \( A \equiv B \) in \( X \) and that \( f : A \to Y \) is continuous. Let \( g : B \to Y \) be defined as follows: for \( b \in B \), let \( g(b) \in f[c(b) \cap A] \). Then \( g : B \to Y \) is continuous.

**Proof.** Let \( E \subseteq Y \), \( E \) closed. Then \( f^{-1}[E] = A \cap F \) for some closed set \( F \). It suffices to show that \( g^{-1}[E] = B \cap F \) or that \( g^{-1}[E] \subseteq F \). Let \( b \in g^{-1}[E] \). Then \( g(b) \in E \) and \( g(b) \in f[c(b) \cap A] \) or \( g(b) = f(a) \) where \( a \in c(b) \cap A \). Then \( f(a) \in E \) and hence \( a \in f^{-1}[E] \subseteq F \). \( b \in c(a) \subseteq F \).
Theorem 8.3. Let \( A \equiv B \) in an \( R_0 \)-space \( X \). If \( A \) is completely regular, then \( B \) is completely regular.

Proof. Let \( b \in O \cap B \), \( O \) being open in \( X \). By Lemma 1.2, \( c(b) \cap A \neq \emptyset \); take \( a \in c(b) \cap A \). It follows that \( a \in O \cap A \). Since \( A \) is completely regular, there exists a continuous map \( f : A \to [0, 1] \) such that \( f(a) = 0 \) and \( f(a^*) = 1 \) for all \( a^* \in A - O \). Let \( g : B \to [0, 1] \) be as in Lemma 8.2, \( g(b) \) being taken as \( f(a) \). Then \( g(b) = f(a) = 0 \). Now let \( b^* \in B - O \). Then \( c(b^*) \subseteq O \) and \( a^* \in c(b^*) \cap A \) which implies that \( a^* \in A - O \) and thus \( g(b^*) = f(a^*) = 1 \). \( g : B \to [0, 1] \) is continuous by Lemma 8.2.

In Example 1.4, \( \{ b \} \equiv X \), \( \{ b \} \) is completely regular, but \( X \) is not completely regular.

Theorem 8.4. Let \( A \equiv B \) in an \( R_0 \)-space \( X \). If \( A \) is normal, then \( B \) is normal.

Proof. Let \( B \cap F \cap E = \emptyset \), \( E \) and \( F \) being closed in \( X \). Then \( B \subseteq C(E \cap F) \), an open set, and hence \( A \subseteq C(E \cap F) \) and \( A \cap E \cap F = \emptyset \).

Since \( A \) is normal, there exist open sets \( O_i \) and \( O_2 \) in \( X \) such that \( A \cap E \subseteq A \cap O_1 \) and \( A \cap E \cap A \cap O_2 \) and \( A \cap O_1 \cap O_2 = \emptyset \). Applying Theorem 2.2, it follows that \( B \cap E \subseteq B \cap O_1 \) and \( B \cap F \subseteq B \cap O_2 \). It remains to show that \( B \cap O_1 \cap O_2 = \emptyset \). Suppose \( b \in B \cap O_1 \cap O_2 \), then \( a \in c(b) \cap A \); \( a \in O_i \cap O_2 \cap A \), a contradiction.

In Example 7.2, \( \{ b, c \} \equiv X \), \( \{ b, c \} \) is normal, but \( X \) is not.

Theorem 8.5. Let \( A \) be a completely normal subspace of an \( R_0 \)-space \( X \). Then \( \cup \{ c(a) : a \in A \} \) is completely normal.

Proof. Let \( B \subseteq \cup \{ c(a) : a \in A \} \). We must show that \( B \) is normal. Let \( A^k = \{ a : c(a) \cap B = \emptyset , \ a \in A \} \). It suffices to show that \( \cup \{ c(b) : b \in B \} = \cup \{ c(a) : a \in A^k \} \) since \( A^k \) is normal and is equivalent to \( \cup \{ c(a) : a \in A^k \} \) and \( B \) is equivalent to \( \cup \{ c(b) : b \in B \} \) (see Theorem 8.4). Let \( x \in c(b) \) for some \( b \in B \). Then \( b \in c(a) \) for some \( a \in A \). Then \( x \in c(b) \subseteq c(a) \) and hence \( a \in A^k \). Thus \( x \in \cup \{ c(a) : a \in A^k \} \). Conversely, let \( y \in c(a) \) for some \( a \in A^k \). Then \( c(a) \cap B = \emptyset \); let \( b \in c(a) \cap B \). They \( y \in c(a) \subseteq c(b) \).

Corollary 8.6. In an \( R_0 \)-space \( X \), let \( A \equiv B \), \( A \) being completely normal. Then \( B \) is completely normal.

Proof. By Theorem 8.5, \( \cup \{ c(a) : a \in A \} \) is completely normal and by Theorem 6.5, \( B \subseteq \cup \{ c(a) : a \in A \} \). Hence \( B \) is completely normal.
9. $R_0$ and conutability

**Theorem 9.1.** Let $A \equiv B$ in an $R_0$-space $X$. If $A$ is separable, then $B$ is separable.

**Proof.** Let $\{a_n : n \geq 1\}$ be dense in $A$. Take $b_n \in c(a_n) \cap B$ for each $n \geq 1$. Then $\{b_n : n \geq 1\}$ is dense in $B$; let $\emptyset \neq O \cap B$ were $O$ is open in $X$. Choose $b \in O \cap B$ and let $a \in c(b) \cap A$. Then $a \in O \cap A$. Since $O \cap A \neq \emptyset$, $a_n \in O \cap A$ for some $n$. It follows then that $b_n \in O \cap B$.

**Example 9.2.** Let $(X, \mathcal{S})$ be an uncountable discrete space and $y \notin X$; let $Y = X \cup \{y\}$ and $U = \mathcal{S} \cup \{Y\}$. Then $\{y\} \equiv Y$, $\{y\}$ is separable, but $Y$ is not separable.

**Theorem 9.3.** In an $R_0$-space $X$, let $A \equiv B$ and let $A$ be a second axiom space. Then $B$ is second axiom.

**Proof.** If $\{A \cap O_i : i \geq 1, O_i \text{open in } X\}$ is a base for $A \cap \mathcal{S}$, then $\{B \cap O_i : i \geq 1\}$ is a base for $B \cap \mathcal{S}$.

In Example 9.2, $\{y\}$ is second axiom, but $Y$ is not.

10. $R_0$ and local compactness, paracompactness

**Lemma 10.1.** In an $R_0$-space $X$, let $A$ be locally compact. Then $\bigcup \{c(a) : a \in A\}$ is locally compact.

**Proof.** Let $x \in \bigcup \{c(a) : a \in A\}$. Then $x \in c(a^*)$ for some $a^* \in A$ and hence there exists an open set $O$ and a compact set $M$ such that $a^* \in O \cap A \subseteq M \subseteq A$. Then $x \in O \cup \{c(a) : a \in A\} \subseteq \bigcup \{c(m) : m \in M\} \subseteq \bigcup \{c(a) : a \in A\}$. Now $M = \bigcup \{c(m) : m \in M\}$ and since $M$ is compact, so is $\bigcup \{c(m) : m \in M\}$ (see Theorems 6.5 and 4.1).

**Lemma 10.2.** In an $R_0$-space $X$, $A$ is locally compact if $\bigcup \{c(a) : a \in A\}$ is locally compact.

**Proof.** Let $a^* \in A$; there exists an open set $O$ and a compact set $M$ such that $a^* \in O \cup \{c(a) : a \in A\} \subseteq M \subseteq \bigcup \{c(a) : a \in A\}$. Then $a^* \in O \cap A \subseteq A \cup \{c(m) : m \in M\} \subseteq A$. We need only show that $A \cap \bigcup \{c(m) : m \in M\}$ is compact. By Theorem 4.1 and Theorem 6.5, it suffices to show that $\bigcup \{c(a') : a' \in A \cap \bigcup \{c(m) : m \in M\}\}$ is compact. The reader can easily verify that this set is merely $\bigcup \{c(m) : m \in M\}$ which is compact.

**Theorem 10.3.** In an $R_0$-space, let $A \equiv B$. If $A$ is locally compact, then $B$ is locally compact (see Theorem 4.3).
Theorem 10.4. In an $R$-space $X$, let $A \equiv B$ and suppose that $A$ is paracompact. Then $B$ is paracompact.

Proof. Suppose that $B = B \cap \bigcup \{O_\alpha : \alpha \in \Delta\}$ where each $O_\alpha$ is open in $X$. Then $B \subseteq \bigcup \{O_\alpha : \alpha \in \Delta\}$ and hence $A \subseteq \bigcup \{O_\alpha : \alpha \in \Delta\}$. There exists then a family of open sets $\{O_\gamma : \gamma \in \Gamma\}$ such that $A = \bigcup \{A \cap O_\gamma : \gamma \in \Gamma\}$, $\{A \cap O_\gamma : \gamma \in \Gamma\}$ is locally finite in $A$ and $\{A \cap O_\gamma : \gamma \in \Gamma\}$ refines $\{A \cap O_\alpha : \alpha \in \Delta\}$. Thus $B = \bigcup \{B \cap O_\gamma : \gamma \in \Gamma\}$, $\{B \cap O_\gamma : \gamma \in \Gamma\}$ is locally finite in $B$ and $\{B \cap O_\gamma : \gamma \in \Gamma\}$ refines $\{B \cap O_\alpha : \alpha \in \Delta\}$. The details are left to the reader.

REFERENCES