Note on results of K. Motose

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Throughout the present note, $R$ will represent a ring with 1, and $\overline{R}$ the residue class ring of $R$ modulo its (Jacobson) radical $J(R)$. Further, $G$ will represent a group, and $RG$ the group ring of $G$ over $R$. Occasionally, we consider the following ring epimorphisms:

$$
\phi_\sigma : RG \rightarrow R \left( \sum_{\sigma \in G} r_\sigma \sigma \rightarrow \sum_{\sigma \in G} r_\sigma \right),
$$

$$
\psi_\sigma : RG \rightarrow \overline{R} \left( \sum_{\sigma \in G} r_\sigma \sigma \rightarrow \sum_{\sigma \in G} \overline{r_\sigma} \sigma \right).
$$

More generally, if $H$ is a normal subgroup and $G^* = G/H$ then we can consider the following ring epimorphisms:

$$
\phi^*_\sigma : RG \rightarrow RG^* \left( \sum_{\sigma \in G} r_\sigma \sigma \rightarrow \sum_{\sigma \in G} r_\sigma \sigma^* \right),
$$

$$
\psi^*_\sigma : RG \rightarrow \overline{RG}^* \left( \sum_{\sigma \in G} r_\sigma \sigma \rightarrow \sum_{\sigma \in G} \overline{r_\sigma} \sigma^* \right),
$$

where $\sigma^*$ is the residue class of $\sigma$ modulo $H$.

In what follows, we shall show that all the results in [3] are still valid without assuming that $R$ is semi-primary. As to notations and terminologies used here without mention, we follow [3].

First, we shall prove a slight modification of [2; Lemma 2].

**Lemma 1.** Let $G$ be a finite $p$-group $|G|=p^n$. If $J(R)=0$ and $pR=0$ then $J(RG)=\text{Ker} \phi_\sigma$ and $(J(RG))^{p^n}=0$.

**Proof.** Evidently, $\text{Ker} \phi_\sigma \supseteq J(RG)$. It remains therefore to prove $(\text{Ker} \phi_\sigma)^{p^n}=0$. We shall prove this by making use of the induction with respect to $n$. In case $n=1$, it is easy to see that $(\text{Ker} \phi_\sigma)^p=(\sum_{\sigma \in G} R(1-\sigma))^p=0$. Next, suppose $n>1$ and that our assertion is true for $n-1$. Choose a normal subgroup $H$ of $G$ such that $(G:H)=p$, and set $G^*=G/H$. Then, by the case $n=1$, we have $\phi^{p^n}_H(\text{Ker} \phi_\sigma)^p=(\text{Ker} \phi_\sigma)^{p^n}=0$, whence it follows $(\text{Ker} \phi_\sigma)^{p^n}=\text{Ker} \phi^{p^n}_H$. Accordingly, by the induction hypothesis, $(\text{Ker} \phi_H)^{p^n}=(\text{Ker} \phi_H)^{p^n-1}=\overline{(\text{Ker} \phi_H)^{p^n-1} G}=0$.

Now, by the light of Lemma 1, we can prove [3; Th. 2] without assuming that $R$ is semi-primary.

**Theorem 1.** Let $G$ be a locally finite $p$-group. If $p\overline{R}=0$ then $J(RG)=\text{Ker} \phi_H$.
Proof. It is enough to prove that \( \text{Ker } \psi_C \subseteq J(\mathcal{R}G) \). Let \( x = \sum r_i \sigma_i \) be an arbitrary element of \( \text{Ker } \psi_C \), where \( r_i \in \mathcal{R} \) and \( \sigma_i \in G \). We set \( K = \langle \sigma_1, \ldots, \sigma_n \rangle \). Then, \( K \) is a finite \( p \)-group and \( x \) is in \( \text{Ker } \psi_K \). By Lemma 1, we have \( \psi_K(\text{Ker } \psi_K) \subseteq J(\mathcal{R}K) \). Hence, \( (\text{Ker } \psi_K)^{\mathcal{R}1} \subseteq \text{Ker } \psi_K = J(\mathcal{R}K) \subseteq J(\mathcal{R}K) \), whence it follows that \( x \) is quasi-regular.

Finally, we shall present the following which contains [3; Ths. 1 and 3] and [2; Th. 2].

Theorem 2. Let \( pR = 0 \). If \( H \) is a normal subgroup of \( G \) such that \( G/H \) is a locally finite \( p' \)-group then \( J(\mathcal{R}G) = J(\mathcal{R}H) G \).

Proof. As \( J(\mathcal{R}H) G \subseteq J(\mathcal{R}G) \) by [3; Lemma 1], it remains only to prove the converse inclusion. Let \( x = \sum a_i \sigma_i \) be an arbitrary element of \( J(\mathcal{R}G) \), where \( a_i \in RH \) and \( \sigma_i \in G \). Let \( K = \langle H, \sigma_1, \ldots, \sigma_n \rangle \). Then, it is evident that \( K/H \) is a finite \( p' \)-group. Now, patternning after the proof of [4; Prop. 1.5], one will easily see that \( J(\mathcal{R}K) = J(\mathcal{R}H) K \). Combining this with [1; Prop. 9], we obtain

\[ x \in J(\mathcal{R}G) \cap \mathcal{R}K \subseteq J(\mathcal{R}K) = J(\mathcal{R}H) K \subseteq J(\mathcal{R}H) G, \]

completing the proof.

References


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(Received April 15, 1972)